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A research on the some properties and distribution of zeros for Stirling polynomials

Jung Yoog Kang*, Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea.

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Abstract

We find some identities of the Stirling polynomials and relations between these polynomials and other numbers and polynomials such as generalized Bernoulli numbers. We also display some properties and figures that are related to the distribution of fixed points in the Stirling polynomials from the Newton dynamical system containing iterated mapping.

Keywords: Stirling polynomials, generalized Bernoulli numbers and polynomials, Euler polynomials of the second kind, Newton dynamical system, fixed point.
2010 MSC: 11B68, 11B75, 12D10.

1. Introduction

Research on the Bernoulli, Euler, and Genocchi polynomials, has been actively conducted by many mathematicians. Ira Gessel, R. P. Stanley, A. Erdel and S. Roman introduced the Stirling polynomials, and these polynomials are continuously researched by many mathematicians (see [5, 6, 7, 9, 10, 11, 13, 14, 15, 16]). Recently, Qi constructed the differentiable of various polynomials that are related to the Stirling polynomials and Ryoo discovered the properties of zeros in the Stirling polynomials(see [6, 9, 10]).

Definition 1.1. For nonnegative integers n, the Stirling polynomials are defined by

$$\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!} = \left(\frac{t}{1-e^{-t}}\right)^{x+1}, \quad |t| < 2\pi.$$

*Corresponding author
Email addresses: rkdwjddnr2002@yahoo.co.kr (Jung Yoog Kang), ryoocs@hnu.kr (Cheon Seoung Ryoo)

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From Definition 1.1, we know the relations between these polynomials and the Stirling numbers of the first and second kinds.

Theorem 1.2 ([2, 3, 8, 11, 12, 15]). *Let $m \in \mathbb{N}_0 \cup \{-1\}$. Then we have*

$$S_n(-m) = \frac{(-1)^k}{\binom{n+m-1}{n}} S(n+m-1, m-1),$$

where $S(m,n)$ is the Stirling numbers of the second kind.

$$S_n(m) = \frac{(-1)^k}{\binom{m}{n}} s(m+1, m+1-n),$$

where $s(m,n)$ is the Stirling numbers of the first kind.

Theorem 1.3. *Explicit representations involving Stirling polynomials can be deduced with Lagrange's interpolation formula:*

$$S_n(x) = n! \sum_{j=0}^{n} (-1)^{n-j} \sum_{m=j}^{n} \binom{x+m}{m} \binom{m}{j} L_{n+m}^{(-n-j)}(-j),$$

where $L_n^{(\alpha)}$ are Laguerre polynomials. We also can note that $S_n(0) = (-1)^n B_n$, where B_n are the Bernoulli numbers (*see* [4, 5, 7]).

The first seven Stirling polynomials are (*see* [9])*:*

$$\begin{aligned}
S_0(x) &= 1, \\
S_1(x) &= \frac{1}{2}(1+x), \\
S_2(x) &= \frac{1}{12}\left(2+5x+3x^2\right), \\
S_3(x) &= \frac{1}{8}\left(x+2x^2+x^3\right), \\
S_4(x) &= \frac{1}{240}\left(-8-18x+5x^2+30x^3+15x^4\right), \\
S_5(x) &= \frac{1}{96}\left(-6x-13x^2-5x^3+5x^4+3x^5\right), \\
S_6(x) &= \frac{1}{4032}\left(96+236x-84x^2-539x^3-315x^4+63x^5+63x^6\right).
\end{aligned}$$

The Stirling polynomials are a family of polynomials with important applications in the branch of mathematics dealing with combinatorics, number theory and numerical analysis. This paper is organized as follows. In Section 2, we investigate some basic properties of Stirling polynomials and find some relations between Stirling polynomials and various numbers and polynomials. In Section 3, we study some properties of zeros for Stirling polynomials from Newton's method. In section 4, we find distribution of fixed point for Stirling polynomials by using iterating map.

2. Some properties of the Stirling polynomials

In this section, from the definition of the Stirling polynomials, we find some basic properties of these polynomials, such as the recurrence formula and addition theorem. We also investigate the relations between the Stirling polynomials, the generalized Bernoulli numbers and Euler polynomials of the second kind.

