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Several improvements of Mitrinović-Adamović and Lazarević's inequalities with applications to the sharpening of Wilker-type inequalities

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Abstract

In this paper, we give several improvements of Mitrinović-Adamović's inequality and Lazarević's inequality. Our results show some interesting relationships between Mitrinović-Adamović's inequality and Lazarević's inequality. At the end of the paper, the improved Lazarević's inequality is applied to the sharpening of Wilker-type inequalities for hyperbolic functions.

Keywords: Mitrinović-Adamović's inequality, Lazarević's inequality, Wilker-type inequalities, hyperbolic functions, improvement.
2010 MSC: 26D05, 26D15, 33B10.

1. Introduction

In 1965, Mitrinović and Adamović [5] proved that the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \tag{1.1}$$

holds for all $x \in (0, \pi/2)$ and showed that the exponent 3 is the largest possible. A year later, a hyperbolic analogue of inequality (1.1) was presented by Lazarević in [4], as follows

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$$\cosh x < \left(\frac{\sinh x}{x}\right)^3, \tag{1.2}$$

where $x \neq 0$, and the exponent 3 is the least possible.

During the past several years there has been a great deal of interest in inequalities of Mitrinović-Adamović type and Lazarević type. Some generalizations, improvements and variants of the Mitrinović-Adamović and Lazarević's inequalities can be found in the literature [3, 6, 7, 8, 9, 13, 17, 18, 19, 20, 21, 23].

The main purpose of this paper is to improve Mitrinović-Adamović's inequality (1.1) and Lazarević's inequality (1.2). As special cases of our results, the following improved versions of inequalities (1.1) and (1.2) will be obtained respectively.

$$\cos x < \left(\frac{\sin x}{x}\right)^3, \tag{1.3}$$

where $0 < |x| < x_0$ and $x_0 \approx 4.70277543$ is the unique real root of the equation $\cos x - ((\sin x)/x)^3 = 0$ in the interval $(\pi, 2\pi)$.

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|, \tag{1.4}$$

where $x \neq 0$.

Moreover, some new inequalities of Mitrinović-Adamović and Lazarević type are established, which reveals some interesting relationships between Mitrinović-Adamović's inequality and Lazarević's inequality. Several complementary inequalities which are related to inequalities (1.1) and (1.2) are also considered. In Section 4, the improved Lazarević's inequality is applied to the sharpening of Wilker-type inequalities for hyperbolic functions.

2. Lemmas

In order to prove the main results in Sections 3 and 4, we first introduce the following lemmas.

Lemma 2.1 ([2])**.** *If* $x_i > 0$, $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ *and* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, *then*

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}. \tag{2.1}$$

Lemma 2.2. *For any positive real numbers x, the following inequality holds*

$$\sin x < \sinh x. \tag{2.2}$$

Proof. Consider the function

$$f(x) = \sin x - \sinh x, \quad x \in (0, +\infty).$$

Differentiating $f(x)$ with respect to x gives

$$f'(x) = \cos x - \cosh x.$$

It is easy to observe that

$$\cos x \leq 1$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2} > \sqrt{e^x e^{-x}} = 1.$$

Hence $f'(x) < 0$ for $x \in (0, +\infty)$, it follows that $f(x)$ is decreasing on $(0, +\infty)$. Hence, by $f(0) = 0$, we obtain $f(x) < 0$ for $x \in (0, +\infty)$, which implies the desired inequality (2.2). The Lemma 2.2 is proved.

□

3. Improvements of Mitrinović-Adamović's Inequality

Throughout this paper, let $\mathbb{R}$ and $\mathbb{Z}^+$ denote respectively the set of real numbers and the set of positive integers. We first give a generalization of the Mitrinović-Adamović's inequality (1.1), as follows

Theorem 3.1. *Let* $x_0 = 0$, *and let* $x_n (n \in \mathbb{Z}^+)$ *be the unique real root of the equation* $\cos x - ((\sin x)/x)^3 = 0$ *in* $(n\pi, (n+1)\pi)$.

