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Strong convergence of hybrid Halpern processes in a Banach space

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Abstract

The purpose of this paper is to investigate convergence of a hybrid Halpern process for fixed point and equilibrium problems. Strong convergence theorems of common solutions are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property.

Keywords: Asymptotically nonexpansive mapping, fixed point, quasi-ϕ-nonexpansive mapping, equilibrium problem, generalized projection.
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1. Introduction and Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let C be a nonempty subset of a E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall that the following equilibrium problem [4]: Find $\bar{x} \in C$ such that

$$g(\bar{x}, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

We use $Sol(g)$ to denote the solution set of equilibrium problem (1.1). That is,

$$Sol(g) = \{x \in C : g(x, y) \geq 0 \quad \forall y \in C\}.$$

Given a mapping $A : C \to E^*$, let

$$G(x, y) = \langle Ax, y - x \rangle \quad \forall x, y \in C.$$

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Then $\bar{x} \in Sol(g)$ iff $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0 \quad \forall y \in C. \tag{1.2}$$

The following restrictions (R-a), (R-b), (R-c) and (R-d) imposed on g are essential in this paper.

(R-a) $g(y, x) + g(x, y) \leq 0\ \forall x, y \in C$;

(R-b) $g(x, x) = 0\ \forall x \in C$;

(R-c) $y \mapsto g(x, y)$ is weakly lower semi-continuous and convex $\forall x \in C$;

(R-d) $g(x, y) \geq \limsup_{t\downarrow 0} g(tz + (1 - t)x, y), \forall x, y, z \in C$.

Equilibrium problem (1.1) is a bridge between nonlinear functional analysis and convex optimization theory. Many problems arising in economics, medicine, engineering and physics can be studied via the problem; see [3, 7, 8, 9, 10, 12, 14, 18, 19, 25] and the references therein.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx := \{x^* \in E^* : \|x^*\|^2 = \langle x, x^*\rangle = \|x\|^2\}.$$

Let S^E be the unit sphere of E. Recall that E is said to be a strictly convex space iff $\|x + y\| < 2$ for all $x, y \in S^E$ and $x \neq y$. Recall that E is said to have a Gâteaux differentiable norm iff $\lim_{t\to\infty}(\|tx+y\|-t\|x\|)$ exists $\forall x, y \in S^E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm if for every $y \in S^E$, the limit is attained uniformly for each $x \in S^E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for each $x, y \in S^E$. In this case, we say that E is uniformly smooth. It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is a strictly convex Banach space, then J is strictly monotone; if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$; if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto; if E is a uniformly smooth, then it is smooth and reflexive. It is also known that E^* is uniformly convex if and only if E is uniformly smooth. From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively. Recall that E is said to have the Kadec-Klee property if $\lim_{n\to\infty} \|x_n - x\| = 0$ as $n \to \infty$ for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach spaces, then E has the Kadec-Klee property; see [11] and the references therein.

Let f be a mapping on C. Recall that a point p is said to be a fixed point of f if and only if $p = fp$. p is said to be an asymptotic fixed point of f if and only if C contains a sequence $\{x_n\}$, where $x_n \rightharpoonup p$ such that $x_n - fx_n \to 0$. From now on, We use $Fix(f)$ to stand for the fixed point set and $\widetilde{Fix}(f)$ to stand for the asymptotic fixed point set. f is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} fx_n = y'$, then $fx' = y'$.

Next, we assume that E is a smooth Banach space. Consider the functional defined on E by

$$\phi(x, y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \quad \forall x, y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [2], Alber studied a new mapping Π_C in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$. It is obvious from the definition of function ϕ that

$$(\|y\| + \|x\|)^2 \geq \phi(x, y) \geq (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \tag{1.3}$$

and

$$\phi(x,y) - 2\langle z-x, Jz-Jy\rangle = \phi(x,z) + \phi(z,y) \quad \forall x,y,z \in E. \tag{1.4}$$

Remark 1.1. If E is a strictly convex, reflexive and smooth Banach space, then $\phi(x,y)=0$ iff $x=y$.

