Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2394–2409
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Strong convergence of an iterative algorithm for accretive operators and nonexpansive mappings

Jong Soo Jung

Department of Mathematics, Dong-A University, Busan 49315, Korea.

Communicated by Y. J. Cho

Abstract

In this paper, an iterative algorithm for finding a common point of the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping is considered in a uniformly convex Banach space having a weakly continuous duality mapping. Under suitable control conditions, strong convergence of the sequence generated by proposed algorithm to a common point of two sets is established. The main theorems develop and complement the recent results announced by researchers in this area.

Keywords: Iterative algorithm, accretive operator, resolvent, zeros, nonexpansive mappings, fixed points, variational inequality, weakly continuous duality mapping, uniformly convex, contractive mapping, weakly contractive mapping.
2010 MSC: 47H06, 47H09, 47H10, 47J25, 49M05, 65J15.

1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$ and the dual space E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^* \rangle$ and the normalized duality mapping $\mathcal{J}$ from E into 2^{E^*} is defined by

$$\mathcal{J}(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x\| = \|x^*\|\}, \quad \forall x \in E.$$

Recall that a (possibly multivalued) operator $A \subset E \times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if , for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,\ 2)$, there exists a $j \in \mathcal{J}(x_1 - x_2)$ such that

Email address: `jungjs@dau.ac.kr` (Jong Soo Jung)

Received 2015-11-16

$\langle y_1 - y_2, j\rangle \geq 0$. (Here $\mathcal{J}$ is the normalized duality mapping.) In a Hilbert space, an accretive operator is also called monotone operator. The set of zero of A is denoted by $A^{-1}0$, that is,

$$A^{-1}0 := \{z \in D(A) : 0 \in Az\}.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Ax$ is solvable.

Iterative methods has extensively been studied over the last forty years for constructions of zeros of accretive operators (see, for instance, [4, 5, 6, 12, 13, 15, 17] and the references therein). In particular, in order to find a zero of an accretive operator, Rockafellar [17] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal point algorithm: for any initial point $x_0 \in E$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}(x_n + e_n), \quad \forall n \geq 0,$$

where $J_r = (I + rA)^{-1}$ for all $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in a Hilbert space E. Bruck [6] proposed the following iterative algorithm in a Hilbert space E: for any fixed point $u \in E$,

$$x_{n+1} = J_{r_n}(u). \ \ \forall n \geq 0.$$

Xu [23] in 2006 and Song and Yang [20] in 2009 obtained the strong convergence of the following regularization method for Rockafellar's proximal point algorithm in a Hilbert space E: for any initial point $x_0 \in E$

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n + e_n), \quad \forall n \geq 0, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset E$ and $\{r_n\} \subset (0,\infty)$. In 2009, Song [18] introduced an iterative algorithm for finding a zero of an accretive operator A in a reflexive Banach space E with a uniformly Gâteaux differentiable norm satisfying that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings: for any initial point $x_0 \in E$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n), \quad \forall n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $\{r_n\} \subset (0,\infty)$. Zhang and Song [24] considered the iterative method (1.1) for finding a zero of an accretive operator A in a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm (or with a weakly sequentially continuous normalized duality mapping $\mathcal{J}$). In order to obtain strong convergence of the sequence generated by algorithm (1.1) to a zero of an accretive operator A together with weaker conditions on $\{\beta_n\}$ and $\{r_n\}$ than ones in [18], they used the well-known inequality in uniformly convex Banach spaces (see Xu [21]). In 2013, Jung [10] extended the results of [18, 24] to viscosity iterative algorithms along with different conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$. Very recently, Jung [11] introduced the following iterative algorithm for finding a common point of the set of zeros of accretive operator A and the set of fixed points of a nonexpansive mapping S in a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1 - \alpha_n)Sx_n), \quad \forall n \geq 0, \tag{1.3}$$

where $x_0 \in C$, which is a closed convex subset of E; $f : C \to C$ is a contractive mapping; and $\{\alpha_n\} \subset (0, 1)$; $\{r_n\} \subset (0,\infty)$.

In this paper, as a continuation of study in this direction, we consider the iterative algorithm (1.3) for finding a common point in $A^{-1}0 \cap Fix(S)$ in a uniformly convex Banach space E having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ, where $A^{-1}0$ is the set of zeros of an accretive operator A and $Fix(S)$ is the fixed point set of a nonexpansive mapping S. Under suitable control conditions, we prove that the sequence generated by proposed iterative algorithm converges strongly to a common point in $A^{-1}0 \cap Fix(S)$, which is a solution of a certain variational inequality. As an application, we study the iterative algorithm (1.3) with a weak contractive mapping. The main results improve, develop and supplement the corresponding results of Song [18], Zhang and Song [24], Jung [10, 11] and Song et al [19], and some recent results in the literature.

2. Preliminaries and lemmas

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be its dual. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Recall that a mapping $f : E \to E$ is said to be *contractive* on E if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x,\ y \in E$. An accretive operator A is said to satisfy *the range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where I is an identity operator of E and $\overline{D(A)}$ denotes the closure of the domain $D(A)$ of A. An accretive operator A is called *m-accretive* if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the resolvent of A. We know that J_r is nonexpansive (i.e., $\|J_r x - J_r y\| \leq \|x - y\|$, $\forall x, y \in R(I + rA)$) and $A^{-1}0 = Fix(J_r) = \{x \in D(J_r) : J_r x = x\}$ for all $r > 0$. Moreover, for $r > 0$, $t > 0$ and $x \in E$,

$$J_r x = J_t\left(\frac{t}{r}x + \left(1 - \frac{t}{r}\right)J_r x\right), \tag{2.1}$$

which is referred to as the *Resolvent Identity* (see [1, 7], where more details on accretive operators can be found).

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space.

