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β_1-paracompact spaces

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Abstract

We introduce the class of β_1-paracompact spaces in topological spaces and give characterizations of such spaces. We study subsets and subspaces of β_1-paracompact spaces and discuss their properties. Also, we investigate the invariants of β_1-paracompact spaces by functions.

Keywords: Paracompact, β_1-paracompact, β-open set, locally finite collection.
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1. Introduction and preliminaries

Throughout this work a space will always mean a topological space with no separation axioms assumed, unless otherwise stated. If (X, τ) is a given space, then $Int(A)$ and $Cl(A)$ denotes the interior of A and the closure of A, respectively in (X, τ). Let (X, τ) be a space and A a subset of X. A subset A is said to be *preopen* [16] (resp., *semi-open* [13], α*-open* [18], *regular open* [21]) if $A \subset Int(Cl(A))$ (resp., $A \subset Cl(Int(A))$, $A \subset Int(Cl(Int(A)))$, $A = Int(Cl(A))$). The family of α-sets of a space (X, τ), denoted by τ^α, forms a topology on X, finer than τ [18]. For a space (X, τ), if (X, τ^α) is normal, then $\tau = \tau^\alpha$ [10].

In 1983, Abd El-Monsef et al. [1] introduced and studied the concept of β-open sets in topological spaces. They define a subset A of a space (X, τ) is said to be β-open if $A \subset Cl(Int(Cl(A)))$. The complement of a β-open set is said to be β-closed [1]. The collection of all β-open (resp., β-closed) subsets of X is denoted by $\beta O(X, \tau)$ (resp., $\beta C(X, \tau)$). The union of all β-open sets of X contained in A is called β-interior of A and is denoted by $\beta Int(A)$ and the intersection of all β-closed sets of X containing A is called the β-closure of A and is denoted by $\beta Cl(A)$. A set A is called β-regular [20] if it is both β-open and β-closed. A space (X, τ) is said to be β-regular [2] if for each β-open set U and each $x \in U$, there exists a β-open set V such that $x \in V \subseteq \beta Cl(V) \subseteq U$. For any space, one has $\beta O(X, \tau^\alpha) = \beta O(X, \tau)$ [4]. A collection $\Im = \{F_\alpha : \alpha \in \Delta\}$ of

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subsets of a space (X, τ) is said to be locally finite if for each $x \in X$, there exists an open set U containing x and U intersects at most finitely many members of $\Im$.

A space (X, τ) is said to be paracompact if every open cover of X has a locally finite open refinement. α-paracompact [5] (resp., P_1-paracompact [15], S_1-paracompact [3]) spaces are defined by replacing the open cover in original definition by α-open (resp., preopen, semiopen) cover. A subset A of space X is said to be N-closed relative to X (briefly, N-closed) [9] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by open sets of X, there exists a finite subfamily $\Delta_\circ$ of Δ such that $A \subset \bigcup\{Int(Cl(U_\alpha)) : \alpha \in \Delta_\circ\}$. In [11], it was shown that every compact T_2-space is regular.

In this paper, we follow a similar line and introduce β_1-paracompact spaces by utilizing the β-open cover. We provide several characterizations of β_1-paracompact spaces and study subsets and subspaces of β_1-paracompact spaces and discuss their properties. Finally, we investigate the invariants of β_1-paracompact spaces by functions.

Now we recall some known definitions, lemmas, and theorems, which will be used in the work.

Theorem 1.1 ([17])**.** *Let (X, τ) be a space, $A \subset Y \subset X$ and Y β-open in (X, τ). Then A is β-open in (X, τ) if and only if A is β-open in the subspace (Y, τ_Y).*

Definition 1.2. A space (X, τ) is said to be α-paracompact [5] (resp.,P_1-paracompact [15], S_1-paracompact[3]), if every α-open (resp., preopen, semiopen) cover of X has a locally finite open refinement.

