Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1497–1504
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

q-Durrmeyer operators based on Pólya distribution

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Communicated by Y. J. Cho

Abstract

We introduce a q analogue of Durrmeyer type modification of Bernstein operators based on Pólya distributions. We study the ordinary approximation properties of operators using modulus of continuity and Peetre K-functional of second order. Further, we establish the weighted approximation properties for these operators.

Keywords: Pólya distribution, q-integers, q-Bernstein operators, modulus of continuity, Peetre K-functional, weighted modulus of continuity.
2010 MSC: 41A25, 41A35.

1. Introduction

The quantum calculus (q-Calculus) is an important branch of mathematics and physics. In the theory of approximation for linear positive operators, q-calculus was applied first by Lupaş [12], who introduced the q-analogue of the classical Bernstein polynomials. In 1997 Phillips [15] proposed a new q analogue of the Bernstein polynomials, which became popular amongst researchers. Recently Aral, Gupta and Agarwal [1] presented some results on convergence of several different operators. We use the notations of q calculus as mentioned in [1].

Also, Nowak [13] introduced the q-variant of the Lupaş operators, which is based on Pólya distribution. For $f \in C[0,1]$, $0 < q < 1$ and $\alpha \geq 0$, the operators considered in [13] are defined by

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} f([k]_q/[n]_q)p_{n,k}^{q,\alpha}(x), \ x \in [0,1], \tag{1.1}$$

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Received 2015-01-16

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ K \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+[i]_q\alpha)\prod_{j=0}^{n-k-1}(1-q^jx+[j]_q\alpha)}{\prod_{l=0}^{n-1}(1+[l]_q\alpha)}.$$

Note that empty product in the above basis function $p_{n,k}^{q,\alpha}(x)$ is denoted by 1. Nowak in [13] and Nowak and Gupta in [14] estimated the moments and established some direct estimates for the operators (1.1).

In the year 2008 Gupta [6] introduced q-Durrmeyer operators. Some other results and forms of q-Durrmeyer type operators were discussed in [2, 5, 7, 10, 11] and [8] etc. We now introduce the q-analogue of Lupaş Durrmeyer operators for $f \in C[0,1]$ and $0 < q < 1$ by

$$D_n^{q,1/[n]_q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}^{q,1/[n]_q}(x) \int_0^1 p_{n,k}^q(qt)f(t)d_qt, \quad x \in [0,1], \tag{1.2}$$

where

$$p_{n,k}^{q,1/[n]_q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\frac{[i]_q}{[n]_q})\prod_{i=0}^{n-k-1}(1-x+\frac{[i]_q}{[n]_q})}{\prod_{i=0}^{n-1}(1+\frac{[i]_q}{[n]_q})}$$

and

$$p_{n,k}^q(qt) = \begin{bmatrix} n \\ k \end{bmatrix}_q q^kt^k \prod_{j=0}^{n-k-1}(1-q^{j+1}t).$$

As a special case when $q \to 1$, we get the operators due to Gupta and Rassias [9].

2. Lemmas

Lemma 2.1. *For $e_i = t^i$, $i = 0,1,2,3,4$, the moments of q-Lupaş operators based on Pólya distribution are*

$$\begin{aligned}
B_n^{q,\alpha}(e_0;x) &= 1,\\
B_n^{q,\alpha}(e_1;x) &= x,\\
B_n^{q,\alpha}(e_2;x) &= \frac{1}{1+\alpha}\left(x(x+\alpha)+\frac{x(1-x)}{[n]_q}\right),\\
B_n^{q,\alpha}(e_3;x) &= \frac{1}{\prod_{i=0}^{2}(1+[i]_q\alpha)}\sum_{k=0}^{2}\frac{\overline{W}_k(q,\alpha;x)}{[n]_q^k},\\
B_n^{q,\alpha}(e_4;x) &= \frac{1}{\prod_{i=0}^{3}(1+[i]_q\alpha)}\sum_{k=0}^{3}\frac{W_k(q,\alpha;x)}{[n]_q^k},
\end{aligned}$$