Recall that a mapping f is said to be relatively nonexpansive ([5]) iff

$$\phi(p,x) \geq \phi(p,fx) \quad \forall x \in C, \forall p \in \widetilde{Fix}(f) = Fix(f) \neq \emptyset.$$

f is said to be relatively asymptotically nonexpansive ([1]) iff

$$\phi(p,f^n x) \leq (1+\mu_n)\phi(p,x) \quad \forall x \in C, \forall p \in \widetilde{Fix}(f) = Fix(f) \neq \emptyset, \forall n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.2. The class of relatively asymptotically nonexpansive mappings, which include the class of relatively nonexpansive mappings ([5]) as a special case, were first considered in [1] and [26]; see the references therein.

f is said to be quasi-ϕ-nonexpansive ([21]) iff

$$\phi(p,x) \geq \phi(p,fx) \quad \forall x \in C, \forall p \in Fix(f) \neq \emptyset.$$

f is said to be asymptotically quasi-ϕ-nonexpansive ([22]) iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p,f^n x) \leq (1+\mu_n)\phi(p,x) \quad \forall x \in C, \forall p \in Fix(f) \neq \emptyset, \forall n \geq 1.$$

Remark 1.3. The class of asymptotically quasi-ϕ-nonexpansive mappings, which include the class of quasi-ϕ-nonexpansive mappings ([21]) as a special case, were first considered in [20] and [22]; see the references therein.

Remark 1.4. The class of asymptotically quasi-ϕ-nonexpansive mappings is more desirable than the class of asymptotically relatively nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require $Fix(f) = \widetilde{Fix}(f)$.

Recently, Qin and Wang ([23]) introduced the asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense, which is a generalization of asymptotically quasi-nonexpansive mapping in the intermediate sense in Banach spaces. Recall that f is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense iff $Fix(f) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in Fix(f), x\in C} \big(\phi(p,f^n x) - \phi(p,x)\big) \leq 0.$$

The so called convex feasibility problems which capture lots of applications in various subjects are to find a special point in the intersection of convex (solution) sets. Recently, many author studied fixed points of nonexpansive mappings and equilibrium (1.1); see [6], [13], [15]-[17], [24], [27]-[33] and the references therein. The aim of this paper is to investigate convergence of a hybrid Halpern process for fixed point and the equilibrium problem. Strong convergence theorems of common solutions are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. In order to our main results, we also need the following lemmas.

Lemma 1.5 ([2])**.** *Let E be a strictly convex, reflexive and smooth Banach space and let C be a convex and closed subset of E. Let $x \in E$. Then*

$$\phi(y,\Pi_C x) \leq \phi(y,x) - \phi(\Pi_C x, x) \quad \forall y \in C.$$

Lemma 1.6 ([4])**.** *Let C be a convex and closed subset of a smooth Banach space E and let $x \in E$. Then $\langle y - x_0, Jx - Jx_0\rangle \leq 0$ $\forall y \in C$ iff $x_0 = \Pi_C x$.*

Lemma 1.7 ([4], [21])**.** *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c) and (R-d). Let $r > 0$ and $x \in E$. Then*

(a) *There exists $z \in C$ such that*

$$\langle y - z, Jz - Jx\rangle + rg(z, y) \geq 0 \quad \forall y \in C.$$

(b) *Define a mapping $\tau_r : E \to C$ by*

$$\tau_r x = \{z \in C : \langle y - z, Jz - Jx\rangle + rg(z, y) \geq 0 \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *τ_r is single-valued;*

(2) *τ_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle \tau_r x - \tau_r y, Jx - Jy\rangle \geq \langle \tau_r x - \tau_r y, J\tau_r x - J\tau_r y\rangle;$$

(3) *$Fix(\tau_r) = Sol(g)$;*

(4) *τ_r is quasi-ϕ-nonexpansive;*

(5) *$\phi(q, \tau_r x) \leq \phi(q, x) - \phi(\tau_r x, x) \quad \forall q \in F(\tau_r)$;*

(6) *$Sol(g)$ is convex and closed.*

Lemma 1.8 ([23])**.** *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. Let C be a nonempty closed and convex subset of E. Let $f : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Then $Fix(f)$ is a convex closed subset of C.*