Lemma 1.3 ([6])**.** *The union of a finite family of locally finite collection of sets in a space is a locally finite family of sets.*

Theorem 1.4 ([7])**.** *If $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is a locally finite family of subsets in a space X and if $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ for each $\alpha \in \Delta$, then the family $\{\mathcal{V}_\alpha : \alpha \in \Delta\}$ is a locally finite in X.*

Lemma 1.5 ([12])**.** *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous surjective function and $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is locally finite in Y, then $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is locally finite in X.*

Lemma 1.6 ([19])**.** *Let $f : (X, \tau) \to (Y, \sigma)$ be almost closed surjection with N-closed point inverse. If $\{U_\alpha : \alpha \in \Delta\}$ is a locally finite open cover of X, then $\{f(U_\alpha) : \alpha \in \Delta\}$ is a locally finite cover of Y.*

2. β_1-paracompact spaces

In this section we introduce and study a new class of spaces, namely β_1-paracompact spaces, and we provide several characterizations of them.

Definition 2.1. A space (X, τ) is called β_1-paracompact if every β-open cover of X has a locally finite open refinement.

The following diagram shows the relations among the mentioned properties.

$$\begin{tikzcd} \beta_1\text{-paracompact} \arrow[r] \arrow[dr] & P_1\text{-paracompact} \arrow[r] & \alpha\text{-paracompact} \arrow[r] & \text{paracompact} \\ & S_1\text{-paracompact} \arrow[ur] & & \end{tikzcd}$$

The converses need not be true as shown by the following examples.

Example 2.2. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, X, \{1\}\}$. Then (X, τ) is paracompact but it is not β_1-paracompact, since $\{\{1, x\} : x \in X\}$ is a β-open cover of X which admits no locally finite open refinement.

Example 2.3. Let $X = \{1, 2, 3\}$ with the topology $\tau = \{\phi, X, \{1\}, \{2, 3\}\}$. Then (X, τ) is S_1-paracompact since $SO(X, \tau) = \tau$, but it is not β_1-paracompact since $\{\{1\}, \{2\}, \{3\}\}$ is a β-open cover of X which admits no locally finite open refinement.

Example 2.4. Let $X = \{1,2,3\}$ with the topology $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then (X,τ) is P_1-paracompact since $PO(X,\tau) = \tau$ but it is not β_1-paracompact since $\{\{1,2\},\{2,3\}\}$ is a β-open cover of X which admits no locally finite open refinements.

Theorem 2.5. *If (X,τ) is a β_1-paracompact T_1-space, then $\tau = \beta O(X,\tau) = \tau^\alpha$.*

Proof. Let U be a β-open set in (X,τ). For each $x \in U$, we have $\mathcal{U} = \{U\} \cup \{X - \{x\}\}$ is a β-open cover for (X,τ) and so it has a locally finite open refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $x \in U$, there exist an $\alpha_\circ \in \Delta$ such that $x \in V_{\alpha_\circ} \subseteq U$ where $V_{\alpha_\circ}$ is open and so U is open. Now, we know that $\tau \subset \tau^\alpha \subset \beta O(X,\tau)$ and we show $\tau = \beta O(X,\tau)$, so $\tau = \tau^\alpha$. □

The proof of the following corollary follows immediately from Definition 2.1 and Theorem 2.5.

Corollary 2.6. *Let (X,τ) be a T_1-space. Then (X,τ) is β_1-paracompact if and only if (X,τ) is paracompact and $\tau = \beta O(X,\tau)$.*

Recall that, a space (X,τ) is said to be extremally disconnected (briefly e.d.) if the closure of every open set in (X,τ) is open.

Proposition 2.7. *Let (X,τ) be a β_1-paracompact space. Then:*

i. *If (X,τ) is T_1, then it is extremally disconnected.*
ii. *If (X,τ) is T_2, then it is β-regular.*

Proof. i) Let U be an open set in (X,τ); then $Cl(U)$ is a β-open set and by Theorem 2.5, $Cl(U)$ is an open set in (X,τ).
ii) Let U be a β-open set in (X,τ) and $x \in U$. By Theorem 2.5, U is an open set. Since (X,τ) is regular, there exists an open set V such that $x \in V \subseteq Cl(V) \subset U$. Thus $x \in V \subseteq \beta Cl(V) \subseteq U$. It follows that (X,τ) is β-regular. □

Theorem 2.8. *Let (X,τ) be a space. Then:*

i. *If (X,τ^α) is β_1-paracompact, then (X,τ) is paracompact.*
ii. *If (X,τ) is β_1-paracompact, then (X,τ^α) is β_1-paracompact; the converse true if the space is T_2.*