where

$$\begin{aligned}
\overline{W}_0(q,\alpha;x) &= x(x+\alpha)(x+[2]_q\alpha),\\
\overline{W}_1(q,\alpha;x) &= x(1-x)(x+\alpha)(2+q),\\
\overline{W}_2(q,\alpha;x) &= x(1-x)(1-[2]_qx),\\
W_0(q,\alpha;x) &= x(x+\alpha)(x+[2]_q\alpha)(x+[3]_q\alpha),\\
W_1(q,\alpha;x) &= x(1-x)(x+\alpha)(x+[2]_q\alpha)(q^2+2q+3),\\
W_2(q,\alpha;x) &= x(1-x)(x+\alpha)\left((q^2+3q+3)x(x+\alpha)-[2]_q^2(x+\alpha-[2]_qx(1+[3]_q\alpha))\right),\\
W_3(q,\alpha;x) &= x(1-x)(x+\alpha)([2]_qx([3]_qx-q-2)+1-q\alpha).
\end{aligned}$$

Lemma 2.2. *For all $x \in [0,1]$, $n \in \mathbb{N}$ and $q \in (0,1)$, we have*

$$\begin{aligned}
D_n^{q,1/[n]_q}(e_0;x) &= 1,\\
D_n^{q,1/[n]_q}(e_1;x) &= \frac{q[n]_q x + 1}{[n+2]_q},\\
D_n^{q,1/[n]_q}(e_2;x) &= \frac{[n]_q^3 q^3}{(1+[n]_q)[n+2]_q[n+3]_q}\left(x(x+1/[n]_q) + \frac{x(1-x)}{[n]_q}\right)\\
&\quad + \frac{q(2q+1)[n]_q x}{[n+2]_q[n+3]_q} + \frac{q+1}{[n+2]_q[n+3]_q},\\
D_n^{q,1/[n]_q}(e_3;x) &= \frac{a_1(q)[n]_q^3}{[n+2]_q[n+3]_q[n+4]_q}\frac{1}{\prod_{i=0}^2(1+[i]_q/[n]_q)}\sum_{k=0}^{2}\frac{\overline{W}_k(q,1/[n]_q;x)}{[n]_q^k}\\
&\quad + \frac{a_2(q)[n]_q^3}{(1+[n]_q)[n+2]_q[n+3]_q[n+4]_q}\left(x(x+1/[n]_q) + \frac{x(1-x)}{[n]_q}\right)\\
&\quad + \frac{a_3(q)[n]_q x}{[n+2]_q[n+3]_q[n+4]_q} + \frac{a_4(q)}{[n+2]_q[n+3]_q[n+4]_q},\\
D_n^{q,1/[n]_q}(e_4;x) &= \frac{b_1(q)[n]_q^4}{[n+2]_q[n+3]_q[n+4]_q[n+5]_q}\frac{1}{\prod_{i=0}^3(1+[i]_q/[n]_q)}\sum_{k=0}^{3}\frac{W_k(q,1/[n]_q;x)}{[n]_q^k}\\
&\quad + \frac{b_2(q)[n]_q^3}{[n+2]_q[n+3]_q[n+4]_q[n+5]_q}\frac{1}{\prod_{i=0}^2(1+[i]_q/[n]_q)}\sum_{k=0}^{2}\frac{\overline{W}_k(q,1/[n]_q;x)}{[n]_q^k}\\
&\quad + \frac{b_3(q)[n]_q^3}{([n]_q+1)[n+2]_q[n+3]_q[n+4]_q[n+5]_q}\left(x(x+1/[n]_q) + \frac{x(1-x)}{[n]_q}\right)\\
&\quad + \frac{b_4(q)[n]_q x}{[n+2]_q[n+3]_q[n+4]_q[n+5]_q} + \frac{a_4(q)}{[n+2]_q[n+3]_q[n+4]_q[n+5]_q}.
\end{aligned}$$

We have

$$\int_0^1 p_{n,k}^q(qt)t^s d_q t = q^k \frac{[s+k]_q![n]_q!}{[s+n+1]_q![k]_q!},$$

using the methods described in [6] and Lemma 2.1. The proof of the lemma follows immediately and thus we omit the details.