Theorem 2.1. *Let $x \in \mathbb{N}_0 \cup \{-1\}$. Then we have*

$$\sum_{k=0}^{x} \binom{x}{k} (-1)^k \left(S(x-1) - k\right)^n = \begin{cases} n! & \text{if } n = x \\ 0 & \text{if } n \neq x \end{cases}.$$

Proof. When $x \in \mathbb{N}_0 \cup \{-1\}$, we can express the Definition 1.1 as follows.

$$\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}(1-e^{-t})^{x+1} = t^{x+1}.$$

From the binomial formula, we can find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{x+1}\binom{x+1}{k}(-1)^k\left(S_n(x)-k\right)^n\frac{t^n}{n!} = t^{x+1}.$$

Therefore, we complete the proof of Theorem 2.1. □

Theorem 2.2. *For integers $n \geq 0$, we obtain*

$$S_n(x+y) = \frac{1}{n+1}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l+1}{m}\binom{n+1}{l+1}(-1)^{l-m}S_m(x)S_{n-l}(y)\right).$$

Proof. From the generating function, we have

$$\sum_{n=0}^{\infty} S_n(x+y)\frac{t^n}{n!} = \left(\frac{1-e^{-t}}{t}\right)\left(\frac{t}{1-e^{-t}}\right)^{x+y+2}.$$

We can transform the above equation as follows.

$$\sum_{n=0}^{\infty} S_n(x+y)\frac{t^{n+1}}{n!} = (1-e^{-t})\left(\frac{t}{1-e^{-t}}\right)^{x+1}\left(\frac{t}{1-e^{-t}}\right)^{y+1}.$$

We can note that

$$1-e^{-t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{n+1}}{(n+1)!}. \tag{2.1}$$

From equation (2.1), we find

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)S_n(x+y)\frac{t^{n+1}}{(n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{n+1}}{(n+1)!}\sum_{n=0}^{\infty}S_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_n(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}S_m(x)\frac{t^{n+1}}{(n+1)!}\sum_{n=0}^{\infty}S_n(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l+1}{m}\binom{n+1}{l+1}(-1)^{l-m}S_m(x)S_{n-l}(y)\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

By comparing the coefficients of both sides in the above equation, we can complete Theorem 2.2. □

Theorem 2.3. *For any nonnegative integer n, one has*

(i) $S_n(x) = (x+1-S(x))^n$,

(ii) $S_n(x-1) = \frac{1}{n+1}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m+1}{l}\binom{n+1}{m+1}(-1)^{m-n}x^l S_{n-m}(x)\right)$,

(iii) *let $x \leq n$ and $x \in \mathbb{N}_0 \cup \{-1\}$. Then*

$$\binom{n}{x}(-1)^{n-x}x!S_n(-x+1) = \sum_{k=0}^{x-1}\binom{x-1}{k}(-1)^{x-1-k}k^n.$$

Proof. From Definition 1.1, we can obtain the following equation by substituting t with $-t$.

$$\begin{aligned}\sum_{n=0}^{\infty} S_n(x)\frac{(-t)^n}{n!} &= \left(\frac{te^{-t}}{1-e^{-t}}\right)^{x+1} \\ &= \sum_{n=0}^{\infty}(-1)^n(x+1)^n\frac{t^n}{n!}\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}S_k(x)(-1)^{n-k}(x+1)^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Since

$$(-1)^n S_n(x) = (S(x)-(x+1))^n,$$

we complete Theorem 2.3 (i). From now on, we will find Theorem 2.3 (ii). By substituting t and x with $-t$ and $x-1$ respectively, we can find

$$\sum_{n=0}^{\infty} S_n(x-1)\frac{(-t)^n}{n!} = \left(\frac{1-e^{-t}}{te^{-t}}\right)e^{-t(x+1)}\left(\frac{t}{1-e^{-t}}\right)^{x+1}.$$

From the above equation, we can investigate the below equation.

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n S_n(x-1)\frac{t^{n+1}}{n!} &= \left(1-e^{-t}\right)e^{-tx}\left(\frac{t}{1-e^{-t}}\right)^{x+1} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^n x^l\frac{t^{n+1}}{(n+1)!}\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m+1}{l}\binom{n+1}{m+1}(-1)^m x^l S_{n-m}(x)\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

The left side of the above equation is transformed as follows:

$$\sum_{n=0}^{\infty}(-1)^n S_n(x-1)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty}(n+1)(-1)^n S_n(x-1)\frac{t^{n+1}}{(n+1)!}.$$

From coefficient comparison method, we derive the following equation.

$$(n+1)(-1)^n S_n(x-1) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m+1}{l}\binom{n+1}{m+1}(-1)^m x^l S_{n-m}(x).$$

Hence, we finish the proof of Theorem 2.3 (ii).
We also obtain the below equation by substituting t and x with $-t$ and $-x$, respectively.