Proof. i) Let $\mathcal{U}$ be an open cover of (X,τ). Then $\mathcal{U}$ is an open cover of the β_1-paracompact space (X,τ^α) and so it has a locally finite open refinement $\mathcal{V}$ in (X,τ^α). Now for every $V \in \mathcal{V}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. One can easily show that the collection $\{U_V \cap Int(Cl(Int(V))) : V \in \mathcal{V}\}$ is a locally finite open refinement of $\mathcal{U}$ in (X,τ).
ii) Let $\mathcal{U}$ be a β-open cover of (X,τ^α). Then $\mathcal{U}$ is a β-open cover of the β_1-paracompact space (X,τ) and so it has a locally finite open refinement $\mathcal{V}$ in (X,τ). Since $\tau \subseteq \tau^\alpha$, then $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$ in (X,τ^α) and so (X,τ^α) is β_1-paracompact. To prove the converse, let (X,τ^α) be β_1-paracompact. Then (X,τ^α) is a paracompact T_2-space and so it is normal [11]. Therefore, $\tau = \tau^\alpha$. □

The following examples show that the converse of (i) in the above theorem need not be true in general and the condition T_2 on the space (X,τ) in (ii) is essential.

Example 2.9. Let (X,τ) be as in Example 2.4. Then (X,τ) is paracompact, but (X,τ^α) is not β_1-paracompact.

Example 2.10. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}\}$. Then $\tau^\alpha = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\} = \beta O(X,\tau^\alpha)$. Therefore (X,τ^α) is a β_1-paracompact space. On the other hand, (X,τ) is not β_1-paracompact since $\{\{1,2\},\{1,3\}\}$ is a β-open cover of (X,τ) which admits no locally finite open refinement.

Theorem 2.11. *If each β-open cover of a space (X,τ) has an open σ-locally finite refinement, then each β-open cover of X has a locally finite refinement.*

Proof. Let $\mathcal{U}$ be a β-open cover of X. Let $\mathcal{V} = \bigcup_{n\in N} \mathcal{V}_n$ be an open σ-locally finite refinement of $\mathcal{U}$ where $\mathcal{V}_n$ is locally finite. For each $n \in N$ and each $V \in \mathcal{V}_n$, let $\mathcal{V}'_n = \mathcal{V} - \bigcup_{k\prec n} \mathcal{V}^*_k$ where $\mathcal{V}^*_k = \bigcup\{V : V \in \mathcal{V}_k\}$ and put $\mathcal{V}'_n = \{V'_n : V \in \mathcal{V}_n\}$. Now, put $\mathcal{W} = \{V'_n : n \in N,\ V \in \mathcal{V}_n\} = \bigcup\{\mathcal{V}'_n : n \in N\}$. We show that $\mathcal{W}$ is a locally finite refinement of $\mathcal{U}$. Let $x \in X$ and let n be the first positive integer such that $x \in \mathcal{V}^*_n$. Therefore $x \in V'$ for some $V' \in \mathcal{V}'_n$. Thus $\mathcal{W}$ is a cover of X. To show that $\mathcal{W}$ is locally finite, let $x \in X$ and n be the first positive integer such that $x \in \mathcal{V}^*_n$. Then $x \in V$ for some $V \in \mathcal{V}_n$. Now, $V \cap V' = \phi$ for each $V' \in \mathcal{V}_k$ and for each $k \succ n$. Therefore, V can intersect at most the elements of $\mathcal{V}'_k$ for $k \leq n$. Since $\mathcal{V}'_k$ is locally finite for each $k \leq n$, so we choose an open set $O_{x(k)}$ containing x such that $O_{x(k)}$ meets at most finitely many members of $\mathcal{V}'_k$. Finally, put $O_x = V \cap (\bigcap_{k=1}^{n} O_{x(k)})$. Then O_x is an open set containing x such that O_x meets at most finitely many members of $\mathcal{W}$. □

Theorem 2.12. *Let (X, τ) be a β-regular space. If each β-open cover of the space X has a locally finite refinement, then each β-open cover of X has a locally finite β-closed refinement.*