(i) If $x \in (-x_{2k+1}, -x_{2k}) \cup (x_{2k}, x_{2k+1})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, *then*

$$\cos x < \left(\frac{\sin x}{x}\right)^3. \tag{3.1}$$

(ii) If $x \in (-x_{2k+2}, -x_{2k+1}) \cup (x_{2k+1}, x_{2k+2})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, *then*

$$\cos x > \left(\frac{\sin x}{x}\right)^3. \tag{3.2}$$

Proof. Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \cos x - (\dfrac{\sin x}{x})^3 & \text{if } \ x \neq 0, \\ 0 & \text{if } \ x = 0. \end{cases}$$

Furthermore, it is easy to see that $f(x)$ is a continuous even function. Hence, to prove the validity of inequalities (3.1) and (3.2) in the given intervals in Theorem 3.1, it is sufficient to prove that the inequality (3.1) holds for $x \in (x_{2k}, x_{2k+1})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, and the inequality (3.2) holds for $x \in (x_{2k+1}, x_{2k+2})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, respectively. So, we need only consider the case of $x \in (0, +\infty)$ in the following discussion.

Differentiating $f(x)$ with respect to x gives

$$\begin{aligned} f'(x) &= \frac{1}{4x^4}(-4x^4 \sin x + 12x(\cos x)(-\sin^2 x) - 9\cos^2 x \sin x + 3\sin^3 x + 9\sin x) \\ &= \frac{1}{2x^4}(\sin x)(3 - 3x\sin 2x - 2x^4 - 3\cos 2x) \\ &= \frac{1}{2x^4}(\sin x) g_1(x), \end{aligned}$$

where $g_1(x) = 3 - 3x\sin 2x - 2x^4 - 3\cos 2x$. Further, computing the derivative of $g_1(x)$ gives

$$\begin{aligned} g_1'(x) =& 3\sin 2x - 6x\cos 2x - 8x^3 \\ g_1''(x) =& 12x\,(\sin 2x - 2x) = 12x g_2(x), \end{aligned}$$

and

$$g_2'(x) = 2\,(\cos 2x - 1) \leq 0.$$

From $g_2'(x) \leq 0$ for $x \in (0, +\infty)$ and $g_2(0) = 0$, we conclude that the function $g_2(x)$ is decreasing on $(0, +\infty)$ and $g_2(x) < 0$, which leads to $g_1''(x) < 0$.

Further, it follows that $g_1'(x)$ is decreasing on $(0, +\infty)$ and $g_1'(x) < g_1'(0) = 0$. Consequently, we conclude that $g_1(x)$ is decreasing on $(0, +\infty)$, hence, by $g_1(0) = 0$, we obtain

$$g_1(x) < 0 \quad \text{for} \quad x \in (0, +\infty). \tag{3.3}$$

By $f'(x) = g_1(x)(\sin x)/(2x^4)$, it is easy to observe that $f'(x)$ does not change sign on $(n\pi, (n+1)\pi)$. Thus we infer that $f(x)$ is a monotonous function on $(n\pi, (n+1)\pi)$.

On the other hand, since $f(x)$ is a continuous function on $[n\pi, (n+1)\pi]$ with

$$f(n\pi)f((n+1)\pi) = \cos(n\pi)\cos((n+1)\pi) < 0,$$

which, along with the monotonicity of $f(x)$, implies that the equation $\cos x - ((\sin x)/x)^3 = 0$ has unique real root in $(n\pi, (n+1)\pi)$.

Next, let us consider two cases below

Case 1. $x \in (x_{2k}, x_{2k+1})$ $(k \in \mathbb{Z}^+ \cup \{0\})$.