$$\sum_{n=0}^{\infty} S_n(-x)\frac{(-t)^n}{n!} = \left(\frac{-t}{1-e^t}\right)^{-x+1} = \left(\frac{e^t-1}{t}\right)^{x-1}.$$

When x belongs to nonnegative integers containing -1, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n S_n(-x)\frac{t^{n+x-1}}{n!} &= \left(e^t-1\right)^{x-1} = \sum_{k=0}^{x-1}\binom{x-1}{k}(-1)^{x-1-k}e^{tk} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{x-1}\binom{x-1}{k}(-1)^{x-1-k}k^n\right)\frac{t^n}{n!}.\end{aligned}$$

From the above equation, we can transform such as in the following equation.

$$\sum_{n=0}^{\infty}(-1)^n S_n(-x-1)\frac{t^{n+x}}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{x}\binom{x}{k}(-1)^{x-k}k^n\right)\frac{t^n}{n!}.$$

By calculating both sides in the above equation, we have

$$\sum_{n=x}^{\infty}\binom{n}{x}(-1)^{n-x}x!S_n(-x-1)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{x-1}\binom{x-1}{k}(-1)^{x-1-k}k^n\right)\frac{t^n}{n!}.$$

Therefore, the proof of Theorem 2.3 (iii) is complete. □

Theorem 2.4. *From the generating function we find*

(i) $n \in \mathbb{N}_0$, $\quad n = \sum_{k=0}^{n-1}\binom{n}{k}S_k$,

(ii) $x \leq n \in \mathbb{N}_0$, $\quad \binom{n}{x} = \frac{x^{x-n}}{x!}\sum_{k=0}^{x}(-1)^{x-k}\left(S(x-1)+k\right)^n$.

Proof. From Definition 1.1, we can note that $S_n(0) = S_n$. When $x = 0$, we have

$$\sum_{n=0}^{\infty} S_n\frac{t^n}{n!}\left(e^t-1\right) = te^t. \tag{2.2}$$

The left-hand side of (2.2) can then be transformed as follows.

$$\sum_{n=0}^{\infty} S_n\frac{t^n}{n!}(e^t-1) = \sum_{n=0}^{\infty} S_n\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}S_k\frac{t^{n+1}}{(n+1)!}.$$

Additionally, the right-hand side in (2.2) can be transformed in the following form.

$$te^t = \sum_{n=0}^{\infty}(n+1)\frac{t^{n+1}}{(n+1)!}.$$

From the above two equations, we can see that the following equation holds true.

$$n+1 = \sum_{k=0}^{n}\binom{n+1}{k}S_k.$$

Therefore, we obtain the proof of Theorem 2.4 (i). The proof of Theorem 2.4 (ii) is very similar to that of Theorem 2.4 (i). We can assume $x \leq n \in \mathbb{N}_0$ to show Theorem 2.4 (ii). We then have the Stirling polynomials such as

$$\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}(e^t-1)^{x+1} = \left(te^t\right)^{x+1}.$$

From the above equation, the left-hand side can be changed in the following equation.

$$\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}(e^t-1)^{x+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{x+1}(-1)^{x+1-k}\left(S(x)+k\right)^n\frac{t^n}{n!}.$$

The right-hand side can be changed as shown below.

$$\left(te^t\right)^{x+1} = t^{x+1}\sum_{n=0}^{\infty}(x+1)^n\frac{t^n}{n!} = \sum_{n=x+1}^{\infty}\binom{n}{x+1}(x+1)!\,(x+1)^{n-x-1}\frac{t^n}{n!}.$$

If $x+1=x'$, then the above equation is represented as

$$\sum_{k=0}^{x'}(-1)^{x'-k}\left(S(x'-1)+k\right)^n=\binom{n}{x'}x'!x'^{n-x}.$$

Therefore, we know the above proof of Theorem 2.4 (ii) is clear. □

Theorem 2.5. *For $k\in\mathbb{N}_0\cup\{-1\}$, we derive*

$$S_n(k)=n!\sum_{v_1,v_2,\cdots,v_{k+1}=0}^{v_1+v_2+\cdots+v_{k+1}=n}\frac{S_{v_1}S_{v_2}\cdots S_{v_{k+1}}}{v_1!v_2!\cdots v_{k+1}!},$$

where $S_n=S_n(0)$.