Proof. Let $\mathcal{U}$ be a β-open cover of X. For each $x \in X$, pick a $U_x \in \mathcal{U}$ such that $x \in U_x$. Since (X, τ) is β-regular, there exits a β-open set V_x such that $x \in V_x \subset \beta Cl(V_x) \subset U_x$. The family $\mathcal{V} = \{V_x : x \in X\}$ is a β-open cover of X and so has a locally finite refinement $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$. The collection $\beta Cl(\mathcal{W}) = \{\beta Cl(W_\alpha) : \alpha \in \Delta\}$ is locally finite for each $\alpha \in \Delta$; if $W_\alpha \subset V_\alpha$, then $\beta Cl(W_\alpha) \subset U^*$ for some $U^* \in \mathcal{U}$. Thus $\beta Cl(\mathcal{W})$ is a β-closed locally finite refinement of $\mathcal{U}$. □

Theorem 2.13. *If (X, τ) is β-regular space, then the following are equivalent:*

i. *(X, τ) is β_1-paracompact.*
ii. *Each β-open cover of X has a σ-locally finite open refinement.*
iii. *Each β-open cover of X has a locally finite refinement.*
iv. *Each β-open cover of X has a locally finite β-closed refinement, provided that the space (X, τ) is e.d.*

Proof. The proof follows from Theorems 2.11 and 2.12. □

3. properties of β_1-paracompact spaces

In this section we study some basic properties of β_1-paracompact spaces related to their subsets, subspaces, sums, images, and inverse images under some types of functions.

Definition 3.1. A subset A of a space (X, τ) is called a β_1-paracompact set in (X, τ) if every cover of A by β-open subset of (X, τ) has a locally finite open refinement in (X, τ), and A is called β_1-paracompact if (A, τ_A) is a β_1-paracompact space.

Theorem 3.2. *If A and B are β_1-paracompact relative to a space (X, τ), then $A \cup B$ is β_1-paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of $A \cup B$. Then $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is a β-open cover of A and B. So, there exist open locally finite families $V_A = \{V_{\alpha'} : \alpha' \in \Delta_1\}$ of A and $V_B = \{V_{\alpha''} : \alpha'' \in \Delta_1\}$ of B which refines $\mathcal{U}$ such that $A \subset \bigcup_{\alpha'\in\Delta_1} V_{\alpha'}$ and $B \subset \bigcup_{\alpha''\in\Delta_1} V_{\alpha''}$. Now $A \cup B \subset \bigcup_{\alpha',\alpha''\in\Delta}(V_{\alpha'} \cup V_{\alpha''}) = \mathcal{V}$. By Lemma 1.3, $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. Therefore, $A \cup B$ is β_1-paracompact relative to X. □

Theorem 3.3. *Let A and B be subsets of a space (X, τ). If A is β_1-paracompact relative to X and B is β-closed in X, then $A \cap B$ is β_1-paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta_\circ\}$ be a cover of $A \cap B$ such that U_α is β-open in (X, τ). Since $X - B$ is β-open in X, $\mathcal{U}_1 = \{U_\alpha : \alpha \in \Delta_\circ\} \cup \{X - B\}$ is a β-open cover of A. So, there exists a locally finite open family $\mathcal{V}_1 = \{V_{\alpha'} : \alpha' \in \Delta_1\} \cup V$ ($V_{\alpha'} \subset U_\alpha$ and $V \subset X - B$) which refines $\mathcal{U}_1$ such that $A \subset \bigcup_{\alpha'}\{V_{\alpha'} \mid \alpha' \in \Delta_1\} \cup V$. Now $A \subset \bigcup_{\alpha'}\{V_{\alpha'} \mid \alpha' \in \Delta_1\} \cup V$ implies that $A \cap B \subset (\bigcup_{\alpha'}\{V_{\alpha'} \mid \alpha' \in \Delta_1\} \cup V) \cap B \subseteq \bigcup_{\alpha'}\{V_{\alpha'} \mid \alpha' \in \Delta_1\} \cup V$. Take $\mathcal{V} = \{V_{\alpha'} \mid \alpha' \in \Delta_1\}$. Then $\mathcal{V}$ is a locally finite open family which refines $\mathcal{U}$. Hence $A \cap B$ is β_1-paracompact relative to X. □

Definition 3.4 ([8])**.** A subset A of a space (X,τ) is called βg-closed if $\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is any β-open set in (X,τ).