For any $k \in \mathbb{Z}^+ \cup \{0\}$, it follows from the assumption of Theorem 3.1 that

$$x_{2k} \in (2k\pi, (2k+1)\pi), \quad x_{2k+1} \in ((2k+1)\pi, (2k+2)\pi),$$

where x_{2k}, x_{2k+1} are the real roots of the equation $\cos x - ((\sin x)/x)^3 = 0$.
It is easy to observe that

$$\sin x > 0 \quad \text{for} \quad x \in (2k\pi, (2k+1)\pi)$$

and

$$\sin x < 0 \quad \text{for} \quad x \in ((2k+1)\pi, (2k+2)\pi).$$

Hence, we conclude that

$$f'(x) < 0 \quad \text{for} \quad x \in (x_{2k}, (2k+1)\pi)$$

and

$$f'(x) > 0 \quad \text{for} \quad x \in ((2k+1)\pi,\ x_{2k+1}).$$

This means that $f(x)$ is decreasing on $(x_{2k}, (2k+1)\pi)$, and $f(x)$ is increasing on $((2k+1)\pi,\ x_{2k+1})$. Now, from the assumption that x_{2k}, x_{2k+1} are the real roots of the equation $\cos x - ((\sin x)/x)^3 = 0$, we obtain

$$f(x) < f(x_{2k}) = 0 \ \text{ for } x \in (x_{2k}, (2k+1)\pi)$$

and

$$f(x) < f(x_{2k+1}) = 0 \quad \text{for} \quad x \in ((2k+1)\pi,\ x_{2k+1}).$$

Thus

$$f(x) < 0 \quad \text{for} \quad x \in (x_{2k},\ x_{2k+1}),\ k \in \mathbb{Z}^+ \cup \{0\}.$$

In view of $f(x)$ is an even function, we claim that $f(x) < 0$ for $x \in (-x_{2k+1}, -x_{2k}) \cup (x_{2k}, x_{2k+1})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, which implies the desired inequality (3.1).

Case 2. $x \in (x_{2k+1}, x_{2k+2})$ $(k \in \mathbb{Z}^+ \cup \{0\})$.

For any $k \in \mathbb{Z}^+ \cup \{0\}$, one has

$$x_{2k+1} \in ((2k+1)\pi, (2k+2)\pi), \quad x_{2k+2} \in ((2k+2)\pi, (2k+3)\pi),$$

where x_{2k+1}, x_{2k+2} are the real roots of the equation $\cos x - ((\sin x)/x)^3 = 0$.
Since

$$\sin x < 0 \quad \text{for} \quad x \in ((2k+1)\pi, (2k+2)\pi)$$

and

$$\sin x > 0 \quad \text{for} \quad x \in ((2k+2)\pi, (2k+3)\pi),$$

we deduce that

$$f'(x) > 0 \quad \text{for} \quad x \in (x_{2k+1}, (2k+2)\pi)$$

and

$$f'(x) < 0 \quad \text{for} \quad x \in ((2k+2)\pi,\ x_{2k+2}).$$

Hence, $f(x)$ is increasing on $(x_{2k+1}, (2k+2)\pi)$, and $f(x)$ is decreasing on $((2k+2)\pi,\ x_{2k+2})$, which, along with the fact that x_{2k+1}, x_{2k+2} are the real roots of the equation $\cos x - ((\sin x)/x)^3 = 0$, we get

$$f(x) > f(x_{2k+1}) = 0 \quad \text{for } x \in \ (x_{2k+1}, (2k+2)\pi)$$

and

$$f(x) > f(x_{2k+2}) = 0 \quad \text{for} \quad x \in ((2k+2)\pi,\ x_{2k+2}).$$

Therefore

$$f(x) > 0 \quad \text{for} \quad x \in (x_{2k+1},\ x_{2k+2}),\ k \in \mathbb{Z}^+ \cup \{0\}.$$

Note that $f(x)$ is an even function, we conclude that $f(x) > 0$ for $x \in (-x_{2k+2}, -x_{2k+1}) \cup (x_{2k+1}, x_{2k+2})$ $(k \in \mathbb{Z}^+ \cup \{0\})$, which is the required inequality (3.2). The proof of Theorem 3.1 is completed. □

As a direct consequence of the Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Suppose that* $0 < |x| < x_0$ *and* $x_0 \approx 4.70277543$ *is the unique real root of the equation* $\cos x - ((\sin x)/x)^3 = 0$ *in* $(\pi, 2\pi)$. *Then we have the inequality*

$$\cos x < \left(\frac{\sin x}{x}\right)^3. \tag{3.4}$$

It is worth nothing that the Mitrinović-Adamović's inequality (1.1) would follow as a special case of Corollary 3.2 when $x \in (0, \pi/2)$.