2. Main results

Theorem 2.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $i \in \Lambda$. Assume that f_i is continuous and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ x_1 = \Pi_{C_1 := \cap_{i\in\Lambda} C_{(1,i)}} x_0, \\ y_{(n,i)} = J^{-1}((1 - \alpha_{(n,i)})Jf_i^n z_{(n,i)} + \alpha_{(n,i)}Jx_1), \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) + \alpha_{(n,i)}D + (1 - \alpha_{(n,i)})\xi_{(n,i)} \geq \phi(z, y_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_{(n,i)} = \max\{0, \sup_{p\in Fix(f_i), x\in C} \big(\phi(p, f_i^n x) - \phi(p, x)\big)$, $D = \sup\{\phi(w, x_1) : w \in \cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)\}$, $z_{(n,i)} \in C_n$ such that $r_{(n,i)} g_i(z_{(n,i)}, y) \geq \langle z_{(n,i)} - y, Jz_{(n,i)} - Jx_n\rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$.

Proof. We divide the proof into six steps.

Step 1. We prove that $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is convex and closed.

In the light of Lemma 1.7 and Lemma 1.8, we easily find the conclusion. This shows that the generalized projection onto $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is well defined.

Step 2. We prove that C_n is convex and closed.

$C_{(1,i)} = C$ is convex and closed. Next, we assume that $C_{(k,i)}$ is convex and closed for some $k \geq 1$. For $q_1, q_2 \in C_{(k+1,i)} \subset C_{(k,i)}$, we have $q = tq_1 + (1-t)q_2 \in C_{(k,i)}$, where $t \in (0,1)$. Notice that $\phi(q_1, x_k) + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq \phi(q_1, y_{(k,i)})$ and $\phi(q_2, x_k) + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq \phi(q_2, y_{(k,i)})$. The above inequalities are equivalent to

$$\|x_k\|^2 - \|y_{(k,i)}\|^2 + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq 2\langle q_1, Jx_k - Jy_{(k,i)}\rangle$$

and

$$\|x_k\|^2 - \|y_{(k,i)}\|^2 + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq 2\langle q_2, Jx_k - Jy_{(k,i)}\rangle.$$

Using the above inequalities, we find that

$$\|x_k\|^2 - \|y_{(k,i)}\|^2 + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq 2\langle q, Jx_k - Jy_{(k,i)}\rangle.$$

That is,

$$\phi(q, x_k) + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} \geq \phi(q, y_{(k,i)}),$$

where $q \in C_{(k,i)}$. This finds that $C_{(k+1,i)}$ is convex and closed. We conclude that $C_{(n,i)}$ is convex and closed. This in turn implies that $C_n = \cap_{i\in\Lambda}C_{(n,i)}$ is convex and closed. Hence, $\Pi_{C_{n+1}}x_1$ is well defined.

Step 3. We prove that $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_n$.

$\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_1 = C$ is clear. Suppose that $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_{(k,i)}$ for some positive integer k. For any $w \in \cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_{(k,i)}$, we see that

$$\begin{aligned}
\phi(w, x_k) + \alpha_{(k,i)}D + (1-\alpha_{(k,i)})\xi_{(k,i)} &\geq \phi(w, x_k) + \alpha_{(k,i)}\phi(w, x_1) - \alpha_{(k,i)}\phi(w, x_k) + (1-\alpha_{(k,i)})\xi_{(k,i)} \\
&\geq \alpha_{(k,i)}\phi(w, x_1) + (1-\alpha_{(k,i)})\phi(w, \tau_{(k,i)}x_k) + (1-\alpha_{(k,i)})\xi_{(k,i)} \\
&= \alpha_{(k,i)}\phi(w, x_1) + (1-\alpha_{(k,i)})\phi(w, f_i^k z_{(k,i)}) \\
&\geq \|w\|^2 + \alpha_{(k,i)}\|x_1\|^2 + (1-\alpha_{(k,i)})\|f_i^k z_{(k,i)}\|^2 \\
&\quad - 2(1-\alpha_{(k,i)})\langle w, Jf_i^k z_{(k,i)}\rangle - 2\alpha_{(k,i)}\langle w, Jx_1\rangle \\
&= \|w\|^2 + \|\alpha_{(k,i)}Jx_1 + (1-\alpha_{(k,i)})Jf_i^k z_{(k,i)})\|^2 \\
&\quad - 2\langle w, \alpha_{(k,i)}Jx_1 + (1-\alpha_{(k,i)})Jf_i^k z_{(k,i)})\rangle \\
&= \phi(w, J^{-1}(\alpha_{(k,i)}Jx_1 + (1-\alpha_{(k,i)})Jf_i^k z_{(k,i)})) \\
&= \phi(w, y_{(k,i)}),
\end{aligned} \tag{2.1}$$