Theorem 3.5. *Let (X,τ) be a β_1-paracompact space and $A \subseteq X$. Then:*

i. *If A is regular open, then (A,τ_A) is β_1-paracompact.*
ii. *If A is a $g\beta$-closed set, then A is a β_1-paracompact set in (X,τ).*

Proof. i) Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be a β-open cover of A in (A,τ_A). Since A is open in (X,τ), by Theorem 1.1, V_α is a β-open set in (X,τ) for each $\alpha \in \Delta$. Therefore, the collection $\mathcal{U} = \{V_\alpha : \alpha \in \Delta\} \cup \{X - A\}$ is a β-open cover of the β_1-paracompact space (X,τ) and so it has a locally finite open refinement in (X,τ), say $\mathcal{W} = \{W_{\alpha'} : \alpha' \in \Delta_1\}$. Now the collection $\{A \cap \mathcal{W}_{\alpha'} : \alpha' \in \Delta_1\}$ is an open refinement of $\mathcal{V}$ in (A,τ_A). Therefore, (A,τ_A) is β_1-paracompact.
ii) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of (X,τ). Since $A \subseteq \bigcup\{U_\alpha : \alpha \in \Delta\}$ and A is βg-closed, we have $\beta Cl(A) \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$. For each $x \notin \omega\beta Cl(A)$, there exists a β-open set W_x of (X,τ) such that $A \cap W_x = \phi$. Put $\mathcal{U}' = \{U_\alpha : \alpha \in \Delta\} \cup \{W_x : x \notin \beta Cl(A)\}$. Then $\mathcal{U}'$ is a β-open cover of the β_1-paracompact space (X,τ). Let $\mathcal{H} = \{\mathcal{H}_{\alpha'} : \alpha' \in \Delta_1\}$ be a locally finite open refinement of $\mathcal{U}'$ and put $\mathcal{H}_u = \{H_{\alpha'} : H_{\alpha'} \subseteq U_{\alpha(\alpha')}, \alpha' \in \Delta_1 \text{ and } \alpha(\alpha') \in \Delta\}$. Then $\mathcal{H}_u$ is a locally finite open refinement of $\mathcal{U}$. Therefore A is a β_1-paracompact set. □

Theorem 3.6. *Let A and B be subsets of a space (X,τ) such that $A \subset B \subset X$:*

i. *If A is β_1-paracompact relative to X and B is β-open in (X,τ), then A is β_1-paracompact relative to B.*
ii. *If A is β_1-paracompact relative to B and B is open in (X,τ), then A is β_1-paracompact relative to X.*

Proof. i) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta_\circ\}$ be a cover of A such that U_α is β-open in (B,τ_B). Since B is β-open in (X,τ), by Theorem 1.1, $\mathcal{U}$ is a β-open cover of A in (X,τ). So, there exist a locally finite open family $\mathcal{V}_{\alpha'} = \{V_{\alpha'} : \alpha' \in \Delta_1\}$ which refines $\mathcal{U}$ such that $A \subset \bigcup\{V_{\alpha'} : \alpha' \in \Delta_1\}$. Then $A \cap B \subset \{V_{\alpha'} \cap B : \alpha' \in \Delta_1\}$. Let $x \in B$. Since $\mathcal{V} = \{V_{\alpha'} : \alpha' \in \Delta_1\}$ is locally finite in X, there exists an open set W in (X,τ) such that $W \cap V_{\alpha'} = \phi$ for each $\alpha' \neq {\alpha'}_1, {\alpha'}_2, \ldots, {\alpha'}_n$, which implies $(W \cap V_{\alpha'}) \cap B = \phi$ for $\alpha' \neq {\alpha'}_1, {\alpha'}_2, \ldots, {\alpha'}_n$, which implies $(V_{\alpha'} \cap B) \cap (W \cap B) = \phi$ for $\alpha' \neq {\alpha'}_1, {\alpha'}_2, \ldots, {\alpha'}_n$. Therefore, the family $\mathcal{V}_1 = \{V_{\alpha'} \cap B : \alpha' \in \Delta_1\}$ is a locally finite open refinement of $\mathcal{U}$ in (B,τ_B). Therefore, A is β_1-paracompact relative to B.
ii) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of (X,τ). Then the collection $\mathcal{W} = \{B \cap U_\alpha : \alpha \in \Delta\}$ is a β-open cover of A in (B,τ_B). But A is a β_1-paracompact set in (B,τ_B), so $\mathcal{W}$ has a locally finite open refinement $\mathcal{V}$ in (B,τ_B). Since B is open in (X,τ), by Theorem 1.1, $\mathcal{V}$ is a locally finite open refinement in (X,τ) and so A is a β_1-paracompact set in (X,τ). □