A Banach space E is said to be *uniformly convex* if for all $\varepsilon \in [0, 2]$, there exists $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

Let $q > 1$ and $M > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g; [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|), \tag{2.2}$$

for all $x,\ y \in B_M(0) = \{x \in E : \|x\| \leq M\}$. For more detail, see Xu [21].

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $\mathcal{J}_\varphi : E \to 2^{E^*}$ defined by

$$\mathcal{J}_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}, \quad \forall x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by $\mathcal{J}$, is referred to as the *normalized duality mapping.* The following property of duality mapping is well-known ([7]):

$$\mathcal{J}_\varphi(\lambda x) = \text{sign}\lambda\left(\frac{\varphi(|\lambda| \cdot \|x\|)}{\|x\|}\right)\mathcal{J}(x), \quad \forall x \in E \setminus 0, \quad \lambda \in \mathbb{R},$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $\mathcal{J}(-x) = -\mathcal{J}(x)$, $\forall x \in E$. It is known that E is smooth if and only if the normalized duality mapping $\mathcal{J}$ is single-valued.

We say that a Banach space E has a weakly continuous duality mapping if there exists a gauge function φ such that the duality mapping $\mathcal{J}_\varphi$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $\mathcal{J}_\varphi(x_n) \overset{*}{\rightharpoonup} \mathcal{J}_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$ ([1, 7]). Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0.$$

Then for $0 < k < 1$, $\varphi(kx) \le \varphi(x)$,

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \le k\int_0^t \varphi(x)dx = k\Phi(t),$$

and moreover

$$\mathcal{J}_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E,$$

where ∂ denotes the subdifferential in the sense convex analysis, i.e., $\partial\Phi(\|x\|) = \{x^* \in E^* : \Phi(\|y\|) \ge \Phi(\|x\|) + \langle x^*, y - x\rangle, \ \forall y \in E\}$.

We need the following lemmas for the proof of our main results. We refer to [1, 7] for Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *Let E be a real Banach space, and let φ be a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \in \mathbb{R}^+.$$

Then the following inequality holds:

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \quad \forall x\ y \in E,$$

where $j_\varphi(x + y) \in \mathcal{J}_\varphi(x + y)$. In particular, if E is smooth, then one has

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, \mathcal{J}(x + y)\rangle, \quad \forall x,\ y \in E.$$

Lemma 2.2 (Demiclosedness principle)**.** *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ, let C be a nonempty closed convex subset of E, and let $S : C \to E$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I - S)x_n \to y$ imply that $x \in C$ and $(I - S)x = y$.*

Lemma 2.3 ([14, 22])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^\infty \lambda_n = \infty$*;*
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^\infty \lambda_n|\delta_n| < \infty$;
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^\infty \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Recall that a mapping $g : C \to C$ is said to be *weakly contractive* ([2]) if

$$\|g(x) - g(y)\| \le \|x - y\| - \psi(\|x - y\|), \quad \text{for all } x,\ y \in C,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0, \infty)$ and $\psi(0) = 0$. As a special case, if $\psi(t) = (1 - k)t$ for $t \in [0, +\infty)$, where $k \in (0, 1)$, then the weakly contractive mapping g is a contraction with constant k. Rhodes [16] obtained the following result for the weakly contractive mapping (see also [2]).

Lemma 2.4 ([16])**.** *Let (X, d) be a complete metric space and g be a weakly contractive mapping on X. Then g has a unique fixed point p in X.*

The following Lemma was given in [3].

Lemma 2.5 ([3]). *Let $\{s_n\}$ and $\{\gamma_n\}$ be two sequences of nonnegative real numbers, and let $\{\lambda_n\}$ be a sequence of positive numbers satisfying the conditions:*

(i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\gamma_n}{\lambda_n} = 0$.

Let the recursive inequality

$$s_{n+1} \le s_n - \lambda_n \psi(s_n) + \gamma_n, \quad n \ge 0,$$

be given, where $\psi(t)$ is a continuous and strict increasing function on $[0,\infty)$ with $\psi(0) = 0$. Then $\lim_{n\to\infty} s_n = 0$.

3. Iterative algorithms

Let E be a real Banach space, let C be a nonempty closed convex subset of E, let $A \subset E \times E$ be an accretive operator in E such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$, and let J_r be the resolvent of A for each $r > 0$. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \cap A^{-1}0 \ne \emptyset$, and let $f : C \to C$ be a contractive mapping with a constant $k \in (0, 1)$.

In this section, first we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = J_r(tfx_t + (1-t)Sx_t). \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a point q in $A^{-1}0 \cap Fix(S)$ which is a solution of the following variational inequality:

$$\langle (I - f)q, \mathcal{J}_\varphi(q - p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap Fix(S). \tag{3.2}$$

We also propose the following algorithm which generates a sequence in an explicit way:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \ge 0, \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0,\infty)$ and $x_0 \in C$ is an arbitrary initial guess, and establish the strong convergence of this sequence to a point q in $A^{-1}0 \cap Fix(S)$, which is also a solution of the variational inequality (3.2).

3.1. Strong convergence of the implicit algorithm

Now, for $t \in (0, 1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = J_r(tfx + (1-t)Sx), \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with a constant $1 - (1-k)t$. Indeed, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\le t\|fx - fy\| + \|(1-t)Sx - (I-t)Sy\| \\ &\le tk\|x - y\| + (1-t)\|x - y\| \\ &= (1 - (1-k)t)\|x - y\|. \end{aligned}$$

Hence Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

The following proposition about the basic properties of $\{x_t\}$ and $\{y_t\}$ was given in [11], where $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0, 1)$. We include its proof for the sake of completeness.

