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Some results on asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and Ky Fan inequalities

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Abstract

In this paper, we study asymptotically quasi-ϕ- nonexpansive mappings in the intermediate sense and Ky Fan inequalities. A convergence theorem is established in a strictly convex and uniformly smooth Banach space. The results presented in the paper improve and extend some recent results.

Keywords: Asymptotically nonexpansive mapping, quasi-ϕ-nonexpansive mapping, fixed point, convergence theorem.
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1. Introduction and Preliminaries

Let E be a real Banach space and let C be nonempty closed and convex subset of E. Let $B : C \times C \to \mathbb{R}$ be a function. Recall the following equilibrium problem in the terminology of Blum and Oettli [4].

$$\text{Find } \bar{x} \in C \text{ such that } B(\bar{x}\, y) \geq 0, \forall y \in C.$$

In this paper, we use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$. The following restrictions on function B are essential in this paper.

(A-1) $B(a, a) \equiv 0, \forall a \in C$;

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(A-2) $0 \geq B(b,a) + B(a,b), \forall a, b \in C$;

(A-3) $b \mapsto B(a,b)$ is convex and weakly lower semi-continuous, $\forall a \in C$;

(A-4) $B(a,b) \geq \limsup_{t \downarrow 0} B(tc + (1-t)a, b)$, $\forall a, b, c \in C$.

The equilibrium problem has been extensively studied based on iterative methods because of its applications in nonlinear analysis, optimization, economics, game theory, mechanics, medicine and so forth, see [3], [7]-[11], [14], [17], [18], [25], [27]-[31] and the references therein.

Let E^* be the dual space of E. Let B_E be the unit sphere of E. Recall that E is said to be uniformly convex if for any $a \in (0,2]$ there exists $b > 0$ such that for any $x, y \in B_E$,

$$\|y - x\| \geq a \quad \text{implies} \quad \|y + x\| \leq 2 - 2b.$$

E is said to be a strictly convex space if and only if $\|y + x\| < 2$ for all $x, y \in B_E$ and $x \neq y$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that E is said to have a Gâteaux differentiable norm if and only if $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in B_E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm iff $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ is attained uniformly for $x, y \in B_E$. In this case, we say that E is uniformly smooth. It is known that a uniformly smooth Banach space is reflexive and smooth. Recall that E is said to has the Kadec-Klee property if $\lim_{m \to \infty} \|x_m - x\| = 0$, for any sequence $\{x_m\} \subset E$, and $x \in E$ with $\{x_n\}$ converges weakly to x, and $\{\|x_n\|\}$ converges strongly to $\|x\|$. It is known that every uniformly convex Banach space has the Kadec-Klee property; see [13] and the references therein.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}.$$

It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Let T be a mapping on C. T is said to be closed if for any sequence $\{x_m\} \subset C$ such that $\lim_{m \to \infty} x_m = x'$ and $\lim_{m \to \infty} Tx_m = y'$, then $Tx' = y'$. Let W be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n \to \infty} \sup_{x \in W} \{\|T^n x - T^{n+1} x\|\} = 0$. From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively and use $Fix(T)$ to denote the fixed point set of mapping T.

Next, we assume that E is a smooth Banach space which means mapping J is single-valued. Study the functional

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy \rangle, \quad \forall x, y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive, that is, $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$. In [2], Alber studied a new mapping $Proj_C$ in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$.

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense iff $Fix(T) \neq \emptyset$ and

$$\limsup_{n \to \infty} \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big) \leq 0.$$

Putting $\xi_n = \max\{0, \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$. Hence, we have

$$\phi(p, T^n x) \leq \phi(p, x) + \xi_n, \quad \forall x \in C, \forall p \in Fix(T).$$

T is said to be asymptotically quasi-ϕ-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \geq 1,$$

where $\{u_n\}$ is a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$.

T is said to be quasi-ϕ-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

Recall that p is said to be an asymptotic fixed point of T if and only if C contains a sequence $\{x_n\}$, where $x_n \rightharpoonup p$ such that $x_n - Tx_n \to 0$. Here, we use $\widetilde{Fix}(T)$ to denote the asymptotic fixed point set of T.