where $D := \sup_{w\in\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}\phi(w, x_1)$. This proves $w \in C_{(k+1,i)}$. Hence, we have

$$\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_{(n,i)}.$$

This in turn implies that $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset \cap_{i\in\Lambda}C_{(n,i)}$. It follows that

$$\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i) \subset C_n.$$

Step 4. We prove that $\{x_n\}$ is a bounded sequence.

Using Lemma 1.6, we see

$$\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0 \;\forall z \in C_n.$$

Since $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is subset of C_n, we find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0 \quad \forall w \in \cap_{i\in\Lambda}F(T_i)\bigcap\cap_{i\in\Lambda}EF(f_i). \tag{2.2}$$

Using Lemma 1.5, we get

$$\phi(\Pi_{\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}x_1, x_n) + \phi(x_n, x_1) \leq \phi(\Pi_{\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}x_1, x_1).$$

Hence, we have

$$\phi(x_n, x_1) \leq \phi(\Pi_{\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}x_1, x_1).$$

This implies that $\{\phi(x_n, x_1)\}$ is a bounded sequence. It follows from (1.3) that sequence $\{x_n\}$ is also a bounded sequence.

Step 5. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. We prove $\bar{x}\in\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$.

Since C_n is convex and closed, we have $\bar{x} \in C_n$. Hence, $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\bar{x}, x_1) &\geq \limsup_{n\to\infty} \phi(x_n, x_1)\\ &= \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle)\\ &= \|\bar{x}\|^2 + \|x_1\|^2 - 2\langle \bar{x}, Jx_1\rangle\\ &= \phi(\bar{x}, x_1).\end{aligned}$$

This implies that $\phi(x_n, x_1) \to \phi(\bar{x}, x_1)$ as $n \to \infty$. Hence, we have $\lim_{n\to\infty}\|x_n\| = \|\bar{x}\|$. In view of Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. Since $x_{n+1} \in C_n$, one has $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. So, $\{\phi(x_n, x_1)\}$ is a nondecreasing sequence. Since $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$, one see that $\lim_{n\to\infty}\phi(x_n, x_1)$ exists. This implies that $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} \in C_{n+1}$, we find that

$$\phi(x_{n+1}, x_n) + \alpha_{(n,i)}D + (1 - \alpha_{(n,i)})\xi_{(n,i)} \geq \phi(x_{n+1}, y_{(n,i)}) \geq 0.$$

Using restriction imposed on $\{\alpha_{(n,i)}\}$, on has $\lim_{n\to\infty}\phi(x_{n+1}, y_{(n,i)}) = 0$. Using (1.3), we see that

$$\lim_{n\to\infty}(\|y_{(n,i)}\| - \|x_{n+1}\|) = 0,$$

which in turn finds

$$\lim_{n\to\infty}\|y_{(n,i)}\| = \|\bar{x}\|.$$

That is,

$$\lim_{n\to\infty}\|Jy_{(n,i)}\| = \|J\bar{x}\| = \lim_{n\to\infty}\|y_{(n,i)}\| = \|\bar{x}\|.$$