Proposition 3.1 ([11]). *Let E be a real uniformly convex Banach space, let C be a nonempty closed convex subset of E, let $A \subset E \times E$ be an accretive operator in E such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$, and let J_r be the resolvent of A for each $r > 0$. Let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \cap A^{-1}0 \ne \emptyset$, and let $f : C \to C$ be a contractive mapping with a constant $k \in (0, 1)$. Let the net $\{x_t\}$ be defined via (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0, 1)$. Then*

(1) *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*

(2) *x_t defines a continuous path from $(0,1)$ in C and so does y_t;*

(3) $\lim_{t\to 0}\|y_t - Sx_t\| = 0$*;*

(4) $\lim_{t\to 0}\|y_t - J_r y_t\| = 0$*;*

(5) $\lim_{t\to 0}\|x_t - y_t\| = 0$*;*

(6) $\lim_{t\to 0}\|y_t - Sy_t\| = 0$.

Proof. (1) Let $p \in Fix(S) \cap A^{-1}0$. Observing $p = Sp = J_r p$, we have

$$\begin{aligned}\|x_t - p\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r p\| = \|Sy_t - Sp\| \\ &\le \|y_t - p\| \\ &= \|t(fx_t - fp) + t(fp - p) + (1-t)(Sx_t - Sp)\| \\ &\le tk\|x_t - p\| + t\|fp - p\| + (1-t)\|x_t - p\|.\end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|fp - p\|}{1-k} \text{ and } \|y_t - p\| \le \frac{\|fp - p\|}{1-k}.$$

Hence $\{x_t\}$ and $\{y_t\}$ are bounded and so are $\{fx_t\}$, $\{Sx_t\}$, $\{J_r x_t\}$, $\{Sy_t\}$ and $\{J_r y_t\}$.

(2) Let $t,\ t_0 \in (0,1)$ and calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r(t_0 fx_{t_0} + (1-t_0)Sx_{t_0})\| \\ &\le \|(t-t_0)fx_t + t_0(fx_t - fx_{t_0}) \\ &\qquad - (t-t_0)Sx_t + (1-t_0)Sx_t - (1-t_0)J_r x_{t_0}\| \\ &\le |t-t_0|\|fx_t\| + t_0 k\|x_t - x_{t_0}\| \\ &\qquad + |t-t_0|\|Sx_t\| + (1-t_0)\|x_t - x_{t_0}\|.\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|.$$

This show that x_t is locally Lipschitzian and hence continuous. Also we have

$$\|y_t - y_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|,$$

and hence y_t is a continuous path.

(3) By the boundedness of $\{fx_t\}$ and $\{J_r x_t\}$ in (1), we have

$$\begin{aligned}\|y_t - Sx_t\| &= \|tfx_t + (1-t)Sx_t - Sx_t\| \\ &\le t\|fx_t - Sx_t\| \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

(4) Let $p \in Fix(S)\bigcap A^{-1}0$. Then it follows from Resolvent Identity (2.1) that

$$J_r y_t = J_{\frac{r}{2}}(\frac{1}{2}y_t + \frac{1}{2}J_r y_t).$$

Then we have

$$\|J_r y_t - p\| = \|J_{\frac{r}{2}}(\frac{1}{2}y_t + \frac{1}{2}J_r y_t) - p\| \le \|\frac{1}{2}(y_t - p) + \frac{1}{2}(J_r y_t - p)\|.$$

By the inequality (2.2) ($q = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned} \|J_r y_t - p\|^2 &\le \|J_{\frac{r}{2}}(\frac{1}{2}y_t + \frac{1}{2}J_r y_t) - p\|^2 \\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|J_r y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &= \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|). \end{aligned} \tag{3.4}$$

Thus, from (3.1), the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.4) we have

$$\begin{aligned} \|x_t - p\|^2 &= \|J_r y_t - p\|^2 \\ &\le \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &= \|t(fx_t - p) + (1-t)(Sx_t - p)\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &\le t\|fx_t - p\|^2 + (1-t)\|x_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|), \end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_t - J_r y_t\|)) \le t(\|fx_t - p\|^2 - \|x_t - p\|^2).$$

By boundedness of $\{fx_t\}$ and $\{x_t\}$, letting $t \to 0$ yields

$$\lim_{t\to 0} g(\|y_t - J_r y_t\|) = 0.$$

Thus, from the property of the function g in (2.2) it follows that

$$\lim_{t\to 0} \|y_t - J_r y_t\| = 0.$$

(5) By (4), we have

$$\|x_t - y_t\| \le \|x_t - J_r y_t\| + \|J_r y_t - y_t\| = \|J_r y_t - y_t\| \to 0 \quad (t \to 0).$$

(6) By (3) and (5), we have

$$\begin{aligned} \|y_t - Sy_t\| &\le \|y_t - Sx_t\| + \|Sx_t - Sy_t\| \\ &\le \|y_t - Sx_t\| + \|x_t - y_t\| \to 0 \quad (t \to 0). \end{aligned}$$

□

We establish strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. *Let E be a real uniformly convex Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ, let C be a nonempty closed convex subset of E, let $A \subset E \times E$ be an accretive operator in E such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$, and let J_r be the resolvent of A for each $r > 0$. Let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \cap A^{-1}0 \ne \emptyset$, and let $f : C \to C$ be a contractive mapping with a constant $k \in (0, 1)$. Let $\{x_t\}$ be a net defined via (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0, 1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point q of $A^{-1}0 \cap Fix(S)$ as $t \to 0$, which solves the variational inequality (3.2).*

Proof. Note that the definition of the weak continuity of duality mapping $\mathcal{J}_\varphi$ implies that E is smooth. By (1) in Proposition 3.1, we see that $\{x_t\}$ and $\{y_t\}$ are bounded. Assume $t_n \to 0$. Put $x_n := x_{t_n}$ and $y_n := y_{t_n}$. Since E is reflexive, we may assume that $y_n \rightharpoonup q$ for some $q \in C$. Since $\mathcal{J}_\varphi$ is weakly continuous, $\|y_n - J_r y_n\| \to 0$ and $\|y_n - Sy_n\| \to 0$ by (4) and (6) in Proposition 3.1, respectively, we have by Lemma 2.2, $q = Sq = J_r q$, and hence $q \in A^{-1}0 \cap Fix(S)$.