T is said to be asymptotically relatively quasi-ϕ-nonexpansive iff $Fix(T) = \widetilde{Fix}(T) \neq \emptyset$ and

$$\phi(p, T^n x) \leq (1 + u_n)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) = \widetilde{Fix}(T), \forall n \geq 1,$$

where $\{u_n\}$ is a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$.

T is said to be relatively nonexpansive iff $Fix(T) = \widetilde{Fix}(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) = \widetilde{Fix}(T).$$

Remark 1.1. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense [24] is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered in [5] as a non-Lipschitz continuous mappings, in the framework of Hilbert spaces.

Remark 1.2. The class of quasi-ϕ-nonexpansive mappings [21] is a generalization of relatively nonexpansive mappings [6]. The class of quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

Remark 1.3. The class of asymptotically quasi-ϕ-nonexpansive mappings [22] is more desirable than the class of asymptotically relatively nonexpansive [1] mappings. Asymptotically quasi-ϕ-nonexpansive mappings are reduced to asymptotically quasi-nonexpansive mappings in the framework of Hilbert spaces.

In this paper, we study the equilibrium problem in the terminology of Blum and Oettli [4] and a finite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. With the aid of generalization projections, we establish a strong theorem in a strictly convex and uniformly smooth Banach space. The results obtained in this paper mainly improve the corresponding results in [15], [16], [19], [20], [23], [30]. In order to prove our main results, we also need the following lemmas.

Lemma 1.4 ([2])**.** *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, x) - \phi(\Pi_C x, x) \geq \phi(y, \Pi_C x), \quad \forall y \in C,$$

$0 \geq \langle y - x_0, Jx - Jx_0 \rangle$, $\forall y \in C$ *if and only if* $x_0 = \Pi_C x$.

Lemma 1.5 ([26])**.** *Let r be a positive real number and let E be uniformly convex. Then there exists a convex, strictly increasing and continuous function $cog : [0, 2r] \to \mathbb{R}$ such that $cog(0) = 0$ and*

$$t\|a\|^2 + (1-t)\|b\|^2 \geq \|(1-t)b + ta\|^2 + t(1-t)cog(\|b - a\|)$$

for all $a, b \in B^r := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

Lemma 1.6 ([4], [21])**.** *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closedconvex subset of E. Let B be a function with restrictions (A-1), (A-2), (A-3) and (A-4), from $C \times C$*

to $\mathbb{R}$. *Let* $x \in E$ *and let* $r > 0$. *Then there exists* $z \in C$ *such that* $\langle z - y, Jz - Jx\rangle + rB(z, y) \leq 0$, $\forall y \in C$ *Define a mapping* $K^{B,r}$ *by*

$$K^{B,r}x = \{z \in C : 0 \leq \langle y - z, Jz - Jx\rangle + rB(z, y), \quad \forall y \in C\}.$$

The following conclusions hold:

(1) $K^{B,r}$ *is single-valued quasi-*ϕ*-nonexpansive;*

(2) $Sol(B) = Fix(K^{B,r})$ *is convex and closed.*

Lemma 1.7 ([24])**.** *Let* E *be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let* C *be a convex and closed subset of* E *and let* T *be an asymptotically quasi-*ϕ*-nonexpansive mapping in the intermediate sense on* C. $Fix(T)$ *is convex.*