Corollary 3.7. *Let A and B be subsets of a space (X,τ). If A is β_1-paracompact relative to (X,τ) and B is β-regular, then the following hold:*

i. *$A \cap B$ is β_1-paracompact relative to B.*
ii. *B is β_1-paracompact relative to X, provided that $B \subset A$.*

Proof. i) Let A be β_1-paracompact relative to X and B a β-regular set in (X,τ). By Theorem 3.3, $A \cap B$ is β_1-paracompact relative to X. Since $A \cap B \subset B$ and B is β-open in (X,τ), by Theorem 3.6, $A \cap B$ is β_1-paracompact relative to B.
ii) Since $B \subset A$ and B is a β-regular set, by Theorem 3.3, B is β_1-paracompact relative to X. □

Theorem 3.8. *Let A be a clopen subspace of a space (X,τ). Then A is a β_1-paracompact set if and only if it is β_1-paracompact.*

Proof. To prove necessity, let A be an open β_1-paracompact subset of (X,τ). Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of the subspace (A,τ_A). Since A is open, $\mathcal{V}$ is a cover of A by β-open subsets of (X,τ) and so it has a locally finite open refinement, say $\mathcal{W}$, in (X,τ). Then $\mathcal{W}_A = \{W \cap A : W \in \mathcal{W}\}$ is

a locally finite open refinement of $\mathcal{V}$ in (A, τ_A) and the result follows.
To prove sufficiency, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of (X, τ). Then $\mathcal{U}' = \{A \cap U_\alpha : \alpha \in \Delta\}$ is a β-open cover of the β_1-paracompact subspace (A, τ_A) and so it has a locally finite open refinement $\mathcal{W}$ in (A, τ_A). But A is an open set in (X, τ), so W is an open set for every $W \in \mathcal{W}$. Now $\tau_A \subseteq \tau$ and $X - A$ is an open set in (X, τ) which intersects no member of $\mathcal{W}$. Therefore $\mathcal{W}$ is locally finite in (X, τ). Thus A is a β_1-paracompact set. □

Corollary 3.9. *Every clopen subspace of a β_1-paracompact space is β_1-paracompact.*

Definition 3.10 ([11])**.** Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \phi$ for each $\alpha \neq \beta$. Let $X = \bigcup_{\alpha\in\Delta} X_\alpha$ be topologized by $\tau = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha,\ \alpha \in \Delta\}$. Then (X, τ) is called the sum of space $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \bigoplus_{\alpha\in\Delta} X_\alpha$.

Theorem 3.11. *The topological sum $X = \bigoplus_{\alpha\in\Delta} X_\alpha$ is β_1-paracompact if and only if the space (X_α, τ_α) is β_1-paracompact, for each $\alpha \in \Delta$.*

Proof. Necessity follows from Corollary 3.9, since (X_α, τ_α) is a clopen subspace of the space $\bigoplus_{\alpha\in\Delta} X_\alpha$, for each $\alpha \in \Delta$.
To prove sufficiency, let $\mathcal{U}$ be a β-open cover of $\bigoplus_{\alpha\in\Delta} X_\alpha$. For each $\alpha \in \Delta$ the family $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is a β-open cover of the β-paracompact space (X_α, τ_α). Therefore $\mathcal{U}_\alpha$ has a locally finite open refinement $\mathcal{V}_\alpha$ in (X_α, τ_α). Put $\mathcal{V} = \bigcup_{\alpha\in\Delta} \mathcal{V}_\alpha$. It is clear that $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. Thus $\bigoplus_{\alpha\in\Delta} X_\alpha$ is β_1-paracompact. □

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be β-continuous [1] (resp., β-irresolute [14]) if $f^{-1}(V)$ is β-open in (X, τ) for each open (resp., β-open) set V in (Y, σ).