Theorem 3.3. *Let* x *be a nonzero real number. Then we have the inequality*

$$\left|\cos x - \left(\frac{\sin x}{x}\right)^3\right| \le 1. \tag{3.5}$$

Proof. Consider the function

$$f(x) = \cos x - \left(\frac{\sin x}{x}\right)^3, \quad x \in (-\infty, 0) \cup (0, +\infty).$$

In the proof of Theorem 3.1, it is proved that

$$f'(x) = g_1(x)(\sin x)/(2x^4)$$

and

$$g_1(x) < 0 \quad \text{for} \quad x \in (0, +\infty).$$

This yields, for all $k \in \mathbb{Z}^+ \cup \{0\}$,

$$f'(x) < 0 \quad \text{for} \quad x \in (2k\pi, (2k+1)\pi)$$

and

$$f'(x) > 0 \quad \text{for} \quad x \in ((2k+1)\pi, (2k+2)\pi).$$

Thus, $f(x)$ is decreasing on $(2k\pi, (2k+1)\pi)$, and $f(x)$ is increasing on $((2k+1)\pi, (2k+2)\pi)$, where $k \in \mathbb{Z}^+ \cup \{0\}$. Now, from

$$\lim_{x\to 0} f(x) = 0,\ f(\pi) = -1,\ f(2k\pi) = 1,\ f((2k+1)\pi) = -1,\ f((2k+2)\pi) = 1\ \ (k = 1, 2...),$$

we deduce that

$$-1 \le f(x) \le 1, \quad x \in (0, +\infty).$$

Since $f(x)$ is an even function, we also have

$$-1 \le f(x) \le 1, \quad x \in (-\infty, 0).$$

Therefore, $|f(x)| \le 1$ for $x \in (-\infty, 0) \cup (0, +\infty)$, which implies the desired inequality (3.5). The Theorem 3.3 is proved. □

Theorem 3.4. *If* $0 < \alpha < \beta$ *(or* $\beta < \alpha < 0$*), then*

$$\cosh\beta - \left(\frac{\sinh\beta}{\beta}\right)^3 < \cosh\alpha - \left(\frac{\sinh\alpha}{\alpha}\right)^3. \tag{3.6}$$

Proof. Define a function ψ by

$$\psi(x) = \cosh x - \left(\frac{\sinh x}{x}\right)^3, \quad x \in (-\infty, 0) \cup (0, +\infty).$$

Differentiating $\psi(x)$ with respect to x gives

$$\begin{aligned}\psi'(x) &= \frac{1}{2x^4}(\sinh x)(3\cosh 2x - 3x\sinh 2x + 2x^4 - 3)\\ &= \frac{1}{2x^4}(\sinh x)\omega_1(x),\end{aligned}$$

where $\omega_1(x) = 3\cosh 2x - 3x\sinh 2x + 2x^4 - 3$.
Further, one has

$$\omega_1'(x) = 3\sinh 2x - 6x\cosh 2x + 8x^3$$

and

$$\omega_1''(x) = -12x\,(\sinh 2x - 2x) = -12x\omega_2(x).$$

From

$$\omega_2'(x) = 2\cosh 2x - 2 = e^{2x} + e^{-2x} - 2 > 0,$$

we infer that $\omega_2(x)$ is increasing on $(0, +\infty)$ and $\omega_2(x) > 0$. Thus, we have $\omega_1''(x) < 0$ for $(0, +\infty)$ and $\omega_1'(x)$ is decreasing on $(0, +\infty)$, which yields $\omega_1'(x) < 0$ for $(0, +\infty)$ and $\omega_1(x)$ is decreasing on $(0, +\infty)$. Therefore, $\omega_1(x) < 0$ for $(0, +\infty)$.