2. Main results

Theorem 2.1. *Let* E *be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let* C *be a convex and closed subset of* E *and let* B *be a function with restrictions (A-1), (A-2), (A-3) and (A-4). Let* $\{T_m\}_{m=1}^N$, *where* N *is some positive integer, be a sequence of asymptotically quasi-*ϕ*-nonexpansive mappings in the intermediate sense on* C. *Assume that every* T_m *is uniformly asymptotically regular and closed and* $Sol(B)\bigcap\cap_{m=1}^N Fix(T_m)$ *is nonempty. Let* $\{\alpha_{(n,0)}\}, \{\alpha_{(n,1)}\}, \cdots, \{\alpha_{(n,N)}\}$ *be real sequences in (0,1) such that* $\sum_{m=0}^N \alpha_{(n,m)} = 1$ *and* $\liminf_{n\to\infty} \alpha_{(n,0)}\alpha_{(n,m)} > 0$ *for any* $1 \leq m \leq N$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily},\\ C_1 = C,\\ x_1 = Proj_{C_1}x_0,\\ r_nB(u_n, u) \geq \langle u_n - u, Ju_n - Jx_n\rangle, \forall u \in C_n,\\ Jy_n = \Big(\sum_{m=1}^N \alpha_{(n,m)}JT_m^n x_n + \alpha_{(n,0)}Ju_n\Big),\\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq (1 - \alpha_{(n,0)})\xi_n + \phi(z, x_n)\},\\ x_{n+1} = Proj_{C_{n+1}}x_1, \end{cases}$$

where $\xi_n = \max\big\{\max\{\sup_{p\in Fix(T_m),x\in C}\big(\phi(p, T_m^n x) - \phi(p, x)\big), 0\} : 1 \leq m \leq N\big\}$, *and* $\{r_n\}$ *is a real sequence such that* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $Proj_{Sol(B)\bigcap\cap_{m=1}^N Fix(T_m)}x_1$.

Proof. The proof is split into seven steps.

Step 1. Prove that $Sol(B)\bigcap\cap_{m=1}^N Fix(T_m)$ is convex and closed.

Using Lemmas 1.6 and 1.7, we find that $Fix(T_m)$ is convex and $Sol(B)$ is convex and closed. Since T_m is closed, we find that $Fix(T_m)$ is also closed. So, $Proj_{Sol(B)\cap\cap_{m=1}^N Fix(T_m)}x$ is well defined, for any element x in E.

Step 2. Prove that C_n is convex and closed.

It is obvious that $C_1 = C$ is convex and closed. Assume that C_i is convex and closed for some $i \geq 1$. Let $p_1, p_2 \in C_{i+1}$. It follows that $p = sp_1 + (1 - s)p_2 \in C_i$, where $s \in (0, 1)$. Since

$$(1 - \alpha_{(i,0)})\xi_i + \phi(p_1, x_i) \geq \phi(p_1, y_i),$$

and

$$(1 - \alpha_{(i,0)})\xi_i + \phi(p_2, x_i) \geq \phi(p_2, y_i),$$

one has

$$(1-\alpha_{(i,0)})\xi_i \geq 2\langle p_1, Jx_i - Jy_i\rangle - \|x_i\|^2 + \|y_i\|^2,$$

and

$$(1-\alpha_{(i,0)})\xi_i \geq 2\langle p_2, Jx_i - Jy_i\rangle - \|x_i\|^2 + \|y_i\|^2.$$

Using the above two inequalities, one has

$$\phi(p, y_i) - \phi(p, x_i) \leq (1-\alpha_{(i,0)})\xi_i.$$

This shows that C_{i+1} is closed and convex. Hence, C_n is a convex and closed set.

Step 3. Prove $\cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_n$.

It is obvious

$$\cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_1 = C.$$

Suppose that $\cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_i$ for some positive integer i. For any $z \in \cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_i$, we see that

$$\begin{aligned}
&\phi(z, x_i) + (1-\alpha_{(i,0)})\xi_i \\
&\geq \sum_{m=1}^N \alpha_{(i,m)}\phi(z, T_m^i x_i) + \alpha_{(i,0)}\phi(z, u_i) \\
&\geq \|z\|^2 + \sum_{m=1}^N \alpha_{(i,m)}\|T_m^i x_i\|^2 + \alpha_{(i,0)}\|Ju_i\|^2 \\
&\quad - 2\alpha_{(i,0)}\langle z, Ju_i\rangle - 2\sum_{m=1}^N \alpha_{(i,m)}\langle z, JT_m^i x_i\rangle \\
&\geq \|z\|^2 + \|\sum_{m=1}^N \alpha_{(i,m)} JT_m^i x_i + \alpha_{(i,0)} Ju_i\|^2 \\
&\quad - 2\langle z, \sum_{m=1}^N \alpha_{(i,m)} JT_m^i x_i + \alpha_{(i,0)} Ju_i\rangle \\
&= \phi(z, y_i),
\end{aligned}$$

where

$$\xi_i = \max\big\{\max\{\sup_{p\in Fix(T_m), x\in C}\big(\phi(p, T_m^i x) - \phi(p, x)\big), 0\} : 1 \leq m \leq N\big\}.$$

This shows that $z \in C_{i+1}$. This implies that $\cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_n$.

