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A fixed point approach to the stability of an AQCQ-functional equation in RN-spaces

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Abstract

Using the fixed point method, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$$

in random normed spaces.

Keywords: Random normed space, fixed point, Hyers-Ulam stability, additive-quadratic-cubic-quartic functional equation.
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1. Introduction

Fuzzy set theory is a powerful tool set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics [4], chaos control [19], computer programming [22], nonlinear operators [39], etc. Recently, the fuzzy topology has proved to be a very useful tool to deal with such situations where the use of classical theories breaks down.

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In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [9, 33, 34, 52, 53]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1 ([52])**.** A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm). Recall (see [23, 24]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T^n_{i=1}x_i$ is defined recurrently by $T^1_{i=1}x_i = x_1$ and $T^n_{i=1}x_i = T(T^{n-1}_{i=1}x_i, x_n)$ for $n \ge 2$. $T^\infty_{i=n}x_i$ is defined as $T^\infty_{i=1}x_{n+i-1}$. It is known ([24]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)^\infty_{i=1}x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1 - x_n) < \infty.$$

Definition 1.2 ([53])**.** A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:
(RN_1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN_2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \ne 0$;
(RN_3) $\mu_{x+y}(t + s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \ge 0$.

Every normed space $(X, \|.\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \ge N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \ge m \ge N$.
(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 ([52])**.** *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.
We recall a fundamental result in fixed point theory.

Theorem 1.5 ([6, 14])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

The stability problem of functional equations originated from a question of Ulam [55] concerning the stability of group homomorphisms. Hyers [25] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [44] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [21] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [1, 30]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)).$$

A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [54] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 5, 12, 13, 20, 26, 29, 45, 46, 47, 48, 49, 50, 51]).

In [28], Jun and Kim considered the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [42], W. Park and Bae considered the following quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y) + 6f(y)] - 6f(x). \tag{1.3}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive mapping $M : X^4 \to Y$ such that $f(x) = M(x, x, x, x)$ for all x. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation (see also [11]). In addition, Kim [31] has obtained the Hyers-Ulam stability for a mixed type of quartic and quadratic functional equation.

In 1996, Isac and Rassias [27] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 8, 33, 38, 40, 41, 43]).

The Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [32, 33, 34, 35, 36, 37, 38].

The aim of this paper is to investigate the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.4}$$

in random normed spaces by using the fixed point method.

Recently, M. Eshaghi Gordji et al. established the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [17, 18]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1.4) in RN-spaces for an odd case. In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1.4) in RN-spaces for an even case.

Throughout this paper, assume that X is a real vector space and that $(Y, \mu, T := \min)$ is a complete RN-space.

2. Hyers-Ulam stability of the functional equation (1.4): an odd mapping case

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.4) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x).$$

It was shown in Lemma 2.2 of [16] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.4) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in Lemma 2.1 of [15] that $g(x) := f(2x)-4f(x)$ and $h(x) := f(2x)-16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) :=& f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x) \\ & -f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete RN-spaces: an odd case.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Df(x,y)}(t) \ge \frac{t}{t+\varphi(x,y)} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then

$$C(x) := \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \ge \frac{(8-8L)t}{(8-8L)t + 5L(\varphi(x,x)+\varphi(2x,x))} \tag{2.2}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (2.1), we get

$$\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq \frac{t}{t+\varphi(y,y)} \tag{2.3}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (2.1), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \frac{t}{t+\varphi(2y,y)} \tag{2.4}$$

for all $y \in X$ and all $t > 0$.

By (2.3) and (2.4),

$$\begin{aligned}\mu_{f(4y)-10f(2y)+16f(y)}(4t+t) &\geq \min\left\{\mu_{4(f(3y)-4f(2y)+5f(y))}(4t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right\} \\ &\geq \frac{t}{t+\varphi(y,y)+\varphi(2y,y)}\end{aligned} \tag{2.5}$$

for all $y \in X$ and all $t > 0$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$ in (2.5), we get

$$\mu_{g(x)-8g\left(\frac{x}{2}\right)}(5t) \geq \frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}, \quad \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 of [34].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(L\varepsilon t) =& \mu_{8g\left(\frac{x}{2}\right)-8h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{8}\varepsilon t\right) \\ \geq& \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\frac{L}{8}(\varphi(x,x)+\varphi(2x,x))} \\ =& \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\mu_{g(x)-8g\left(\frac{x}{2}\right)}\left(\frac{5L}{8}t\right) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5L}{8}$.