Since $\sinh x > 0$ for $(0, +\infty)$, we deduce that $\psi'(x) < 0$ for $(0, +\infty)$. This means that $\psi(x)$ is decreasing on $(0, +\infty)$, which implies that the inequality

$$\cosh\beta - \left(\frac{\sinh\beta}{\beta}\right)^3 < \cosh\alpha - \left(\frac{\sinh\alpha}{\alpha}\right)^3$$

holds for $0 < \alpha < \beta$. Also, we can claim that the above inequality holds also for $\beta < \alpha < 0$ since $\psi(x)$ is an even function. This completes the proof of Theorem 3.4. □

As a consequence of Theorem 3.4, taking a limit as $\alpha \to 0$ in (3.6) yields immediately the Lazarević's inequality (1.2).

Theorem 3.5. *For any nonzero real numbers* x*, the following inequality holds*

$$\cosh x - \left(\frac{\sinh x}{x}\right)^3 < \cos x - \left(\frac{\sin x}{x}\right)^3. \tag{3.7}$$

Proof. Define a function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\phi(x) = \begin{cases} \cosh x - \left(\dfrac{\sinh x}{x}\right)^3 - \cos x + \left(\dfrac{\sin x}{x}\right)^3 & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

It is obviously that $\phi(x)$ is a continuous even function. Hence, to prove the validity of inequality (3.7) for $x \in (-\infty, 0) \cup (0, +\infty)$, it is enough to prove that the inequality (3.7) holds for $x \in (0, +\infty)$. So, we consider the case of $x \in (0, +\infty)$ in the following discussion.

Differentiating $\phi(x)$ with respect to x gives

$$\begin{aligned}\phi'(x) &= \frac{d}{dx}\left(\cosh x - \left(\frac{\sinh x}{x}\right)^3\right) - \frac{d}{dx}\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right) \\ &= \frac{1}{2x^4}(\sinh x)(3\cosh 2x - 3x\sinh 2x + 2x^4 - 3) - \frac{1}{2x^4}(\sin x)(3 - 3x\sin 2x - 2x^4 - 3\cos 2x).\end{aligned}$$

It is proved in the proof of Theorem 3.1 that

$$3 - 3x\sin 2x - 2x^4 - 3\cos 2x < 0 \ \text{ for } \ x \in (0, +\infty),$$

(see the inequality (3.3)). Now, using Lemma 2.2 gives

$$\begin{aligned}\phi'(x) &\leq \frac{1}{2x^4}(\sinh x)(3\cosh 2x - 3x\sinh 2x + 2x^4 - 3) - \frac{1}{2x^4}(\sinh x)(3 - 3x\sin 2x - 2x^4 - 3\cos 2x) \\ &= \frac{1}{2x^4}(\sinh x)(3\cosh 2x - 3x\sinh 2x + 3\cos 2x + 3x\sin 2x + 4x^4 - 6) \\ &= \frac{1}{2x^4}(\sinh x)h_1(x),\end{aligned}$$

where $h_1(x) = 3\cosh 2x - 3x\sinh 2x + 3\cos 2x + 3x\sin 2x + 4x^4 - 6$.
Computing the derivative of $h_1(x)$ with respect to x yields

$$\begin{aligned}h_1'(x) &= 3\sinh 2x - 3\sin 2x + 6x\cos 2x - 6x\cosh 2x + 16x^3, \\ h_1''(x) &= 12x\,(4x - \sinh 2x - \sin 2x) = 12xh_2(x).\end{aligned}$$

Further, we have

$$\begin{aligned}h_2'(x) &= 4 - 2\cos 2x - 2\cosh 2x, \\ h_2''(x) &= 4(\sin 2x - \sinh 2x).\end{aligned}$$

Again, using the Lemma 2.2, we get

$$h_2''(x) < 0 \ \text{ for } \ x \in (0, +\infty).$$

Thus, we claim that the function $h_2'(x)$ is decreasing on $(0, +\infty)$ and $h_2'(x) < h_2'(0) = 0$, which implies that the function $h_2(x)$ is decreasing on $(0, +\infty)$ and $h_2(x) < h_2(0) = 0$. Hence, we have $h_1''(x) < 0$ for $(0, +\infty)$.