Step 4. Prove that $\{x_n\}$ is bounded.

Now, we have $\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0$, for any $z \in C_n$. It follows that

$$0 \leq \langle x_n - z, Jx_1 - Jx_n\rangle, \quad \forall z \in \cap_{m=1}^N Fix(T_m) \cap Sol(B) \subset C_n.$$

On the other hand, we find from Lemma 1.4,

$$\begin{aligned}
&\phi(Proj_{\cap_{m=1}^N Fix(T_m)\cap Sol(B)} x_1, x_1) \\
&\geq \phi(Proj_{\cap_{m=1}^N Fix(T_m)\cap Sol(B)} x_1, x_1) - \phi(Proj_{\cap_{m=1}^N Fix(T_m)\cap Sol(B)} x_1, x_n) \\
&\geq \phi(x_n, x_1),
\end{aligned}$$

which shows that $\{\phi(x_n, x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Without loss of generality, we assume $x_n \rightharpoonup \bar{x}$. Since every C_n is convex and closed. So $\bar{x} \in C_n$.

Step 5. Prove $\bar{x} \in \cap_{m=1}^N Fix(T_m)$.

Since $\bar{x} \in C_n$, one has $\phi(x_n, x_1) \le \phi(\bar{x}, x_1)$. This implies that

$$\phi(\bar{x}, x_1) \le \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle) = \limsup_{n\to\infty} \phi(x_n, x_1) \le \phi(\bar{x}, x_1).$$

Hence, one has

$$\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1).$$

It follows that

$$\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|.$$

Using the Kadec-Klee property, one obtains that $\{x_n\}$ converges strongly to $\bar{x}$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we find that

$$\phi(x_{n+1}, x_1) \ge \phi(x_n, x_1),$$

which shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Since

$$\phi(x_{n+1}, x_1) - \phi(x_n, x_1) \ge \phi(x_{n+1}, x_n) \ge 0,$$

one has $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Using the fact $x_{n+1} \in C_{n+1}$, one sees

$$\phi(x_{n+1}, y_n) - \phi(x_{n+1}, x_n) \le (1 - \alpha_{(n,0)})\xi_n.$$

Since

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = \lim_{n\to\infty} \xi_n = 0,$$

one has

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

Therefore, one has

$$\lim_{n\to\infty} (\|y_n\| - \|x_{n+1}\|) = 0.$$

This implies that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Jy_n\}$ is bounded. Without loss of generality, we assume that $\{Jy_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n\rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has $0 \ge \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2 = \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy\rangle = \phi(\bar{x}, y) \ge 0$. That is, $\bar{x} = y$, which in turn implies that $J\bar{x} = y^*$. Hence, $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E is uniformly smooth. Hence, E^* is uniformly convex and it has the Kadec-Klee property, we obtain

$$\lim_{n\to\infty} Jy_n = J\bar{x}.$$

Since $J^{-1} : E^* \to E$ is demi-continuous and E has the Kadec-Klee property, one gets that $y_n \to \bar{x}$, as $n \to \infty$. Using the fact

$$(\|x_n\| + \|y_n\|)\|y_n - x_n\| + 2\langle z, Jy_n - Jx_n\rangle \ge \phi(z, x_n) - \phi(z, y_n),$$