By Theorem 1.5, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.7}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.7) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5L}{8-8L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$\mu_{8^n Dg\left(\frac{x}{2^n},\frac{y}{2^n}\right)}(8^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{8^n Dg\left(\frac{x}{2^n},\frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{DC(x,y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{2.8}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := \lim_{n\to\infty} 8^n\left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(2^p-8)t}{(2^p-8)t+5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). Then

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \ge \frac{(8-8L)t}{(8-8L)t + 5\varphi(x,x) + 5\varphi(2x,x)} \tag{2.9}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{8}g(2x)-\frac{1}{8}h(2x)}(L\varepsilon t) = \mu_{g(2x)-h(2x)}(8L\varepsilon t)\\
&\ge \frac{8Lt}{8Lt + \varphi(2x,2x) + \varphi(4x,2x)}\\
&\ge \frac{8Lt}{8Lt + 8L(\varphi(x,x) + \varphi(2x,x))}\\
&= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le L d(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that

$$\mu_{g(x)-\frac{1}{8}g(2x)}\left(\frac{5}{8}t\right) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \le \frac{5}{8}$.

By Theorem 1.5, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x) \tag{2.10}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.10) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} g(2^n x) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5}{8 - 8L}.$$

This implies that the inequality (2.9) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.8)*. Then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} \left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8 - 2^p)t}{(8 - 2^p)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1)*. Then*

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + 5L(\varphi(x, x) + \varphi(2x, x))} \tag{2.11}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.5), we get

$$\mu_{h(x)-2h\left(\frac{x}{2}\right)}(5t) \geq \frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \tag{2.12}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(L\varepsilon t) =&\mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{2}\varepsilon t\right)\\ \geq&\frac{\frac{Lt}{2}}{\frac{Lt}{2}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2}+\frac{L}{2}(\varphi(x,x)+\varphi(2x,x))}\\ =&\frac{t}{t+\varphi(x,x)+\varphi(2x,x)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g,h \in S$.

It follows from (2.12) that

$$\mu_{h(x)-2h\left(\frac{x}{2}\right)}\left(\frac{5L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(h,Jh) \leq \frac{5L}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.13}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.13) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu_{h(x)-A(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(h, A) \leq \frac{1}{1-L} d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{5L}{2-2L}.$$

This implies that the inequality (2.11) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (2.8). Then $A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2^p-2)t}{(2^p-2)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). Then

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t + 5\varphi(x, x) + 5\varphi(2x, x)} \tag{2.14}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{2}g(2x)-\frac{1}{2}h(2x)}(L\varepsilon t) = \mu_{g(2x)-h(2x)}(2L\varepsilon t)\\ &\geq \frac{2Lt}{2Lt + \varphi(2x, 2x) + \varphi(4x, 2x)}\\ &\geq \frac{2Lt}{2Lt + 2L(\varphi(x, x) + \varphi(2x, x))}\\ &= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.12) that

$$\mu_{h(x)-\frac{1}{2}h(2x)}\left(\frac{5}{2}t\right) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(2x) = 2A(x) \tag{2.15}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.15) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{h(x)-A(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} h(2^n x) = A(x)$$

for all $x \in X$;

(3) $d(h, A) \leq \frac{1}{1-L} d(h, Jh)$, which implies the inequality

$$d(h, A) \leq \frac{5}{2-2L}.$$

This implies that the inequality (2.14) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.8). *Then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2-2^p)t}{(2-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.7 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3. Hyers-Ulam stability of the functional equation (1.4): an even mapping case

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete random normed spaces: an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{16}\varphi(2x,2y)$$

for all $x,y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \ge \frac{(16-16L)t}{(16-16L)t + 5L(\varphi(x,x) + \varphi(2x,x))} \tag{3.1}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y$ in (2.1), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \ge \frac{t}{t + \varphi(y,y)} \tag{3.2}$$

for all $y \in X$ and all $t > 0$.

Replacing x by $2y$ in (2.1), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \ge \frac{t}{t + \varphi(2y,y)} \tag{3.3}$$

for all $y \in X$ and all $t > 0$.

By (3.2) and (3.3),

$$\begin{aligned}\mu_{f(4x)-20f(2x)+64f(x)}(4t+t) &\ge \min\left\{\mu_{4(f(3x)-6f(2x)+15f(x))}(4t), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x))}(t)\right\} \\ &\ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}\end{aligned} \tag{3.4}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-16g\left(\frac{x}{2}\right)}(5t) \ge \frac{t}{t + \varphi\left(\frac{x}{2},\frac{x}{2}\right) + \varphi\left(x,\frac{x}{2}\right)} \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(L\varepsilon t) =& \mu_{16g\left(\frac{x}{2}\right)-16h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{16}\varepsilon t\right)\\ \geq& \frac{\frac{Lt}{16}}{\frac{Lt}{16}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)} \geq \frac{\frac{Lt}{16}}{\frac{Lt}{16}+\frac{L}{16}(\varphi(x,x)+\varphi(2x,x))}\\ =& \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$\mu_{g(x)-16g\left(\frac{x}{2}\right)}\left(\frac{5L}{16}t\right) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g,Jg) \leq \frac{5L}{16}$.