Corollary 2.3. *For central moments, denoted by $\phi_{n,m}^q(x) = D_n^{q,1/[n]_q}((t-x)^m, x)$, $m = 1,2$, we have*

$$\begin{aligned}
\phi_{n,1}^q(x) &= \frac{(q[n]_q - [n+2]_q)x + 1}{[n+2]_q},\\
\phi_{n,2}^q(x) &= \left(1 + \frac{([n]_q-1)[n]_q^2 q^3}{(1+[n]_q)[n+2]_q[n+3]_q} - \frac{2q[n]_q}{[n+2]_q}\right)x^2\\
&\quad + \left(\frac{2[n]_q^2 q^3}{(1+[n]_q)[n+2]_q[n+3]_q} + \frac{q(2q+1)[n]_q}{[n+2]_q[n+3]_q} - \frac{2}{[n+2]_q}\right)x\\
&\quad + \frac{q+1}{[n+2]_q[n+3]_q}.
\end{aligned}$$

Lemma 2.4. *For $q \in (0,1)$ and $n > 3$, we have*

$$\phi_{n,2}^q(x) = \frac{3}{[n+2]_q}\delta_n^2(x), \tag{2.1}$$

where $\delta_n^2(x) = \left(\varphi^2(x) + \frac{1}{[n+2]_q}\right)$ and $\varphi^2(x) = x(1-x)$.

Proof. By using Corollary 2.3 and simple computation, we obtain

$$\begin{aligned}\phi^q_{n,2}(x) =& x(1-x)\left(\frac{[n]_q^2 q(2q+1)-[n]_q(2q^3+q+2)-2(q^2+q+1)}{(1+[n]_q)[n+2]_q[n+3]_q}\right)\\ &+x^2\left(\frac{[n]_q^3(q^2-2q+1)+[n]_q^2 q(q^4-q^2+q-1)+[n]_q(2q^4-q^3+q^2-q-1)+q^3-1}{(1+[n]_q)[n+2]_q[n+3]_q}\right)\\ &+\frac{q+1}{[n+2]_q[n+3]_q}.\end{aligned}$$

Clearly for $n>3$, we have

$$[n]_q^2 q(2q+1)-[n]_q(2q^3+q+2)-2(q^2+q+1)>0 \tag{2.2}$$

and

$$\frac{[n]_q^2 q(2q+1)-[n]_q(2q^3+q+2)-2(q^2+q+1)}{(1+[n]_q)[n+2]_q[n+3]_q}\leq\frac{3}{[n+3]_q}. \tag{2.3}$$

Also, as $q\in(0,1)$, we obtain

$$\begin{aligned}&[n]_q^3(q^2-2q+1)+[n]_q^2 q(q^4-q^2+q-1)+[n]_q(2q^4-q^3+q^2-q-1)+q^3-1\\ &\quad=[n]_q^3(q-1)^2+[n]_q^2 q(q-1)(q^3+q^2+1)+[n]_q(q-1)(q^2+1)(2q+1)+(q-1)(q^2+q+1)\\ &\quad=(q-1)\left([n]_q^3(q-1)+[n]_q^2 q(q^3+q^2+1)+[n]_q(q^2+1)(2q+1)+(q^2+q+1)\right)\\ &\quad\leq 0.\end{aligned}$$

Finally, by using the above inequality, (2.2) and (2.3), we get

$$\begin{aligned}\phi^q_{n,2}(x)&\leq x(1-x)\frac{3}{[n+3]_q}+\frac{q+1}{[n+2]_q[n+3]_q}\\ &=\frac{3}{[n+2]_q}\left(x(1-x)+\frac{1}{[n+2]_q}\right).\end{aligned}$$

□

3. Local Approximation

Let us consider:

$$W^2=\{g\in C[0,1]:g',g''\in C[0,1]\}.$$

The Peetre's K-functional for $\delta>0$ is defined by

$$K_2(f,\delta)=\inf_{g\in C_B^2[0,\infty)}\left\{||f-g||+\delta||g''||:g\in W^2\right\},$$

where $||.||$ is the uniform norm on $C[0,1]$. There exists a constant $C>0$, due to [3], such that for $\delta>0$, we have

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}), \tag{3.1}$$

where the second order modulus of continuity for $f\in C[0,1]$ is defined by

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1]}|f(x+2h-2f(x+h)+f(x)|.$$

We prove below the following local direct result:

Theorem 3.1. *Let $0 < q < 1$. Then*

$$\left| D_n^{q,1/[n]_q}(f;x) - f(x) \right| \leq K\omega_2\left(f, \sqrt{\phi_{n,2}^q(x) + {\phi_{n,1}^q}^2(x)}\right) + \omega\left(f, \phi_{n,1}^q(x)\right)$$

for every $x \in [0,1]$ and $f \in C[0,1]$, where K is a positive constant. Here $\phi_{n,1}^q(x)$ and $\phi_{n,2}^q(x)$ are the first and second central moments of the operator $D_n^{q,1/[n]_q}$.

Proof. We consider modified operators $\overline{D}_n^{q,1/[n]_q}$ defined by

$$\overline{D}_n^{q,1/[n]_q}(f;x) = D_n^{q,1/[n]_q}(f;x) - f\left(\frac{q[n]_q x + 1}{[n+2]_q}\right) + f(x), \tag{3.2}$$

where $x \in [0,1]$. The operators $\overline{D}_n^{q,1/[n]_q}$ preserve linear functions:

$$\overline{D}_n^{q,1/[n]_q}(t-x;x) = 0. \tag{3.3}$$

Let $g \in C^2[0,1]$ and $t \in [0,1]$. Using Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Then using (3.3), we get

$$\overline{D}_n^{q,1/[n]_q}(g;x) = g(x) + \overline{D}_n^{q,1/[n]_q}\left(\int_x^t (t-u)g''(u)du; x\right).$$

Therefore from (3.2), we have

$$\begin{aligned}
&|\ \overline{D}_n^{q,1/[n]_q}(g;x) - g(x)\ | \\
&\leq \left| D_n^{q,1/[n]_q}\left(\int_x^t (t-u)g''(u)du; x\right)\right| + \left| \int_x^{\frac{q[n]_q x+1}{[n+2]_q}} \left(\frac{q[n]_q x + 1}{[n+2]_q} - u\right) g''(u)du \right| \\
&\leq D_n^{q,1/[n]_q}\left(\int_x^t |t-u|\,|g''(u)|\,du; x\right) + \int_x^{\frac{q[n]_q x+1}{[n+2]_q}} \left|\frac{q[n]_q x + 1}{[n+2]_q} - u\right| |g''(u)|\,du \\
&\leq \left[D_n^{q,1/[n]_q}\left((t-x)^2; x\right) + \left(\frac{q[n]_q x + 1}{[n+2]_q} - x\right)^2 \right] \|g''\|.
\end{aligned} \tag{3.4}$$

On using Equation (3.2), we see that

$$\left|\overline{D}_n^{q,1/[n]_q}(f;x)\right| \leq \left| D_n^{q,1/[n]_q}(f;x)\right| + 2\|f\| \leq \|f\|\, D_n^{q,1/[n]_q}(1;x) + 2\|f\| = 3\|f\|. \tag{3.5}$$

Now using Equations (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned}
\left| D_n^{q,1/[n]_q}(f;x) - f(x)\right| &\leq \left|\overline{D}_n^{q,1/[n]_q}(f-g;x)\right| + |(f-g)(x)| + \left|\overline{D}_n^{q,1/[n]_q}(g;x) - g(x)\right| \\
&\quad + \left| f\left(\frac{q[n]_q x + 1}{[n+2]_q}\right) - f(x)\right| \\
&\leq 4\|f-g\| + (\phi_{n,2}^q(x) + {\phi_{n,1}^q}^2(x))\|g''\| + \left| f\left(\frac{q[n]_q x + 1}{[n+2]_q}\right) - f(x)\right|.
\end{aligned}$$

Thus taking infimum on the right-hand side over all $g \in W^2$, we get

$$\left| D_n^{q,1/[n]_q}(f;x) - f(x)\right| \leq 4K_2\left(f, \phi_{n,2}^q(x) + {\phi_{n,1}^q}^2(x)\right) + \omega\left(f, \phi_{n,1}^q(x)\right).$$