Proof. We have the following equation by using the generating function when k is nonnegative integers containing -1.

$$\begin{aligned}\sum_{n=0}^{\infty}S_n(k)\frac{t^n}{n!}&=\left(\frac{t}{1-e^{-t}}\right)^{k+1}\\&=\left(\frac{t}{1-e^{-t}}\right)\left(\frac{t}{1-e^{-t}}\right)\cdots\left(\frac{t}{1-e^{-t}}\right)\\&=\sum_{v_1=0}^{\infty}S_{v_1}\frac{t^{v_1}}{v_1!}\sum_{v_2=0}^{\infty}S_{v_2}\frac{t^{v_2}}{v_2!}\cdots\sum_{v_{k+1}=0}^{\infty}S_{v_{k+1}}\frac{t^{v_{k+1}}}{v_{k+1}!}\\&=\sum_{n=0}^{\infty}n!\sum_{v_1,v_2,\cdots,v_{k+1}=0}^{v_1+v_2+\cdots+v_{k+1}=n}\frac{S_{v_1}S_{v_2}\cdots S_{v_{k+1}}}{v_1!v_2!\cdots v_{k+1}!}\frac{t^n}{n!}.\end{aligned}$$

Hence, we complete the proof of Theorem 2.5. □

We reduce Theorem 2.5 as follows:

Corollary 2.6. *If $k=1$ in* Theorem 2.5, *then we have*

$$S_n(1)=n!\sum_{v_1,v_2=0}^{v_1+v_2=n}\frac{S_{v_1}S_{v_2}}{v_1!v_2!}.$$

Theorem 2.7. *For a real or complex parameter x, one has*

$$S_n(x-1)=\sum_{k=0}^{n}\binom{n}{k}x^kB_{n-k}^{(x)},$$

where $B_{n-k}^{(x)}$ is generalized Bernoulli numbers.

Proof. Through the definition of Stirling polynomials and generalized Bernoulli numbers, we can look for the below relation.

$$\begin{aligned}\sum_{n=0}^{\infty}S_n(x)\frac{t^n}{n!}&=\left(\frac{t}{1-e^{-t}}\right)^{x+1}\\&=\left(\frac{t}{e^t-1}\right)^{x+1}e^{t(x+1)}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(x+1)^kB_{n-k}^{(x+1)}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we prove Theorem 2.7. □

Theorem 2.8. *For $x \in \mathbb{C}$, we have*

$$\begin{aligned} S_n(x) &= \frac{1}{2}\left[\left(\widetilde{E}(x) + B^{(x+1)}(2)\right)^n + \left(\widetilde{E}(x) + B^{(x+1)}\right)^n\right] \\ &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(B_k^{(x+1)}(2) + B_k^{(x+1)}\right)\widetilde{E}_{n-k}(x), \end{aligned}$$

where $B^{(x)}$ is generalized Bernoulli polynomials and $\widetilde{E}_n(x)$ is Euler polynomials of the second kind (see [8]).

Proof. Note that

$$x^n = \frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\widetilde{E}_k(x-1) + \widetilde{E}_n(x-1)\right]. \tag{2.3}$$

We obtain the following equation by combining Theorem 2.7 and Equation (2.3):

$$\begin{aligned} S_n(x-1) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k^{(x)}\left[\sum_{i=0}^{n-k}\binom{n-k}{i}2^{n-k-i}\widetilde{E}_i(x-1) + \widetilde{E}_{n-k}(x-1)\right] \\ &= \frac{1}{2}\left[\sum_{i=0}^{n}\binom{n}{i}\widetilde{E}_i(x-1)\sum_{k=0}^{n-i}\binom{n-i}{k}2^{n-k-i}B_k^{(x)} + \sum_{k=0}^{n}\binom{n}{k}B_k^{(x)}\widetilde{E}_{n-k}(x-1)\right]. \end{aligned} \tag{2.4}$$

From the generalized Bernoulli polynomials, we can note that

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}x^{n-k} \quad \text{(see [14])}.$$

Using the above equation we can change the equation (2.4).

$$\begin{aligned} S_n(x-1) &= \frac{1}{2}\left[\sum_{i=0}^{n}\binom{n}{i}\widetilde{E}_i(x-1)B_{n-i}^{(x)}(2) + \sum_{k=0}^{n}\binom{n}{k}B_k^{(x)}\widetilde{E}_{n-k}(x-1)\right] \\ &= \frac{1}{2}\left[\left(\widetilde{E}(x-1) + B^{(x)}(2)\right)^n + \left(\widetilde{E}(x-1) + B^{(x)}\right)^n\right] \\ &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(B_k^{(x)}(2) + B_k^{(x)}\right)\widetilde{E}_{n-k}(x-1). \end{aligned}$$

Therefore, we complete prove of Theorem 2.8. □

3. The observation of scattering zeros of the Stirling polynomials

In this section, we can see a certain phenomenon of zeros which is related to the fractal using the Newton dynamical system. Ryoo found the approximate zeros of the Stirling polynomials and some interesting properties of these polynomials (see [7]). By using the Mathematica software, we can see a certain structure of the zeros of the Stirling polynomials in an iterated map.