Now we prove that $\{x_t\}$ and $\{y_t\}$ converge strongly to a point in $A^{-1}0 \cap Fix(S)$ provided it remains bounded when $t \to 0$.

Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup q$ as $n \to \infty$. By (5) in Proposition, $y_{t_n} \rightharpoonup q$ as $n \to \infty$ too. Then argument above shows that $q \in A^{-1}0 \cap Fix(S)$. We next show that $x_{t_n} \to q$. As a matter of fact, we have by Lemma 2.1,

$$\begin{aligned}\Phi(\|x_{t_n} - q\|) &\le \Phi(\|y_{t_n} - q\|)\\ &= \Phi(\|t_n(fx_{t_n} - fq) + (1 - t_n)(Sx_{t_n} - q) + t_n(fq - q)\|)\\ &\le \Phi(\|t_n k\|x_{t_n} - q\| + (1 - t_n)\|x_{t_n} - q\|) + t_n\langle fq - q, \mathcal{J}_\varphi(y_{t_n} - q)\rangle\\ &= \Phi((1 - (1 - k)t_n)\|x_{t_n} - q\|) + t_n\langle fq - q, \mathcal{J}_\varphi(y_{t_n} - q)\rangle\\ &\le (1 - (1 - k)t_n)\Phi(\|x_{t_n} - q\|) + t_n\langle fq - q, \mathcal{J}_\varphi(y_{t_n} - q)\rangle.\end{aligned}$$

This implies that

$$\Phi(\|x_{t_n} - q\|) \le \frac{1}{1-k}\langle fq - q, \mathcal{J}_\varphi(y_{t_n} - q)\rangle.$$

Observing that $y_{t_n} \rightharpoonup q$ implies $\mathcal{J}_\varphi(y_{t_n} - q) \to 0$, we conclude from the last inequality

$$\Phi(\|x_{t_n} - q\|) \to 0.$$

Hence $x_{t_n} \to q$ and $y_{t_n} \to q$ by (5) in Proposition 3.1.

We prove that the entire net $\{x_t\}$ and $\{y_t\}$ converge strongly to q. To this end, we assume that two sequences $\{t_n\}$ and $\{s_n\}$ in (0,1) are such that

$$x_{t_n} \to q, \quad y_{t_n} \to q \quad \text{and} \quad x_{s_n} \to \overline{q}, \quad y_{s_n} \to \overline{q}.$$

We have to show that $q = \overline{q}$. Indeed, for $p \in A^{-1}0 \cap Fix(S)$, it is easy to see that

$$\begin{aligned}\langle y_t - Sx_t, \mathcal{J}_\varphi(x_t - p)\rangle &= \langle y_t - x_t, \mathcal{J}_\varphi(x_t - p)\rangle + \langle x_t - p + p - Sx_t, \mathcal{J}_\varphi(x_t - p)\rangle\\ &\ge \langle y_t - x_t, \mathcal{J}_\varphi(x_t - p)\rangle + \Phi(\|x_t - p\|) - \langle Sx_t - p, \mathcal{J}_\varphi(x_t - p)\rangle\\ &\ge \langle y_t - x_t, \mathcal{J}_\varphi(x_t - p)\rangle + \Phi(\|x_t - p\|) - \|x_t - p\|\|\mathcal{J}_\varphi(x_t - p)\|\\ &\ge \langle y_t - x_t, \mathcal{J}_\varphi(x_t - p)\rangle + \Phi(\|x_t - p\|) - \Phi(\|x_t - p\|)\\ &= \langle y_t - x_t, \mathcal{J}_\varphi(x_t - p)\rangle.\end{aligned}$$

On the other hand, since

$$y_t - Sx_t = -\frac{t}{1-t}(y_t - fx_t),$$

we have for $t \in (0, 1)$ and $p \in F(S) \bigcap A^{-1}0$,

$$\begin{aligned}\langle y_t - fx_t, \mathcal{J}_\varphi(x_t - p)\rangle &\le \frac{1-t}{t}\langle x_t - y_t, \mathcal{J}_\varphi(x_t - p)\rangle\\ &\le (1 - \frac{1}{t})\|x_t - y_t\|\|\mathcal{J}_\varphi(x_t - p)\|\\ &\le \|x_t - y_t\|\|\mathcal{J}_\varphi(x_t - p)\|.\end{aligned} \tag{3.5}$$

In particular, we obtain

$$\langle y_{t_n} - fx_{t_n}, \mathcal{J}_\varphi(x_{t_n} - p)\rangle \le \|x_{t_n} - y_{t_n}\|\|\mathcal{J}_\varphi(x_{t_n} - p)\|$$

and

$$\langle y_{s_n} - fx_{s_n}, \mathcal{J}_\varphi(x_{s_n} - p)\rangle \le \|x_{s_n} - y_{s_n}\|\|\mathcal{J}_\varphi(x_{s_n} - p)\|.$$

Letting $n \to \infty$ in above inequalities, we deduce by (5) in Proposition 3.1,

$$\langle q - fq, \mathcal{J}_\varphi(q - p)\rangle \le 0, \quad \text{and} \quad \langle \overline{q} - f\overline{q}, \mathcal{J}_\varphi(\overline{q} - p)\rangle \le 0.$$

In particular, we have

$$\langle q - fq, \mathcal{J}_\varphi(q - \overline{q})\rangle \le 0, \quad \text{and} \quad \langle \overline{q} - f\overline{q}, \mathcal{J}_\varphi(\overline{q} - q)\rangle \le 0.$$

Adding up these inequalities yields

$$\begin{aligned}\|q - \overline{q}\|\|\mathcal{J}_\varphi(q - \overline{q})\| &= \langle q - \overline{q}, \mathcal{J}_\varphi(q - \overline{q})\rangle \\ &\le \langle fq - f\overline{q}, \mathcal{J}_\varphi(q - \overline{q})\rangle \le \ k\|q - \overline{q}\|\|\mathcal{J}_\varphi(q - \overline{q})\|.\end{aligned}$$

This implies that $(1 - k)\|q - \overline{q}\|\|\mathcal{J}_\varphi(q - \overline{q})\| \le 0$. Hence $q = \overline{q}$ and $\{x_t\}$ and $\{y_t\}$ converge strongly to q.