we find

$$\lim_{n\to\infty} \big(\phi(z, x_n) - \phi(z, y_n)\big) = 0. \tag{2.1}$$

It follows from Lemma 1.5, that

$$
\begin{aligned}
&\phi(z,x_n)+(1-\alpha_{(n,0)})\xi_n-\alpha_{(n,0)}\alpha_{(n,m)}g(\|JT_m^n x_n-Ju_n\|)\\
&\geq \sum_{m=1}^{N}\alpha_{(n,m)}\phi(z,T_m^n x_n)+\alpha_{(n,0)}\phi(z,u_n)-\alpha_{(n,0)}\alpha_{(n,m)}g(\|JT_m^n x_n-Ju_n\|)\\
&\geq \sum_{m=0}^{N}\alpha_{(n,m)}\|z\|^2+\sum_{m=1}^{N}\alpha_{(n,m)}\|T_m^n x_n\|^2+\alpha_{(n,0)}\|Ju_n\|^2\\
&\quad -2\alpha_{(n,0)}\langle z,Ju_n\rangle-2\sum_{m=1}^{N}\alpha_{(n,m)}\langle z,JT_m^n x_n\rangle\\
&\quad -\alpha_{(n,0)}\alpha_{(n,m)}g(\|JT_m^n x_n-Ju_n\|)\\
&\geq \phi(z,y_n).
\end{aligned}
$$

This implies

$$0\leq \alpha_{(n,0)}\alpha_{(n,m)}g(\|JT_m^n x_n-Ju_n\|)\leq \big(\phi(z,x_n)-\phi(z,y_n)\big)+(1-\alpha_{(n,0)})\xi_n.$$

Since $\liminf_{n\to\infty}\alpha_{(n,0)}\alpha_{(n,m)}>0$, one sees from 2.1

$$\lim_{n\to\infty}\|Ju_n-JT_m^n x_n\|=0$$

for any $1\leq m\leq N$. Using the fact

$$\sum_{m=1}^{N}\alpha_{(n,m)}(JT_m^n x_n-Ju_n)=Jy_n-Ju_n,$$

one has $\{Ju_n\}$ converges strongly to $J\bar{x}$. It follows that $JT_m^n x_n\to J\bar{x}$ as $n\to\infty$. Since $J^{-1}:E^*\to E$ is demi-continuous, one has $T_m^n x_n\rightharpoonup\bar{x}$. Using the fact

$$|\|T_m^n x_n\|-\|\bar{x}\||=|\|JT_m^n x_n\|-\|J\bar{x}\||\leq\|JT_m^n x_n-J\bar{x}\|,$$

one has $\|T_m^n x_n\|\to\|\bar{x}\|$ as $n\to\infty$. Since E has the Kadec-Klee property, one has

$$\lim_{n\to\infty}\|\bar{x}-T_m^n x_n\|=0.$$

Since T_m is also uniformly asymptotically regular, one has

$$\lim_{n\to\infty}\|\bar{x}-T_m^{n+1}x_n\|=0.$$

That is, $T_m(T_m^n x_n)\to\bar{x}$. Using the closedness of T_m, we find $T_m\bar{x}=\bar{x}$. This proves $\bar{x}\in Fix(T_m)$, that is, $\bar{x}\in\cap_{m=1}^{N}Fix(T_m)$.

Step 6. Prove $\bar{x}\in Sol(B)$.

Since B is a monotone bifunction, one has

$$r_nB(u,u_n)\leq\|u-u_n\|\|Ju_n-Jx_n\|.$$

Since $\liminf_{n\to\infty}r_n>0$, we may assume there exists $\lambda>0$ such that $r_n\geq\lambda$. It follows that

$$B(u,u_n)\leq\|u-u_n\|\frac{\|Ju_n-Jx_n\|}{\lambda}.$$

Hence, one has $B(u,\bar{x})\leq 0$. For $0<s<1$, define $u^s=(1-s)\bar{x}+su$. This implies that $0\geq B(u^s,\bar{x})$. Hence, we have

$$sB(u^s,u)\geq B(u^s,u^s)=0.$$

It follows that $B(\bar{x}, u) \geq 0$, $\forall u \in C$. This implies that $\bar{x} \in Sol(B)$.