In Figure 1, the x-axis means the numbers of real zeros and the y-axis means the numbers of complex zeros in the Stirling polynomials. For example, the numbers of real zero are $1, 2, 3, 4$ and 5 and the numbers of complex zero correspond to $0, 0, 0, 0$ and 0 for $1 \le n \le 5$ in the Stirling polynomials, respectively. Our numerical results, which are the numbers of real and complex zeros of $S_n(x)$ for $1 \le n \le 60$, are displayed in Figure 1.

Let $f : D \to D$ be a complex function, with D as a subset of $\mathbb{C}$. We define the iterated maps of the complex function as the following:

$$f_r : z_0 \mapsto \underbrace{f(f(\cdots(f}_{r}(z_0)\cdots)).$$

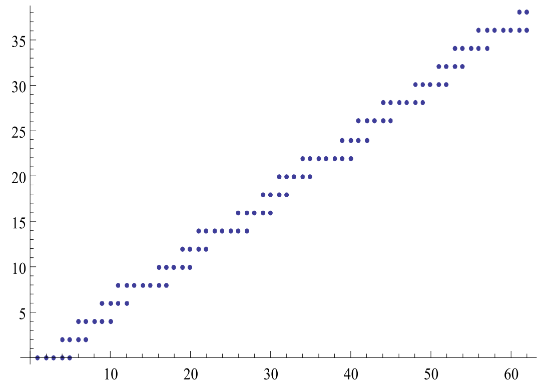

Figure 1: Numbers of real and complex zeros of $S_n(x)$

The iterates of f are the functions $f, f \circ f, f \circ f \circ f, \ldots$, which are denoted by $f^1, f^2, f^3, \ldots$. If $z \in \mathbb{C}$ and then the orbit of z_0 under f is the sequence $< z_0, f(z_0), f(f(z_0)), \cdots >$.

We consider the Newton's dynamical system as follows [1, 15]:

$$\left\{ \mathbb{C}_\infty : R(x) = x - \frac{S(x)}{S'(x)} \right\}.$$

R is called the Newton iteration function of S. It can be considered that the fixed points of R are the zeros of S and all the fixed points of R are attracting. R may also have one or more attracting cycles.

For $x \in \mathbb{C}$, we consider $S_4(x)$ and then this polynomial has four distinct complex numbers, $a_i (i = 1, 2, 3, 4)$ such that $S_4(a_i) = 0$. Using a computer, we obtain the approximate zeros as follows:

$$\begin{aligned}
a_1 &= -1,\\
a_2 &= -\frac{1}{3} + \frac{1}{\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}}} + \frac{1}{3}\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}} \approx 0.768956,\\
a_3 &= -\frac{1}{3} - \frac{1 + i\sqrt{3}}{2\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}}} - \frac{1}{6}(1 - i\sqrt{3})\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}} \approx -1.18234,\\
a_4 &= -\frac{1}{3} - \frac{1 - i\sqrt{3}}{2\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}}} - \frac{1}{6}(1 + i\sqrt{3})\left(\frac{1}{5}\left(16 + i\sqrt{419}\right)\right)^{\frac{1}{3}} \approx -0.58662.
\end{aligned}$$

In Newton's method, the generalized expectation is that a typical orbit $\{R(x)\}$ will converge to one of the roots of $S_4(x)$ for $x_0 \in \mathbb{C}$. If we choose x_0, which is sufficiently close to a_i, then this proves that

$$\lim_{r \to \infty} R(x_0) = a_i \text{ for } i = 1, 2, 3, 4.$$

When it is given a point x_0 in the complex plane, we want to determine whether the orbit of x_0 under the action of $R(x)$ converges to one of the roots of the equation. The orbit of x_0 under the action of R also appears by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}.