Finally we show that q is the unique solution of the variational inequality (3.2). Indeed, since $x_t,\ y_t \to q$ by (5) in Proposition 3.1 and $fx_t \to fq$ as $t \to 0$, letting $t \to 0$ in (3.5), we have

$$\langle (I - f)q, \mathcal{J}_\varphi(q - p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap Fix(S).$$

This implies that q is a solution of the variational inequality (3.2). If $\widetilde{q} \in A^{-1}0 \cap Fix(S)$ is other solution of the variational inequality (3.2), then

$$\langle (I - f)\widetilde{q}, \mathcal{J}_\varphi(\widetilde{q} - q)\rangle \le 0. \tag{3.6}$$

Interchanging $\overline{q}$ and q, we obtain

$$\langle (I - f)q, \mathcal{J}_\varphi(q - \widetilde{q})\rangle \le 0. \tag{3.7}$$

Adding up (3.6) and (3.7) yields

$$(1 - k)\|\widetilde{q} - q\|\|\mathcal{J}_\varphi(\widetilde{q} - q)\| \le 0.$$

That is, $q = \widetilde{q}$. Hence q is the unique solution of the variational inequality (3.2). This completes the proof. □

3.2. Strong convergence of the explicit algorithm

Now, using Theorem 3.2, we show the strong convergence of the sequence generated by the explicit algorithm (3.3) to a point $q \in A^{-1}0 \cap Fix(S)$, which is also a solution of the variational inequality (3.2).

Theorem 3.3. *Let E be a real uniformly convex Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ, let C be a nonempty closed convex subset of E, let $A \subset E \times E$ be an accretive operator in E such that $A^{-1}0 \ne \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{r>0} R(I + rA)$, and let J_{r_n} be the resolvent of A for each $r_n > 0$. Let $r > 0$ be any given positive number, and let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \cap A^{-1}0 \ne \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\} \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition);*
(C4) $\lim_{n\to\infty} r_n = r$ *and* $r_n \ge \varepsilon > 0$ *for* $n \ge 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $f : C \to C$ be a contractive mapping with a constant $k \in (0, 1)$ and $x_0 = x \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = J_{r_n}(\alpha_n fx_n + (1 - \alpha_n)Sx_n), \quad \forall n \ge 0, \tag{3.8}$$

and let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n fx_n + (1 - \alpha_n)J_{r_n}x_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap Fix(S)$, where q is the unique solution of the variational inequality (3.2).

Proof. First, we note that by Theorem 3.2, there exists the unique solution q of the variational inequality

$$\langle (I-f)q, \mathcal{J}_\varphi(q-p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap Fix(S),$$

where $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ with x_t and y_t being defined by $x_t = J_r(tfx_t + (1-t)Sx_t)$ and $y_t = tfx_t + (1-t)Sx_t$ for $0 < t < 1$, respectively.

We divide the proof into the several steps.

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p)-p\|\}$ for all $n \ge 0$ and all $p \in A^{-1}0 \cap Fix(S)$, and so $\{x_n\}$, $\{y_n\}$, $\{J_{r_n}x_n\}$, $\{Sx_n\}$, $\{J_{r_n}y_n\}$, $\{Sy_n\}$ and $\{f(x_n)\}$ are bounded. Indeed, let $p \in A^{-1}0 \cap Fix(S)$. From $A^{-1}0 = Fix(J_r)$ for each $r > 0$, we know $p = Sp = J_{r_n}p$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \|y_n - p\| \\
&= \|\alpha_n(fx_n - p) + (1-\alpha_n)(Sx_n - Sp)\| \\
&\le \alpha_n\|fx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n(\|fx_n - fp\| + \|fp - p\|) + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n k\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|x_n - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{1}{1-k}\|f(p) - p\|\right\}.
\end{aligned}$$

Using an induction, we obtain

$$\begin{aligned}
&\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\right\} \text{ and} \\
&\|y_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\right\}, \quad \forall n \ge 0.
\end{aligned}$$

Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{Sx_n\}$, $\{J_{r_n}x_n\}$, $\{Sy_n\}$, $\{J_{r_n}y_n\}$ and $\{fx_n\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Sx_n\| = \alpha_n\|f(x_n) - Sx_n\| \to 0 \quad (n \to \infty). \tag{3.9}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. First, from the resolvent identity (2.1) we observe that

$$\begin{aligned}
&\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \\
&= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}y_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - J_{r_{n-1}}y_{n-1}\right\| \\
&\le \left\|\frac{r_{n-1}}{r_n}y_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - y_{n-1}\right\| \\
&\le \|y_n - y_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|(\|y_n - y_{n-1}\| + \|J_{r_n}y_n - y_{n-1}\|) \\
&\le \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.10}$$

where $M_1 = \sup_{n\ge 0}\{\|J_{r_n}y_n - y_{n-1}\| + \|y_n - y_{n-1}\|\}$. Since