Step 7. Prove $\bar{x} = Proj_{\cap_{m=1}^{N} Fix(T_m) \cap Sol(B)} x_1$.

Using Lemma 1.5, we find

$$0 \leq \langle x_n - z, Jx_1 - Jx_n \rangle, \forall z \in \cap_{m=1}^{N} Fix(T_m) \cap Sol(B).$$

Let $n \to \infty$, one has

$$0 \leq \langle \bar{x} - z, Jx_1 - J\bar{x} \rangle.$$

It follows that $\bar{x} = Proj_{\cap_{m=1}^{N} Fix(T_m) \cap Sol(B)} x_1$. This completes the proof. □

If $N = 1$, we have the following result.

Corollary 2.2. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (A-1), (A-2), (A-3) and (A-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed and $Sol(B) \bigcap Fix(T)$ is nonempty. Let $\{\alpha_{(n,0)}\}$ be a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_{(n,0)}(1 - \alpha_{(n,0)}) > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ r_n B(u_n, u) \geq \langle u_n - u, Ju_n - Jx_n \rangle, \forall u \in C_n, \\ y_n = J^{-1}\Big((1 - \alpha_{(n,0)}) JT^n x_n + \alpha_{(n,0)} Ju_n\Big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq (1 - \alpha_{(n,0)})\xi_n + \phi(z, x_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\}$, and $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $Proj_{Sol(B)\cap Fix(T)} x_1$.

If T is the identity mapping, we have the following results on the equilibrium problem.

Corollary 2.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (A-1), (A-2), (A-3) and (A-4). Let $N \geq 1$ be some positive integer and assume $Sol(B) \neq \emptyset$. Let $\{\alpha_{(n,0)}\}, \{\alpha_{(n,1)}\}, \cdots, \{\alpha_{(n,N)}\}$ be real sequences in (0,1) such that $\sum_{m=0}^{N} \alpha_{(n,m)} = 1$ and $\liminf_{n\to\infty} \alpha_{(n,0)}\alpha_{(n,m)} > 0$ for any $1 \leq m \leq N$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ r_n B(u_n, u) \geq \langle u_n - u, Ju_n - Jx_n \rangle, \forall u \in C_n, \\ y_n = J^{-1}\Big(\sum_{m=1}^{N} \alpha_{(n,m)} Jx_n + \alpha_{(n,0)} Ju_n\Big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $Proj_{Sol(B)} x_1$.

In the framework of Hilbert spaces, $\sqrt{\phi(x, y)} = \|x - y\|$, $\forall x, y \in E$. The generalized projection is reduced to the metric projection and the class of asymptotically-ϕ-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense.

Corollary 2.4. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let B be a function with (A-1), (A-2), (A-3) and (A-4). Let $\{T_m\}_{m=1}^N$, where N is some positive integer, be a sequence of asymptotically quasi-nonexpansive mappings in the intermediate sense on C. Assume that every T_m is uniformly asymptotically regular and closed and $Sol(B)\bigcap\cap_{m=1}^N Fix(T_m)$ is nonempty. Let $\{\alpha_{(n,0)}\}, \{\alpha_{(n,1)}\}, \cdots, \{\alpha_{(n,N)}\}$ be real sequences in (0,1) such that $\sum_{m=0}^N \alpha_{(n,m)} = 1$ and*

$$\liminf_{n\to\infty} \alpha_{(n,0)}\alpha_{(n,m)} > 0$$

for any $1 \le m \le N$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = P_{C_1}x_0, \\ r_n B(u_n, u) \ge \langle u_n - u, u_n - x_n\rangle, \forall u \in C_n, \\ y_n = \sum_{m=1}^N \alpha_{(n,m)} T_m^n x_n + \alpha_{(n,0)} u_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le (1 - \alpha_{(n,0)})\xi_n + \|z - x_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \max\big\{\max\{\sup_{p\in Fix(T_m), x\in C}\big(\|p - T_m^n x\|^2 - \|p - x\|^2\big), 0\} : 1 \le m \le N\big\}$, and $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{Sol(B)\bigcap\cap_{m=1}^N Fix(T_m)}x_1$.

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