The output in the Figure 2 is the last calculated orbit value. We construct a function, which assigns one of four colors for each point according to the outcome of R in the plane. If an orbit of x_0 converges to $-1, 0.768956, -1.18234$ and -0.58662, then we denote the red, yellow, blue and sky-blue, respectively. For example, the yellow region represents the part of the basin of attraction of $a_3 = -1.18234$. The range in the left figure is $\{(x, y) : -4 \le x \le 4, -4 \le y \le 4\}$, the middle figure is $\{(x, y) : -0.2 \le x \le 0.6, -0.5 \le y \le 0.5\}$ and the right figure is $\{(x, y) : -0.2 \le x \le 0.6, -0.5 \le y \le 0.5\}$. From Figure 2, we make the following conjecture.

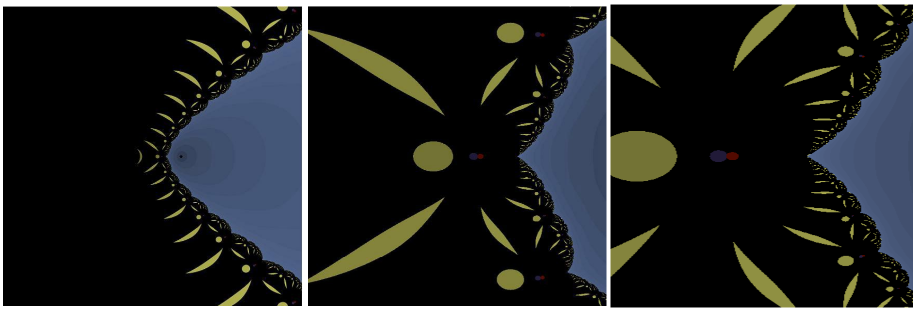

Figure 2: Orbit of x_0 under the action of R for $S_4(x)$

Conjecture 3.1. The orbit of x_0 under the action of R has self-similarity for $n \geq 4$.

4. The distribution of fixed points for the Stirling polynomials

In this section, we find the distribution of fixed points and period points using the definition of fixed point. From Newton's method we construct orbits of points under the action of a complex function using Mathematica to generate graphic images.

Definition 4.1. The orbit of the point $z_0 \in \mathbb{C}$ under the action of the function f is said to be bounded if there exists $M \in \mathbb{R}$ such that $|f^n(z_0)| < M$ for all $n \in \mathbb{N}$. If the orbit is not bounded, it is said to be unbounded.

Definition 4.2. Let $f : D \to D$ be a transformation on a metric space. A point $z_0 \in D$ such that $f(z_0) = z_0$ is called a fixed point of the transformation.

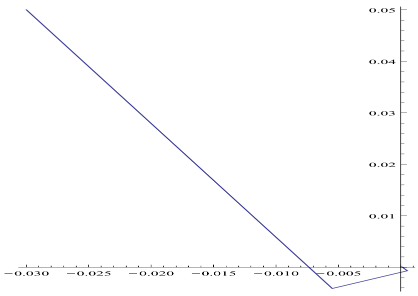

Figure 3: Attracting fixed point of $S_3(x)$

This Figure 3 is an attracting fixed point of the third Stirling polynomials that is bounded. We know that the fixed point is divided as follows. Suppose that the complex function f is analytic in a region D of $\mathbb{C}$ and f has a fixed point at $z_0 \in D$. Then z_0 is said to be:

(1) an attracting fixed point if $|f'(z_0)| < 1$;

(2) a repelling fixed point if $|f'(z_0)| > 1$;

(3) a neutral fixed point if $|f'(z_0)| = 1$.

For example, $S_3(x)$ has three points satisfying $S_3(x) = x$. That is, $x_0 = 0, -3.82843, 1.82843$. Since

$$\left|\frac{d}{dt} S_3(0)\right| = 0 < 1.$$

Theorem 4.3. *The third Stirling polynomial, $S_3(x)$, has only one attracting fixed point, $x_0 = 0$.*

Using Mathematica, we can separate the numerical results for fixed points of $S_n(x)$. From Table 1, we know that $S_n(x)$ have no neutral fixed point for $2 \leq n \leq 7$. We can also reach Conjecture 4.4.

degree n	attractor	repellor	neutral
2	1	1	0
3	1	2	0
4	1	3	0
5	1	4	0
6	1	5	0
7	1	6	0

Table 1: Numbers of attracting, repelling and neutral fixed points of $S_n(x)$

Conjecture 4.4. The Stirling polynomials $S_n(x)$ for $n \geq 2$ have the only one attracting fixed point except for infinity.