Further, we deduce that $h_1'(x)$ is decreasing on $(0, +\infty)$ and $h_1'(x) < h_1'(0) = 0$. Therefore, we infer that $h_1(x)$ is decreasing on $(0, +\infty)$, finally, by $h_1(0) = 0$, we obtain

$$h_1(x) < 0 \ \text{ for } \ x \in (0, +\infty).$$

On the other hand, it is evident that

$$\sinh x = \frac{e^x - e^{-x}}{2} > 0 \quad \text{for} \quad x \in (0, +\infty),$$

which leads to $\phi'(x) < 0$ for $x \in (0, +\infty)$. Consequently, the function $\phi(x)$ is decreasing on $(0, +\infty)$. Now, by $\phi(0) = 0$, we get

$$\phi(x) < 0 \quad \text{for} \quad x \in (0, +\infty).$$

Since $\phi(x)$ is an even function, we deduce that $\phi(x) < 0$ for $x \in (-\infty, +\infty)$, which implies the desired inequality (3.7). This completes the proof of Theorem 3.5. □

Theorem 3.6. *For any nonzero real numbers x, the following inequality holds*

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|. \tag{3.8}$$

Proof. The inequality (3.8) is equivalent to

$$\left|\cos x - \left(\frac{\sin x}{x}\right)^3\right| < \left(\frac{\sinh x}{x}\right)^3 - \cosh x,$$

that is,

$$\cosh x - \left(\frac{\sinh x}{x}\right)^3 < \cos x - \left(\frac{\sin x}{x}\right)^3 < \left(\frac{\sinh x}{x}\right)^3 - \cosh x.$$

The left-hand side inequality of (3.8) is just the result of Theorem 3.5. We need now to show the validity of right-hand side inequality of (3.8).

Case 1. If $0 < x \leq \pi$, then, by Corollary 3.2, one has

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < 0,$$

which, along with the Lazarević's inequality (1.2) gives

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < 0 < \left(\frac{\sinh x}{x}\right)^3 - \cosh x.$$

Case 2. If $x > \pi$, then, form Theorem 3.3, it follows that

$$\cos x - \left(\frac{\sin x}{x}\right)^3 \leq 1.$$

On the other hand, using Theorem 3.4 gives

$$\left(\frac{\sinh x}{x}\right)^3 - \cosh x > \left(\frac{\sinh \pi}{\pi}\right)^3 - \cosh \pi > 1.$$

Hence

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < \left(\frac{\sinh x}{x}\right)^3 - \cosh x.$$

Combining the Cases 1 and 2 leads us to

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < \left(\frac{\sinh x}{x}\right)^3 - \cosh x$$

for $x \in (0, +\infty)$.

If $x \in (-\infty, 0)$, then we have $-x > 0$. A straightforward application of the above result yields that

$$\cos(-x) - \left(\frac{\sin(-x)}{-x}\right)^3 < \left(\frac{\sinh(-x)}{-x}\right)^3 - \cosh(-x),$$

that is,

$$\cos x - \left(\frac{\sin x}{x}\right)^3 < \left(\frac{\sinh x}{x}\right)^3 - \cosh x.$$

The right-hand side inequality of (3.8) is proved. The proof of Theorem 3.6 is completed. □

4. Sharpening Wilker-type Inequality for Hyperbolic Functions

In 1989, Wilker proposed the following inequality as an open problem (see [11])

Prove that, if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{4.1}$$

Sumner et al. [10] proved the inequality (4.1). Guo et al. [1], Zhu [24], Zhang and Zhu [22] showed different proofs of the Wilker's inequality. Wu and Srivastava [12, 15, 16] gave some refinements of Wilker's inequality.