Consequently, we get

$$\left| D_n^{q,1/[n]_q}(f;x) - f(x)\right| \leq K\omega_2\left(f, \sqrt{\phi_{n,2}^q(x) + {\phi_{n,1}^q}^2(x)}\right) + \omega\left(f, \phi_{n,1}^q(x)\right).$$

This completes the proof of the theorem. □

4. Global Approximation

The Ditzian–Totik moduli of the first order is given by

$$\overrightarrow{\omega}_{\psi}(f,\eta)=\sup_{0\leq h\leq\sqrt{\eta}}\ \sup_{x+h\varphi(x)\in[0,1]}|f(x+h\varphi(x))-f(x)|,$$

where ψ is an admissible step-weight function on [0,1]. The second order Ditzian–Totik modulus of smoothness, for $f\in C[0,1]$, $\varphi(x)=\sqrt{x(1-x)}$ and $x\in[0,1]$ is defined by

$$\omega_2^{\varphi}(f,\sqrt{\eta})=\sup_{0\leq h\leq\sqrt{\eta}}\ \sup_{x\pm h\varphi(x)\in[0,1]}|f(x+h\varphi(x))+2f(x)+f(x-h\varphi(x))|.$$

For $W^2(\varphi)=\{g\in C[0,1]:g'\in AC_{loc}[0,1],\varphi^2g''\in C[0,1]\}$ and $g'\in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b]\subset[0,1]$. K-functional corresponding to the second order Ditzian–Totik modulus of smoothness is defined by

$$\overline{K}_{2,\varphi}(f,\eta)=\inf_{g\in W^2(\varphi)}\{\|f-g\|+\eta\|\varphi^2g''\|+\eta^2\|g''\|\}.$$

It is well known that (see [4])

$$\overline{K}_{2,\varphi}(f,\eta)\leq C\omega_2^{\varphi}(f,\sqrt{\eta}). \tag{4.1}$$

Theorem 4.1. *For $f\in C[0,1]$, $q\in(0,1)$ and $\psi(x)=2x$, $x\in[0,1]$, there exists an absolute constant $C>0$ such that*

$$\left|D_n^{q,1/[n]_q}(f;x)-f(x)\right|\leq C\omega_2^{\varphi}\left(f,\sqrt{\frac{1}{[n+2]_q}}\right)+\overrightarrow{\omega}_{\psi}\left(f,\frac{1}{[n+2]_q}\right).$$

Proof. Again, consider modified operators $\overline{D}_n^{q,1/[n]_q}$ defined by

$$\overline{D}_n^{q,1/[n]_q}(f;x)=D_n^{q,1/[n]_q}(f;x)-f\left(\frac{q[n]_qx+1}{[n+2]_q}\right)+f(x), \tag{4.2}$$

where $x\in[0,1]$. Using Taylor's expansion for $g\in W^2(\varphi)$, we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du.$$

Then, using (3.3), we get

$$\overline{D}_n^{q,1/[n]_q}(g;x)=g(x)+\overline{D}_n^{q,1/[n]_q}\left(\int_x^t(t-u)g''(u)du;x\right).$$

Therefore from (4.2), we have

$$\begin{aligned}|\ \overline{D}_n^{q,1/[n]_q}(g;x)-g(x)\ |&\leq\left|D_n^{q,1/[n]_q}\left(\int_x^t(t-u)g''(u)du;x\right)\right|+\left|\int_x^{\frac{q[n]_qx+1}{[n+2]_q}}\left(\frac{q[n]_qx+1}{[n+2]_q}-u\right)g''(u)du\right|\\&\leq D_n^{q,1/[n]_q}\left(\int_x^t|t-u|\left|g''(u)\right|du;x\right)+\int_x^{\frac{q[n]_qx+1}{[n+2]_q}}\left|\frac{q[n]_qx+1}{[n+2]_q}-u\right|\left|g''(u)\right|du.\end{aligned}$$

As the function δ_n^2 is a concave function on $[0,1]$, we have for $u=t+\tau(x-t)$, $\tau\in[0,1]$, that the following estimate holds true

$$\begin{aligned}\frac{|t-u|}{\delta_n^2(u)}&=\frac{\tau|x-t|}{\delta_n^2(t+\tau(x-t))}\\&\leq\frac{\tau|x-t|}{\delta_n^2(t)+\tau(\delta_n^2(x)-\delta_n^2(t))}\leq\frac{|t-x|}{\delta_n^2(x)}.\end{aligned}$$