In Table 2, we denote $R_{S_n^r(x)}$ as the numbers of real zeros for r-th iteration. From this table, we can know that the fixed points of $S_3^r(x)$ using iterated function are less than 3^r. Here, we can know that the r-th iterating Stirling polynomial has some fixed points, but each Stirling polynomials using iteration have one fixed point and some period points in $S_3(x)$ position such as in Table 2. From Table 2, we can suggest Conjecture 4.5.

r	$R_{S_n^r(x)}$	numbers of real fixed point
1	3	3
2	5	3
3	7	3
4	9	11
5	11	2
6	13	2
7	15	2

Table 2: The numbers of real roots and real fixed points of $S_3^r(x)$ for $1 \leq r \leq 7$

Conjecture 4.5. The Stirling polynomials that are iterated, $S_3^r(x)$, have two real fixed points, $\alpha = 1.82843$, and 0. This polynomials always have two fixed points containing α for $r \geq 5$ and $R_{S_n^r(x)}$ has $2r+1$ for $r \geq 1$.

In the top-left of Figure 4, we can see the forms of 3D structure related to stacks of fixed points of $S_3^r(x)$ for $1 \leq r \leq 7$. When we look at the top-left of Figure 4 in the below position, we can draw the top-right figure. The bottom-left of Figure 4 shows that image and n axes exist but not real axis in three dimensions. In three dimensions, the bottom-right of Figure 4 is the right orthographic viewpoint for the top-left figure,- that is, there exist real and n axes but there is no image axis. We can also assume the distribution of the fixed points for each $S_3^r(x)$ contains the distribution of some period points if we consider $S_3^1(x)$.

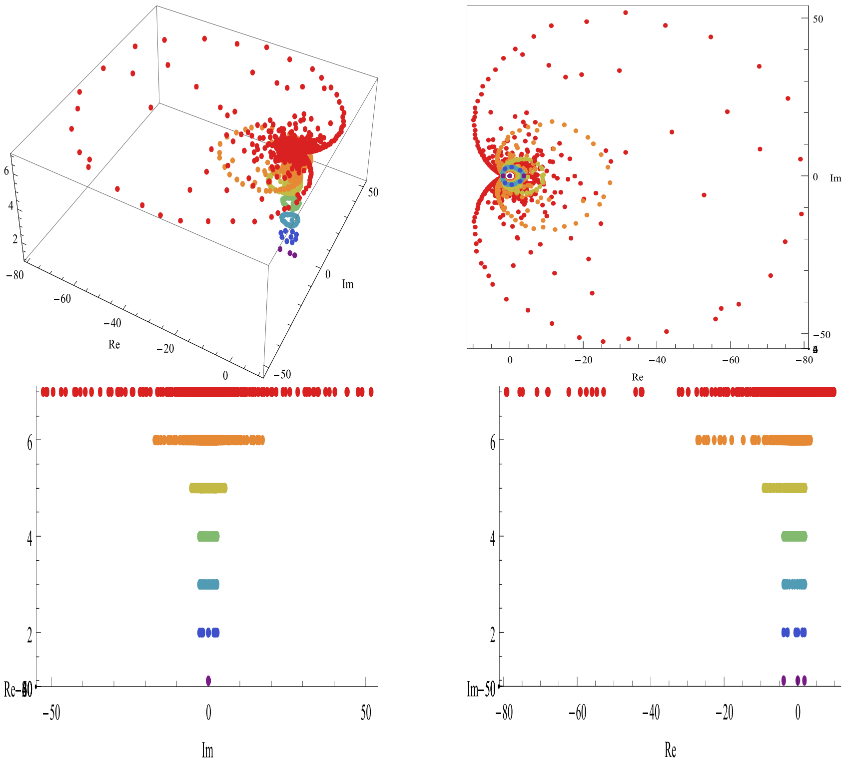

Figure 4: Stacks of fixed point of $S_3^r(x)$ for $1 \leq r \leq 7$

Especially, we assume $S_3^2(x)$ for $x \in \mathbb{C}$. The $S_3^2(x)$ is nine distinct complex numbers, $a_i^*(i=1,2,3,4,5,6,7,8,9)$ such that $S_3^2(a_i^*) = a_i^*$. That is,

$$\begin{aligned}
a_1 &= -3.82843,\\
a_2 &\approx -2.79544 - 1.74178i,\\
a_3 &\approx -2.79544 + 1.74178i,\\
a_4 &\approx -0.5 - 2.78388i,\\
a_5 &\approx -0.5 + 2.78388i,\\
a_6 &= 0,\\
a_7 &\approx 1.29544 - 2.22684i,\\
a_8 &\approx 1.29544 + 2.22684i,\\
a_9 &= 1.82843.
\end{aligned}$$