In 2007, an inequality of Wilker-type for hyperbolic functions was established by Zhu [25]

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 \quad (x \neq 0). \tag{4.2}$$

Wu and Debnath [14] gave the following sharpened version of inequality (4.2)

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x \quad (x \neq 0). \tag{4.3}$$

In this section, we present a new generalized and sharpened form of inequality (4.2), as follows

Theorem 4.1. *Let $0 < \theta < 1$ and $p(1-\theta) \geq 2q\theta > 0$. Then, for all nonzero real numbers x, the following inequality holds*

$$(1-\theta)\left(\frac{\sinh x}{x}\right)^p + \theta\left(\frac{\tanh x}{x}\right)^q > \left(1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{q\theta}. \tag{4.4}$$

Proof. By using Lemma 2.1 and Theorem 3.6, together with the assumption conditions of $0 < \theta < 1$ and $p(1-\theta) \geq 2q\theta > 0$, it follows that

$$\begin{aligned}(1-\theta)\left(\frac{\sinh x}{x}\right)^p &+ \theta\left(\frac{\tanh x}{x}\right)^q \\ &\geq \left(\frac{\sinh x}{x}\right)^{p(1-\theta)}\left(\frac{\tanh x}{x}\right)^{q\theta}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\sinh x}{x}\right)^{p(1-\theta)} \left(\frac{\sinh x}{x}\right)^{q\theta} \left(\frac{1}{\cosh x}\right)^{q\theta} \\
&> \left(\frac{\sinh x}{x}\right)^{p(1-\theta)} \left(\frac{\sinh x}{x}\right)^{q\theta} \left(\left(\frac{\sinh x}{x}\right)^3 - \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{-q\theta} \\
&= \left(\frac{\sinh x}{x}\right)^{p(1-\theta)+q\theta-3q\theta} \left(1 - \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{-q\theta} \\
&= \left(\frac{\sinh x}{x}\right)^{p(1-\theta)-2q\theta} \left(1 - \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{-q\theta} \\
&\geq \left(1 - \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{-q\theta} \\
&= \left(\frac{1}{1 - (\frac{x}{\sinh x})^3 \left|\cos x - (\frac{\sin x}{x})^3\right|}\right)^{q\theta} \\
&= \left(1 + \frac{(\frac{x}{\sinh x})^3 \left|\cos x - (\frac{\sin x}{x})^3\right|}{1 - (\frac{x}{\sinh x})^3 \left|\cos x - (\frac{\sin x}{x})^3\right|}\right)^{q\theta} \\
&\geq \left(1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{q\theta}.
\end{aligned}
$$

This proves the required inequality (4.4), The proof of Theorem 4.1 is completed. □

Next, we give some direct consequences of Theorem 4.1. Setting $p = 2$, $q = 1$ in Theorem 4.1, we get,

Corollary 4.2. *If* $x \neq 0$ *and* $0 < \theta \leq \frac{1}{2}$, *then*

$$(1-\theta)\left(\frac{\sinh x}{x}\right)^2 + \theta\left(\frac{\tanh x}{x}\right) > \left(1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{\theta}. \tag{4.5}$$

Putting $\theta = \frac{1}{3}$ in Theorem (4.1) yields,

Corollary 4.3. *If* $x \neq 0$ *and* $p \geq q > 0$, *then*

$$2\left(\frac{\sinh x}{x}\right)^p + \left(\frac{\tanh x}{x}\right)^q > 3\left(1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{\frac{q}{3}}. \tag{4.6}$$

Putting $\theta = \frac{1}{2}$ in Theorem 4.1 gives,

Corollary 4.4. *If* $x \neq 0$ *and* $p \geq 2q > 0$, *then*

$$\left(\frac{\sinh x}{x}\right)^p + \left(\frac{\tanh x}{x}\right)^q > 2\left(1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|\right)^{\frac{q}{2}}. \tag{4.7}$$

In particular, if we choose $p = 2$ and $q = 1$ in the inequality (4.7), the following Wilker-type inequality for hyperbolic functions is derived.

Corollary 4.5. *For all nonzero real numbers x, the following inequality holds*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2\sqrt{1 + \left(\frac{x}{\sinh x}\right)^3 \left|\cos x - \left(\frac{\sin x}{x}\right)^3\right|}. \tag{4.8}$$

Obviously, the inequality (4.8) is a sharpened version of the Wilker-type inequality (4.2).

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