Hence,

$$
\begin{aligned}
|\overline{D}_n^{q,1/[n]_q}(g;x)-g(x)| &\leq D_n^{q,1/[n]_q}\left(\int_x^t \frac{|t-u|}{\delta_n^2(u)}du;x\right)\|\delta_n^2 g''\| + \int_x^{\frac{q[n]_qx+1}{[n+2]_q}} \frac{\left|\frac{q[n]_qx+1}{[n+2]_q}-u\right|}{\delta_n^2(u)}du\|\delta_n^2 g''\| \\
&\leq \frac{1}{\delta_n^2(x)}\left(D_n^{q,1/[n]_q}\left((t-x)^2;x\right)+\left(\frac{q[n]_qx+1}{[n+2]_q}-x\right)^2\right)\|\delta_n^2 g''\|.
\end{aligned}
$$

Also,

$$
\delta_n^2(x)|g''(x)| = \varphi^2(x)|g''(x)| + \frac{1}{[n+2]_q}|g''(x)| \leq \|\varphi^2 g''\| + \frac{1}{[n+2]_q}\|g''\|,
$$

where $x\in[0,1]$; thus we obtain

$$
|\overline{D}_n^{q,1/[n]_q}(g;x)-g(x)| \leq \frac{10}{[n+2]_q}\left(\|\varphi^2 g''\| + \frac{1}{[n+2]_q}\|g''\|\right). \tag{4.3}
$$

Using (3.2) and (4.3), we get

$$
\begin{aligned}
\left|D_n^{q,1/[n]_q}(f;x)-f(x)\right| &\leq \left|\overline{D}_n^{q,1/[n]_q}(f-g;x)\right| + |(f-g)(x)| + \left|\overline{D}_n^{q,1/[n]_q}(g;x)-g(x)\right| \\
&\quad + \left|f\left(\frac{q[n]_qx+1}{[n+2]_q}\right)-f(x)\right| \\
&\leq 4\|f-g\| + \frac{10}{[n+2]_q}\left(\|\varphi^2 g''\| + \frac{1}{[n+2]_q}\|g''\|\right) + \left|f\left(\frac{q[n]_qx+1}{[n+2]_q}\right)-f(x)\right|.
\end{aligned}
$$

Taking infimum on the right-hand side over $W^2(\varphi)$, we have

$$
\left|D_n^{q,1/[n]_q}(f;x)-f(x)\right| \leq 10\overline{K}_{2,\varphi}\left(f,\frac{1}{[n+2]_q}\right) + \left|f\left(\frac{q[n]_qx+1}{[n+2]_q}\right)-f(x)\right|. \tag{4.4}
$$

Consider

$$
\begin{aligned}
\left|f\left(\frac{q[n]_qx+1}{[n+2]_q}\right)-f(x)\right| &\leq \left|f\left(x+\psi(x).\frac{([n]_q-[n+2]_q)x}{\psi(x)[n+2]_q}\right)-f(x)\right| \\
&\leq \sup_{t,t+\psi(t).\frac{([n]_q-[n+2]_q)x}{\psi(x)[n+2]_q}\in[0,1]}\left|f\left(t+\psi(t).\frac{([n]_q-[n+2]_q)x}{\psi(x)[n+2]_q}\right)-f(t)\right| \\
&\leq \overrightarrow{\omega}_\psi\left(f,\frac{([n]_q-[n+2]_q)x}{\psi(x)[n+2]_q}\right) = \overrightarrow{\omega}_\psi\left(f,\frac{1}{[n+2]_q}\right).
\end{aligned}
$$

Finally, using (4.1), (4.4) and the above inequality, we get

$$
\left|D_n^{q,1/[n]_q}(f;x)-f(x)\right| \leq C\omega_2^\varphi\left(f,\sqrt{\frac{1}{[n+2]_q}}\right) + \overrightarrow{\omega}_\psi\left(f,\frac{1}{[n+2]_q}\right).
$$

□

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