Since both E^* and E are reflexive spaces, we may assume that $Jy_{(n,i)} \rightharpoonup y^{(*,i)} \in E^*$. This shows that there exists an element $y^i \in E$ such that $y^{(*,i)} = Jy^i$. It follows that

$$\begin{aligned}\|x_{n+1}\|^2 + \|Jy_{(n,i)}\|^2 - 2\langle x_{n+1}, Jy_{(n,i)}\rangle &= \|x_{n+1}\|^2 + \|y_{(n,i)}\|^2 - 2\langle x_{n+1}, Jy_{(n,i)}\rangle\\ &= \phi(x_{n+1}, y_{(n,i)}).\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0 \leq \phi(\bar{x}, y^i) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy^i\rangle + \|y^i\|^2\\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy^i\rangle + \|Jy^i\|^2\\ &\leq \|\bar{x}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle + \|y^{(*,i)}\|^2\\ &\leq 0.\end{aligned}$$

This implies $y^i = \bar{x}$. Hence, we have $y^{(*,i)} = J\bar{x}$. It follows that $Jy_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since $\lim_{n\to\infty}\alpha_{(n,i)} = 0$ for every $i \in \Lambda$, we find $\lim_{n\to\infty}\|Jy_{(n,i)} - Jf_i^n z_{(n,i)}\| = 0$. Using the fact

$$\|J\bar{x} - Jf_i^n z_{(n,i)}\| \leq \|Jy_{(n,i)} - J\bar{x}\| + \|Jy_{(n,i)} - Jf_i^n z_{(n,i)}\|,$$

one has $Jf_i^n z_{(n,i)} \to J\bar{x}$ as $n \to \infty$ for every $i \in \lambda$. Since J^{-1} is demicontinuous, we have $f_i^n z_{(n,i)} \rightharpoonup \bar{x}$ for every $i \in \Delta$. Since $|\|f_i^n z_{(n,i)}\| - \|\bar{x}\|| \leq \|J(f_i^n z_{(n,i)}) - J\bar{x}\|$, one has $\|f_i^n z_{(n,i)}\| \to \|\bar{x}\|$, as $n \to \infty$ for every $i \in \Lambda$. Since E has the Kadec-Klee property, one obtains

$$\lim_{n\to\infty} \|f_i^n z_{(n,i)} - \bar{x}\| = 0.$$

On the other hand, we have

$$\|f_i^{n+1} z_{(n,i)} - \bar{x}\| \leq \|f_i^{n+1} z_{(n,i)} - f_i^n z_{(n,i)}\| + \|f_i^n z_{(n,i)} - \bar{x}\|.$$

In view of the uniformly asymptotic regularity of f_i, one has

$$\lim_{n\to\infty} \|f_i^{n+1} z_{(n,i)} - \bar{x}\| = 0,$$

that is, $f_i f_i^n z_{(n,i)} - \bar{x} \to 0$ as $n \to \infty$. Since every f_i is a continuous, we find that $f_i\bar{x} = \bar{x}$ for every $i \in \Lambda$.

Next, we prove $\bar{x} \in \cap_{i\in\Lambda} Sol(g_i)$.

Since f_i is continuous, using (2.1), we find that $\lim_{n\to\infty} \phi(x_{n+1}, z_{(n,i)}) = 0$. Using (1.3), we see that $\lim_{n\to\infty}(\|z_{(n,i)}\| - \|x_{n+1}\|) = 0$, which in turn finds $\lim_{n\to\infty} \|z_{(n,i)}\| = \|\bar{x}\|$. That is,

$$\lim_{n\to\infty} \|Jz_{(n,i)}\| = \|J\bar{x}\| = \lim_{n\to\infty} \|z_{(n,i)}\| = \|\bar{x}\|.$$

Since both E^* and E are reflexive, we may assume that $Jz_{(n,i)} \rightharpoonup z^{(*,i)} \in E^*$. This shows that there exists an element $z^i \in E$ such that $z^{(*,i)} = Jz^i$. It follows that

$$\begin{aligned}\|x_{n+1}\|^2 + \|Jz_{(n,i)}\|^2 - 2\langle x_{n+1}, Jz_{(n,i)}\rangle &= \|x_{n+1}\|^2 + \|z_{(n,i)}\|^2 - 2\langle x_{n+1}, Jz_{(n,i)}\rangle \\ &= \phi(x_{n+1}, z_{(n,i)}).\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}\phi(\bar{x}, z^i) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jz^i\rangle + \|z^i\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jz^i\rangle + \|Jz^i\|^2 \\ &\leq \|\bar{x}\|^2 - 2\langle \bar{x}, z^{(*,i)}\rangle + \|z^{(*,i)}\|^2 \\ &\leq 0.\end{aligned}$$