By Theorem 1.5, there exists a mapping $C : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{3.6}$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\nu \in (0,\infty)$ satisfying

$$\mu_{g(x)-Q(x)}(\nu t) \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(g,Q) \leq \frac{1}{1-L}d(g,Jg)$, which implies the inequality

$$d(g,Q) \leq \frac{5L}{16-16L}.$$

This implies that the inequality (3.1) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.8). Then $Q(x) := \lim_{n\to\infty} 16^n\left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(2^p-16)t}{(2^p-16)t+5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then

$$Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \ge \frac{(16-16L)t}{(16-16L)t + 5\varphi(x,x) + 5\varphi(2x,x)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{16}g(2x)-\frac{1}{16}h(2x)}(L\varepsilon t) = \mu_{g(2x)-h(2x)}(16L\varepsilon t)\\
&\ge \frac{16Lt}{16Lt + \varphi(2x,2x) + \varphi(4x,2x)}\\
&\ge \frac{16Lt}{16Lt + 16L(\varphi(x,x) + \varphi(2x,x))}\\
&= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$\mu_{g(x)-\frac{1}{16}g(2x)}\left(\frac{5}{16}t\right) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \le \frac{5}{16}$.

By Theorem 1.5, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\,(2x) = 16Q(x) \tag{3.8}$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.8) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{16^n} g(2^n x) = Q(x)$$

for all $x \in X$;

(3) $d(g, Q) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, Q) \leq \frac{5}{16 - 16L}.$$

This implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.8). Then $Q(x) := \lim_{n\to\infty} \frac{1}{16^n} \left(f\left(2^{n+1}x\right) - 4f(2^n x)\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(2x)-4f(x)-Q(x)}\,(t) \geq \frac{(16 - 2^p)t}{(16 - 2^p)t + 5(3 + 2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

Theorem 3.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi\,(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then

$$T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right)$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}\,(t) \geq \frac{(4 - 4L)t}{(4 - 4L)t + 5L(\varphi(x, x) + \varphi(2x, x))} \tag{3.9}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $h(x) := f(2x) - 16f(x)$ for all $x \in X$ in (3.4), we get

$$\mu_{h(x)-4h\left(\frac{x}{2}\right)}(5t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \tag{3.10}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} \mu_{Jg(x)-Jh(x)}(L\varepsilon t) =& \mu_{4g\left(\frac{x}{2}\right)-4h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{4}\varepsilon t\right) \\ \geq& \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, x\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}(\varphi(x, x) + \varphi(2x, x))} \\ =& \frac{t}{t + \varphi(x, x) + \varphi(2x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.10) that

$$\mu_{h(x)-4h\left(\frac{x}{2}\right)}\left(\frac{5L}{4}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5L}{4}$.

By Theorem 1.5, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T\left(\frac{x}{2}\right) = \frac{1}{4}T(x) \tag{3.11}$$

for all $x \in X$. Since $h : X \to Y$ is even, $T : X \to Y$ is an even mapping. The mapping T is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that T is a unique mapping satisfying (3.11) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{h(x)-T(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n h\left(\frac{x}{2^n}\right) = T(x)$$

for all $x \in X$;

(3) $d(h,T) \le \frac{1}{1-L} d(h,Jh)$, which implies the inequality

$$d(h,T) \le \frac{5L}{4-4L}.$$

This implies that the inequality (3.9) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.6. *Let $\theta \ge 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.8). Then $T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \ge \frac{(2^p-4)t}{(2^p-4)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.5 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Theorem 3.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \ge \frac{(4-4L)t}{(4-4L)t + 5\varphi(x,x) + 5\varphi(2x,x)} \tag{3.12}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{4} h(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{4}g(2x)-\frac{1}{4}h(2x)}(L\varepsilon t) = \mu_{g(2x)-h(2x)}(4L\varepsilon t) \\
&\ge \frac{4Lt}{4Lt + \varphi(2x,2x) + \varphi(4x,2x)} \\
&\ge \frac{4Lt}{4Lt + 4L(\varphi(x,x) + \varphi(2x,x))} \\
&= \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.10) that

$$\mu_{h(x)-\frac{1}{4}h(2x)}\left(\frac{5}{4}t\right) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \le \frac{5}{4}$.

By Theorem 1.5, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T(2x) = 4T(x) \tag{3.13}$$

for all $x \in X$. Since $h : X \to Y$ is even, $T : X \to Y$ is an even mapping. The mapping T is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that T is a unique mapping satisfying (3.13) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\mu_{h(x)-T(x)}(\mu t) \ge \frac{t}{t + \varphi(x,x) + \varphi(2x,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n h, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{4^n} h(2^n x) = T(x)$$

for all $x \in X$;

(3) $d(h,T) \le \frac{1}{1-L} d(h, Jh)$, which implies the inequality

$$d(h,T) \le \frac{5}{4-4L}.$$

This implies that the inequality (3.12) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.8. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.8). Then $T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$ exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \ge \frac{(4-2^p)t}{(4-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.7 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

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