Let $S_3^2(x) - x = \mathcal{S}_3(x)$. Then we assume the Newton's method for $\mathcal{S}_3(x)$, that is,

$$\left\{ \mathbb{C}_\infty : \mathcal{R}^2(x) = x - \frac{\mathcal{S}_3(x)}{\mathcal{S}_3'(x)} \right\}.$$

From the Newton's method, we obtain the calculated last orbit value such as the Figure 5. We construct the range which is $\{(x,y) : -6 \leq x \leq 2, -4 \leq y \leq 4\}$ containing a_i^*. From Figure 5, we can observe a_6 and a_9. The ocher region represents part of the attracting basin of $a_6 = 0$ and the gray region is part of the basin of attraction for $a_9 = 1.82843$. We can suggest from this figure that these regions are divided into the attracting basins of fixed points in the $S_3^2(x)$ position, but we can also see from this figure that these regions are divided into the attracting basin of one fixed point and the attracting basins of eight period points by $S_3(x)$.

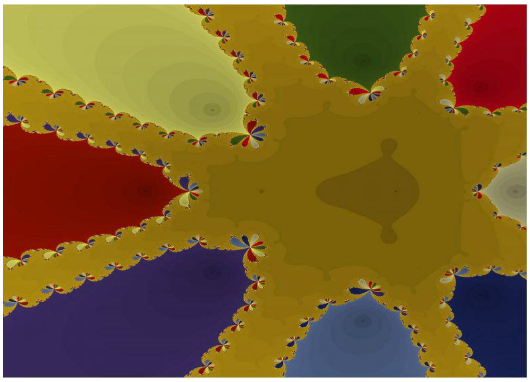

Figure 5: Orbit of x_0 under the action of $\mathcal{R}^2$ for $\mathcal{S}_3(x)$

Figure 6: Palette for escaping points

In Figure 6, we express the coloring for $\mathcal{R}^2(x)$. Using the Newton's method for $\mathcal{S}_3(x)$, a point represents one of nine colors when it approaches a fixed point. That is, we denote the brown, blue, yellow, sky blue, green, ocher, navy blue, red, or gray to x_0 if its orbit converges to $-3.82843, -2.79544-1.74178i, -2.79544+1.74178i, -0.5-2.78388i, -0.5+2.78388i, 0, 1.29544-2.22684i, 1.29544+2.22684i, 1.82843$, respectively.

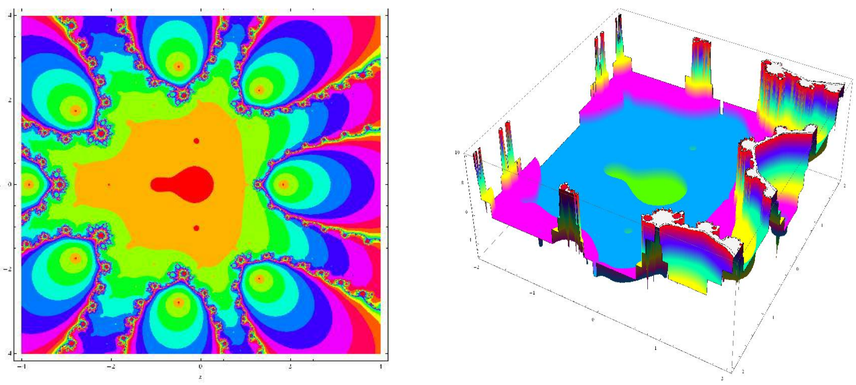

Figure 7: Julia sets of $\mathcal{R}^2(x)$ for $\mathcal{S}_3(x)$

We can illustrate the rapid change by applying the three-dimensional structure to the escape-time function from Figure 7. The range of the left Figure 7 is $\{(x, y) : -4 \le x \le 4, -4 \le y \le 4\}$ and the range of the right Figure 7 is $\{(x, y) : -2 \le x \le 2, -2 \le y \le 2\}$. The orbit of x_0 under the action of R is also appeared by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}.

This color Figure 8 represents the coloring of the above Figure 7. For example, points which escape after 1 to 30 iterations are colored red to green.

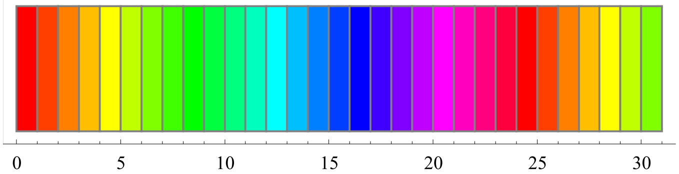

Figure 8: Palette for escaping points

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