This implies $z^i = \bar{x}$. Hence, we have $z^{(*,i)} = J\bar{x}$. It follows that $Jz_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Using the Kadec-Klee property we find that $Jz_{(n,i)} \to J\bar{x} \in E^*$. Since J^{-1} is demicontinuous, we have $z_{(n,i)} \rightharpoonup \bar{x}$. Using the fact that

$$\|Jy_{(n,i)} - Jx_n\| \leq \|Jy_{(n,i)} - J\bar{x}\| + \|Jx_n - J\bar{x}\|,$$

we see that $\lim_{n\to\infty} \|Jy_{(n,i)} - Jx_n\| = 0$. In view of $z_{(n,i)} = \tau_{r_{(n,i)}} x_n$, we see that

$$\|y - z_{(n,i)}\|\|Jz_{(n,i)} - Jx_n\| \geq r_{(n,i)} g_i(y, z_{(n,i)}) \quad \forall y \in C_n.$$

It follows that $g_i(y, \bar{x}) \leq 0\ \forall y \in C_n$. For $0 < t_i < 1$ and $y \in C_n$, define $y_{(t,i)} = t_i y + (1 - t_i)\bar{x}$. It follows that $y_{(t,i)} \in C_n$, which yields that $g_i(y_{(t,i)}, \bar{x}) \leq 0$. Hence, we have

$$0 = g_i(y_{(t,i)}, y_{(t,i)}) \leq t_i g_i(y_{(t,i)}, y) + (1 - t_i) g_i(y_{(t,i)}, \bar{x}) \leq t_i g_i(y_{(t,i)}, y).$$

That is, $g_i(y_{(t,i)}, y) \geq 0$. Letting $t_i \downarrow 0$, we obtain from $(R - d)$ that $g_i(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in Sol(g_i)$ for every $i \in \Lambda$. This shows that $\bar{x} \in \cap_{i\in\Lambda} Sol(g_i)$. This completes the proof that $\bar{x} \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)$.

Step 6. Prove $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$.
Letting $n \to \infty$ in (2.2), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x} \rangle \geq 0 \quad \forall w \in \cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i).$$

In view of Lemma 1.6, we find that that $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$. This completes the proof. □

If f is a asymptotically quasi-ϕ-nonexpansive mapping, we find from Theorem 2.1 the following.

Corollary 2.2. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping for every $i \in \Lambda$. Assume that f_i is continuous and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ x_1 = \Pi_{C_1 := \cap_{i\in\Lambda} C_{(1,i)}} x_0, \\ y_{(n,i)} = J^{-1}((1 - \alpha_{(n,i)}) J f_i^n z_{(n,i)} + \alpha_{(n,i)} J x_1), \\ C_{(n+1,i)} = \{ z \in C_{(n,i)} : \phi(z, x_n) + \alpha_{(n,i)} D \geq \phi(z, y_{(n,i)}) \}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $D = \sup\{\phi(w, x_1) : w \in \cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)\}$, $z_{(n,i)} \in C_n$ such that $r_{(n,i)} g_i(z_{(n,i)}, y) \geq \langle z_{(n,i)} - y, J z_{(n,i)} - J x_n \rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$.

If T is a quasi-ϕ-nonexpansive mapping, we find from Theorem 2.1 the following.

Corollary 2.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be a quasi-ϕ-nonexpansive mapping for every $i \in \Lambda$. Assume that f_i is continuous for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ x_1 = \Pi_{C_1 := \cap_{i\in\Lambda} C_{(1,i)}} x_0, \\ y_{(n,i)} = J^{-1}((1 - \alpha_{(n,i)}) J f_i z_{(n,i)} + \alpha_{(n,i)} J x_1), \\ C_{(n+1,i)} = \{ z \in C_{(n,i)} : \phi(z, x_n) \geq \phi(z, y_{(n,i)}) \}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $z_{(n,i)} \in C_n$ such that $r_{(n,i)} g_i(z_{(n,i)}, y) \geq \langle z_{(n,i)} - y, J z_{(n,i)} - J x_n \rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[r_i, \infty)$, where $\{r_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$.