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\ y_{n-1} = \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}, \quad \forall n \ge 1, \end{cases}$$

by (3.10), we have for $n \geq 1$,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \leq \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1 \\ &= \|(1-\alpha_n)(Sx_n - Sx_{n-1}) + \alpha_n(fx_n - fx_{n-1}) \\ &\quad + (\alpha_n - \alpha_{n-1})(fx_{n-1} - Sx_{n-1})\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1 \\ &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + k\alpha_n\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|M_2 + \left|1 - \frac{r_{n-1}}{r_n}\right| M_1 \\ &\leq (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right| M_1,\end{aligned} \tag{3.11}$$

where $M_2 = \sup\{\|f(x_n) - Sx_n\| : n \geq 0\}$. Thus, by (C3) we have

$$\|x_{n+1} - x_n\| \leq (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + M_2(o(\alpha_n) + \sigma_{n-1}) + M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right|.$$

In (3.11), by taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = (1-k)\alpha_n$, $\lambda_n\delta_n = M_2 o(\alpha_n)$ and

$$\gamma_n = M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right| + M_2\sigma_{n-1},$$

we have

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by conditions (C1), (C2), (C3), (C4) and Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|y_n - J_{r_n} y_n\| = 0$. Indeed, it follows from Resolvent Identity (2.1) that

$$J_{r_n} y_n = J_{\frac{r_n}{2}}(\frac{1}{2}y_n + \frac{1}{2}J_{r_n} y_n).$$

Then we have

$$\|J_{r_n} y_n - p\| = \|J_{\frac{r_n}{2}}(\frac{1}{2}y_n + \frac{1}{2}J_{r_n} y_n) - p\| \leq \|\frac{1}{2}(y_n - p) + \frac{1}{2}(J_{r_n} y_n - p)\|.$$

By the inequality (2.2) ($\lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned}\|J_{r_n} y_n - p\|^2 &\leq \|J_{\frac{r_n}{2}}(\frac{1}{2}x_n + \frac{1}{2}J_{r_n} y_n) - p\|^2 \\ &\leq \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|J_{r_n} y_n - p\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|) \\ &\leq \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|y_n - p\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|) \\ &= \|y_n - p\|^2 - \frac{1}{4}g(\|y_n - J_{r_n} y_n\|).\end{aligned} \tag{3.12}$$

Thus, the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.12), we have for $p \in A^{-1}0 \cap Fix(S)$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|J_{r_n}y_n-p\|^2\\ &\le \|y_n-p\|^2-\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)\\ &\le \|\alpha_n fx_n+(1-\alpha_n)Sx_n-p\|^2-\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)\\ &\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\|Sx_n-p\|^2-\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)\\ &\le \alpha_n\|fx_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\frac{1}{4}g(\|y_n-J_{r_n}y_n\|),\end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)-\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)\le \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Now we consider two cases:

Case 1. When $\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)\le\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)$, by the boundedness of $\{fx_n\}$ and $\{x_n\}$ and condition (C1),

$$\lim_{n\to\infty}g(\|y_n-J_{r_n}y_n\|)=0.$$

Case 2. When $\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)>\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)$, we obtain

$$\sum_{n=0}^{N}[\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)-\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)]\le \|x_0-p\|^2-\|x_N-p\|^2\le\|x_0-p\|^2.$$

Therefore

$$\sum_{n=0}^{\infty}[\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)-\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)]<\infty,$$

and so

$$\lim_{n\to\infty}[\frac{1}{4}g(\|y_n-J_{r_n}y_n\|)-\alpha_n(\|fx_n-p\|^2-\|x_n-p\|^2)]=0.$$

By condition (C1), we have

$$\lim_{n\to\infty}g(\|y_n-J_{r_n}y_n\|)=0.$$

Thus, from the property of the function g in (2.2) it follows that

$$\lim_{n\to\infty}\|y_n-J_{r_n}y_n\|=0.$$

Step 4. We show that $\lim_{n\to\infty}\|x_n-y_n\|=0$. Indeed, from Step 2 and Step 3 it follows that

$$\begin{aligned}\|x_n-y_n\|&\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|\\ &\le\|x_n-x_{n+1}\|+\|J_{r_n}y_n-y_n\|\to 0,\quad (n\to\infty).\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty}\|y_n-Sy_n\|=0$. In fact, by (3.9) and Step 4, we have

$$\begin{aligned}\|y_n-Sy_n\|&\le\|y_n-Sx_n\|+\|Sx_n-Sy_n\|\\ &\le\|y_n-Sx_n\|+\|x_n-y_n\|\to 0\quad (n\to\infty).\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty}\|x_n-Sx_n\|=0$. Indeed, from Step 4 and Step 5 it follows that

$$\begin{aligned}\|x_n-Sx_n\|&\le\|x_n-y_n\|+\|y_n-Sy_n\|+\|Sy_n-Sx_n\|\\ &\le 2\|x_n-y_n\|+\|y_n-Sy_n\|\to\infty\quad (n\to\infty).\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$. Indeed, by Step 3 and Step 4, we obtain

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - y_n\| + \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_{r_n}x_n\|\\ &\le 2\|y_n - x_n\| + \|y_n - J_{r_n}y_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Step 8. We show that $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$ for $r = \lim_{n\to\infty} r_n$. Indeed, from the resolvent identity (2.1) and boundedness of $\{J_{r_n}y_n\}$ we obtain

$$\begin{aligned}\|J_{r_n}y_n - J_r y_n\| &= \left\|J_r\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\right\|\\ &\le \left\|\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - y_n\right\| \\ &\le \left|1-\frac{r}{r_n}\right|\|y_n - J_{r_n}y_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.13}$$

Hence, by Step 3 and (3.13), we have

$$\|y_n - J_r y_n\| \le \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_r y_n\| \to 0 \ \ (n\to\infty).$$