Theorem 3.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be an open, β-irresolute and almost closed surjective function with N-closed point inverse. If (X, τ) is β_1-paracompact, then (Y, σ) is also β_1-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta_\circ\}$ be a β-open cover of (Y, σ). Since f is β-irresolute, $\mathcal{U}_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta_\circ\}$ is a β-open cover of (X, τ). So, there exist a locally finite open refinement, say $\mathcal{W}$. Since f is open and by Lemma 1.6, $f(\mathcal{W})$ is a locally finite open refinement of $\mathcal{V}$ in (Y, σ). □

Since compact sets are N-closed and closed maps are almost closed, the following corollary follows from Theorem 3.12.

Corollary 3.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be an open, β-continuous, closed surjective function with compact point inverse. If (X, τ) is β_1-paracompact, then (Y, σ) is also β_1-paracompact.*

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly β-continuous if the inverse image of each β-open set in (Y, σ) is an open set in (X, τ).

Theorem 3.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be an open, strongly β-continuous, almost closed, surjective function with N-closed point inverse. If (X, τ) is paracompact, then (Y, σ) is β_1-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta_\circ\}$ be a β-open cover of Y. Since f is strongly β-continuous, $\mathcal{U}_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is an open cover of X. Hence, there exists a locally finite open refinement $\mathcal{W}$ of $\mathcal{U}_1$. Since f is open and by Lemma 1.6, $f(\mathcal{W})$ is a locally finite open refinement of $\mathcal{U}$. Therefore, (Y, σ) is β_1-paracompact. □

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be β-open [1] (resp., β-closed [1]) if $f(V)$ is a β-open (resp., β-closed) set in (Y, σ) for each β-open (resp., β-closed) set V in (X, τ).

Proposition 3.15 ([1])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is β-closed if and only if for each $x \in X$ and each β-open set U in (X, τ) containing x, there exists a β-open set V in (Y, σ) containing $f(x)$ such that $f(x) \in V$ and $f^{-1}(V) \subseteq U$.*

Theorem 3.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous β-closed surjective function with compact point inverse. If (Y, σ) is a β_1-paracompact space, then (X, τ) is β_1-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of X. For each $y \in Y$ and for each $x \in f^{-1}(y)$, choose an $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Therefore the collection $\{U_{\alpha(x)} : x \in f^{-1}(y)\}$ is a β-open cover of $f^{-1}(y)$ and so there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \bigcup_{\alpha(x)\in\Delta(y)} U_{\alpha(x)} = U_y$. But f is β-closed, so by Proposition 3.15, there exists a β-open set V_y in (Y, σ) such that $y \in V_y$ and $f^{-1}(V_y) \subseteq U_y$. Thus $\mathcal{V} = \{V_y : y \in Y\}$ is a β-open cover of Y and so it has a locally finite open refinement, say, $\mathcal{W} = \{W_{\alpha'} : \alpha' \in \Delta_\circ\}$. Since f is continuous, the family $\{f^{-1}(W_{\alpha'}) : \alpha' \in \Delta_\circ\}$ is an open locally finite cover of X such that for every $\alpha' \in \Delta_\circ$, we have $f^{-1}(W_{\alpha'}) \subseteq U_y$ for some $y \in Y$. Now, the family $\{f^{-1}(W_{\alpha'}) \cap U_{\alpha(x)} : \alpha' \in \Delta_\circ, \alpha(x) \in \Delta(y)\}$ is an open locally finite refinement of $\mathcal{U}$. Therefore (X, τ) is β_1-paracompact. $\square$

Theorem 3.17. *Let $f : (X, \tau) \to (Y, \sigma)$ be a β-open, continuous, bijective function. If A is β_1-paracompact relative to Y, then $f^{-1}(A)$ is β_1-paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta_\circ\}$ be a β-open cover of A in (X, τ). Since f is β-open, $\mathcal{U}_1 = \{f(U_\alpha) : \alpha \in \Delta_\circ\}$ is a β-open cover of A in (Y, σ). So, there exists a locally finite open refinement of $\mathcal{U}_1$, say $\mathcal{V}_1$. Since f is continuous, by Lemma 1.5, $\mathcal{V} = f^{-1}(\mathcal{V}_1)$ is an open locally finite refinement of $\mathcal{U}$. Therefore, $f^{-1}(A)$ is β_1-paracompact relative to X. $\square$

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