In the Hilbert spaces, we have the following deduced results.

Corollary 2.4. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense for every $i \in \Lambda$. Assume that f_i is continuous and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ x_1 = P_{C_1:=\cap_{i\in\Lambda}C_{(1,i)}}x_0, \\ y_{(n,i)} = (1-\alpha_{(n,i)})f_i^n z_{(n,i)} + \alpha_{(n,i)}x_1, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z-x_n\|^2 + \alpha_{(n,i)}D + (1-\alpha_{(n,i)})\xi_{(n,i)} \ge \|z-y_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda}C_{(n+1,i)}, \\ x_{n+1} = P_{C_{n+1}}x_1, \end{cases}$$

where

$$\xi_{(n,i)} = \max\{0, \sup_{p\in Fix(f_i), x\in C}\left(\|p-f_i^n x\|^2 - \|p-x\|^2\right), D = \sup\{\|w-x_1\|^2 : w \in \cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)\},$$

$z_{(n,i)} \in C_n$ such that $r_{(n,i)}g_i(z_{(n,i)}, y) \ge \langle z_{(n,i)} - y, z_{(n,i)} - x_n\rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[r_i,\infty)$, where $\{r_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}x_1$.

If Tf is an asymptotically quasi-nonexpansive mapping, we find from Theorem 2.1 the following.

Corollary 2.5. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be an asymptotically quasi-nonexpansive mapping for every $i \in \Lambda$. Assume that f_i is continuous and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, x_1 = P_{C_1:=\cap_{i\in\Lambda}C_{(1,i)}}x_0, \\ y_{(n,i)} = (1-\alpha_{(n,i)})f_i^n z_{(n,i)} + \alpha_{(n,i)}x_1, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z-x_n\|^2 + \alpha_{(n,i)}D \ge \|z-y_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda}C_{(n+1,i)}, x_{n+1} = P_{C_{n+1}}x_1, \end{cases}$$

where $D = \sup\{\|w-x_1\|^2 : w \in \cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)\}$, $z_{(n,i)} \in C_n$ such that $r_{(n,i)}g_i(z_{(n,i)}, y) \ge \langle z_{(n,i)} - y, z_{(n,i)} - x_n\rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_{(n,i)} = 0$ and $\{r_{(n,i)}\}$ is a real sequence in $[r_i,\infty)$, where $\{r_i\}$ is a positive real number sequence for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)}x_1$.

If f is a closed quasi-nonexpansive mapping, we find from Theorem 2.1 the following.

Corollary 2.6. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let Λ be an index set. Let g_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-a), (R-b), (R-c), (R-d) and let $f_i : C \to C$ be a quasi-nonexpansive mapping for every $i \in \Lambda$. Assume that f_i is continuous and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda}Fix(f_i)\bigcap\cap_{i\in\Lambda}Sol(g_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, x_1 = P_{C_1:=\cap_{i\in\Lambda} C_{(1,i)}} x_0, \\ y_{(n,i)} = (1-\alpha_{(n,i)}) f_i z_{(n,i)} + \alpha_{(n,i)} x_1, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|z - x_n\|^2 \geq \|z - y_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $z_{(n,i)} \in C_n$ *such that* $r_{(n,i)} g_i(z_{(n,i)}, y) \geq \langle z_{(n,i)} - y, z_{(n,i)} - x_n\rangle$ $\forall y \in C_n$, $\{\alpha_{(n,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ *and* $\{r_{(n,i)}\}$ *is a real sequence in* $[r_i, \infty)$*, where* $\{r_i\}$ *is a positive real number sequence for every* $i \in \Lambda$*. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Lambda} Fix(f_i) \bigcap \cap_{i\in\Lambda} Sol(g_i)} x_1$.

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