Step 9. We show that $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$. Indeed, by Step 4 and Step 8, we have

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - y_n\| + \|y_n - J_r y_n\| + \|J_r y_n - J_r x_n\|\\ &\le 2\|x_n - y_n\| + \|y_n - J_r y_n\| \to \infty \quad (n\to\infty).\end{aligned}$$

Step 10. We show that $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle \le 0$. To prove this, let a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle = \lim_{j\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_{n_j})\rangle$$

and $y_{n_j} \rightharpoonup z$ for some $z\in E$. Then, by Step 5, Step 8 and Lemma 2.2, we have $z\in A^{-1}0\cap Fix(S)$. From the weak continuity of J_φ it follows that

$$w - \lim_{i\to\infty}\mathcal{J}_\varphi(q-y_{n_i}) = w - \mathcal{J}_\varphi(q-z).$$

Hence, from (3.2) we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle &= \lim_{j\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_{n_j})\rangle\\ &= \langle (I-f)q, \mathcal{J}_\varphi(q-z)\rangle \le 0.\end{aligned}$$

Step 11. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (3.8), we have

$$\|x_{n+1}-q\| \le \|y_n - q\| = \|\alpha_n(fx_n - q) + (1-\alpha_n)(Sx_n - q)\|.$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}\Phi(\|x_{n+1}-q\|) &\le \Phi(\|y_n - q\|)\\ &\le \Phi(\|\alpha_n(fx_n - fq) + (1-\alpha_n)(Sx_n-q)\|) + \alpha_n\langle fq - q, \mathcal{J}_\varphi(y_n-q)\rangle\\ &\le \Phi(k\alpha_n\|x_n-q\| + (1-\alpha_n)\|x_n - q\|) + \alpha_n\langle fq-q, \mathcal{J}_\varphi(y_n-q)\rangle\\ &\le (1-(1-k)\alpha_n)\Phi(\|x_n-q\|) + \alpha_n\langle fq-q, \mathcal{J}_\varphi(y_n-q)\rangle.\end{aligned} \tag{3.14}$$

Put

$$\lambda_n = (1-k)\alpha_n \text{ and } \delta_n = \frac{1}{1-k}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle.$$

From conditions (C1), (C2) and Step 8 it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Since (3.14) reduces to

$$\Phi(\|x_{n+1}-q\|) \le (1-\lambda_n)\Phi(\|x_n-q\|) + \lambda_n\delta_n,$$

from Lemma 2.3 with $\gamma_n = 0$ we conclude that $\lim_{n\to\infty}\Phi(\|x_n-q\|) = 0$, and thus $\lim_{n\to\infty} x_n = q$. By Step 4, we also have $\lim_{n\to\infty} y_n = q$. This completes the proof. □

Corollary 3.4. *Let E, C, A, J_{r_n}, S, f and $r > 0$ be as in* Theorem 3.3. *Let $\{\alpha_n\}\in(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfy conditions* (C1) – (C4) *in* Theorem 3.3. *Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n + e_n), \quad \forall n \ge 0,$$

where $\{e_n\}\subset E$ satisfies $\sum_{n=0}^{\infty}\|e_n\| < \infty$ or $\lim_{n\to\infty}\frac{\|e_n\|}{\alpha_n} = 0$, and let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n f x_n + (1-\alpha_n)Sx_n + e_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(S)\cap A^{-1}0$, where q is the unique solution of the variational inequality (3.2).

Proof. Let $z_{n+1} = J_{r_n}(\alpha_n f z_n + (1-\alpha_n)Sz_n)$ for $n \ge 0$. Then, by Theorem 3.3, $\{z_n\}$ converges strongly to a point $q \in A^{-1}0 \cap Fix(S)$, where q is the unique solution of the variational inequality (3.2), and we derive

$$\begin{aligned}\|x_{n+1}-z_{n+1}\| &\le \|\alpha_n f x_n + (1-\alpha_n)Sx_n - (\alpha_n z_n + (1-\alpha_n)Sz_n + e_n)\|\\ &\le \alpha_n\|fx_n - fz_n\| + (1-\alpha_n)\|Sx_n - Sz_n\| + \|e_n\|\\ &\le (1-(1-k)\alpha_n)\|x_n - z_n\| + \|e_n\|.\end{aligned}$$

By Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|x_n - z_n\| = 0,$$

and hence the desired result follows. □

Finally, as in [9], we consider the iterative method with the weakly contractive mapping

Theorem 3.5. *Let E, C, A, J_{r_n}, S, and $r > 0$ be as in* Theorem 3.3. *Let $\{\alpha_n\}\in(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfy the conditions* (C1) – (C4) *in* Theorem 3.3. *Let $g : C \to C$ be a weakly contractive mapping with the function ψ. Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n g x_n + (1-\alpha_n)Sx_n), \quad \forall n \ge 0.$$

and $\{y_n\}$ be a sequence defined by $y_n = \alpha_n g x_n + (1-\alpha_n)Sx_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(S)\cap A^{-1}0$.

Proof. Since E is smooth, there is a sunny nonexpansive retraction Q from C onto $A^{-1}0\cap Fix(S)$. Then Qg is a weakly contractive mapping of C into itself. Indeed, for all $x,\ y \in C$,

$$\|Qgx - Qgy\| \le \|gx - gy\| \le \|x-y\| - \psi(\|x-y\|).$$

Lemma 2.4 assures that there exists a unique element $x^* \in C$ such that $x^* = Qgx^*$. Such a $x^* \in C$ is an element of $A^{-1}0\cap Fix(S)$.

Now we define an iterative scheme as follows:

$$w_{n+1} = J_{r_n}(\alpha_n g x^* + (1-\alpha_n)Sw_n) \quad \forall n \ge 0. \tag{3.15}$$

Let $\{w_n\}$ be the sequence generated by (3.15). Then Theorem 3.3 with a constant $f = gx^*$ assures that $\{w_n\}$ converges strongly to $Qgx^* = x^*$ as $n \to \infty$. For any $n > 0$, we have

$$\begin{aligned}
\|x_{n+1} - w_{n+1}\| &= \|J_{r_n}(\alpha_n g x_n + (1-\alpha_n)Sx_n) - J_{r_n}(\alpha_n g x^* + (1-\alpha_n)Sw_n)\| \\
&\leq \alpha_n(\|gx_n - gx^*\|) + (1-\alpha_n)\|x_n - w_n\| \\
&\leq \alpha_n[\|gx_n - gw_n\| + \|gw_n - gx^*\|] + (1-\alpha_n)\|x_n - w_n\| \\
&\leq \alpha_n[\|x_n - w_n\| - \psi(\|x_n - w_n\|) + \|w_n - x^*\| \\
&\qquad - \psi(\|w_n - x^*\|)] + (1-\alpha_n)\|x_n - w_n\| \\
&\leq \|x_n - w_n\| - \alpha_n\psi(\|x_n - w_n\|) + \alpha_n\|w_n - x^*\|.
\end{aligned}$$

Thus, we obtain for $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} \leq s_n - \alpha_n\psi(s_n) + \alpha_n\|w_n - x^*\|.$$

Since $\lim_{n\to\infty}\|w_n - x^*\| = 0$, from condition (C2) and Lemma 2.5 it follows that $\lim_{n\to\infty}\|x_n - w_n\| = 0$. Hence

$$\lim_{n\to\infty}\|x_n - x^*\| \leq \lim_{n\to\infty}(\|x_n - w_n\| + \|w_n - x^*\|) = 0.$$

By Step 4 in the proof of Theorem 3.3, we also have $\lim_{n\to\infty} y_n = q$. This completes the proof. □

Remark 3.6.

(1) Theorem 3.2, Theorem 3.3 and Theorem 3.5 develop and complement the recent corresponding results studied by many authors in this direction (see, for instance, [10, 11, 18, 20, 24] and the references therein).

(2) The control condition (C3) in Theorem 3.3 can be replaced by the condition $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$; or the condition $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$, which are not comparable ([8]).

Acknowledgements

This study was supported by research funds from Dong-A University.

References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, (2009). 2, 2

[2] Y. I. Alber, S. Guerre-Delabriere, L. Zelenko, *Principle of weakly contractive maps in metric spaces*, Comm. Appl. Nonlinear Anal., **5** (1998), 45–68. 2

[3] Y. I. Alber, S. Reich, J. C. Yao, *Iterative methods for solving fixed-point problems with nonself-mappings in Banach spaces*, Abstr. Appl. Anal., **4** (2003), 193–216. 2, 2.5

[4] T. D. Benavides, G. L. Acedo, H. K. Xu, *Iterative solutions for zeros of accretive operators*, Math. Nachr., **248-249** (2003), 62–71. 1

[5] H. Bréziz, P. L. Lions, *Products infinis de resolvents*, Israel J. Math., **29** (1978), 329–345. 1

[6] R. E. Jr. Bruck, *A strongly convergent iterative solution of* $0 \in U(x)$ *for a maximal monotone operator* U *in Hilbert space*, J. Math. Anal. Appl., **48** (1974), 114–126. 1

[7] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, (1990). 2, 2

[8] J. S. Jung, *Viscosity approximation methods for a family of finite nonexpansive mappings in Banach spaces*, Nonlinear Anal., **64** (2006), 2536–2552. 3.6

[9] J. S. Jung, *Convergence of composite iterative methods for finding zeros of accretive operators*, Nonlinear Anal., **71** (2009), 1736–1746. 3.2

[10] J. S. Jung, *Some results on Rockafellar-type iterative algorithms for zeros of accretive operators*, J. Inequal. Appl., **2013** (2013), 19 pages. 1, 1, 3.6

[11] J. S. Jung, *Iterative algorithms for zeros of accretive operators and fixed points of nonexpansive mappings in Banach spaces*, J. Comput. Anal. Appl., **20** (2016), 688–706. 1, 1, 3.1, 3.1, 3.6

[12] J. S. Jung, W. Takahashi, *Dual convergence theorems for the infinite products of resolvents in Banach spaces*, Kodai Math. J., **14** (1991), 358–365. 1

[13] J. S. Jung, W. Takahashi, *On the asymptotic behavior of infinite products of resolvents in Banach spaces*, Nonlinear Anal., **20** (1993), 469–479. 1

[14] L. S. Liu, *Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194** (1995), 114–125. 2.3

[15] S. Reich, *On infinite products of resolvents*, Atti. Accad. naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **63** (1997), 338–340. 1

[16] B. E. Rhodes, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693. 2, 2.4

[17] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877–898. 1

[18] Y. Song, *New iterative algorithms for zeros of accretive operators*, J. Korean Math. Soc., **46** (2009), 83–97. 1, 1, 1, 3.6

[19] Y. Song, J. I. Kang, Y. J. Cho, *On iterations methods for zeros of accretive operators in Banach spaces*, Appl. Math. Comput., **216** (2010), 1007–1017. 1

[20] Y. Song, C. Yang, *A note on a paper "A regularization method for the proximal point algorithm"*, J. Global Optim., **43** (2009), 115–125. 1, 3.6

[21] H. K. Xu, *Inequality in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138. 1, 2

[22] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256. 2.3

[23] H. K Xu, *A regularization method for the proximal point algorithm*, J. Global Optim., **36** (2006), 115–125. 1

[24] Q. Zhang, Y. Song, *Halpern type proximal point algorithm of accretive operators*, Nonlinear Anal., **75** (2012), 1859–1868. 1, 1, 3.6