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Stationary distribution and pathwise estimation of n-species mutualism system with stochastic perturbation

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Abstract

In this paper, we develop a new stochastic mutualism population model

$$dx_i(t) = x_i(t)\left[\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt + \sigma_i x_i(t)dB_i(t)\right], \quad i = 1, 2, \cdots, n.$$

By constructing suitable Lyapunov functions, we show the system has a stationary distribution. We also discuss the pathwise behaviour of the solution. The conclusions of this paper is very powerful since they are independent both of the system parameters and of the initial value. It is also independent of the noise intensity as long as the noise intensity $\sigma_i^2 > 0$. Computer simulations are used to illustrated our results.

Keywords: Mutualism model, Itô's formula, stationary distribution, ergodicity, pathwise estimation.
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1. Introduction

Consider a n-species Lotka-Volterra mutualism model

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$$dx_i(t) = x_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt, \quad i = 1, 2, \cdots, n, \tag{1.1}$$

where $x_i(t)$ is the ith species population density at time t, r_i is the intrinsic growth rate of species x_i, a_{ii} represents the population decay rate in the competition among the ith species and a_{ij} represents the ith species population increase rate in the mutualism among the other species $x_j(i, j = 1, 2, \cdots, n, i \neq j)$.

In particular, Chen and Song [2] have studied the sufficient conditions for the global stability of positive equilibrium of model (1.1), the sufficient conditions are as follows:

(i) There is a matrix $G = (G_{ij})_{n\times n}$, such that $-a_{ii} \leq G_{ii}, a_{ij} \leq G_{ij}(i \neq j)$ hold for all $i, j = 1, 2, \cdots, n$.

(ii) All of the principal minors of $-G$ are positive.

There are many other researchers who have studied the dynamics of mutualism model (see [4, 10, 16] and references therein).

In fact, mutualism population dynamics is inevitably affected by environmental white noise, which is always present [6, 7, 9, 11, 12, 14, 15]. In practice, we usually estimate parameters by an average value plus errors. We may assume that the errors follow normal distributions, but the standard deviations of the errors, known as the noise intensities, may depend on the population sizes [1, 12]. Therefore, we replace parameter r_i in model (1.1) by

$$r_i \to r_i + \sigma_i x_i(t)\dot{B}(t), \ (i = 1, 2, \cdots, n).$$

Then we get the following new stochastic system

$$dx_i(t) = x_i(t)\left[\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt + \sigma_i x_i(t)dB_i(t)\right], \quad i = 1, 2, \cdots, n, \tag{1.2}$$

where $B_i(t)$ is standard one-dimensional independent Wiener processes, σ_i^2 are the intensity of the white noise and $\sigma_i^2 > 0$.

The organization of this paper is as follows. In next section, we will investigate the pathwise behaviour of the solution of system (1.2). In Section 3, we show that the system has a stationary distribution with no parametric restriction if $\sigma_i^2 > 0$. We illustrate our results through an example in Section 4. Finally, the conclusion is presented in Section 5.

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). We denote by R_+^n the positive cone in R^n, that is $R_+^n = \{x \in R^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$.

In general, we consider a d-dimensional stochastic differential equation [13]

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad \text{ on } t \geq t_0 \tag{1.3}$$

with initial value $x(t_0) = x_0 \in R^d$, where $B(t)$ denotes d-dimensional standard Brownian motion. Define the differential operator L associated with Equation (1.3) by

$$L = \frac{\partial}{\partial t} + \sum_{k=1}^{d} f_k(x,t)\frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,j=1}^{d}[g^T(x,t)g(x,t)]_{kj}\frac{\partial^2}{\partial x_k \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(S_h \times \overline{R}_+; \overline{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[g^T(x,t)V_{xx}g(x,t)],$$

where $C^{2,1}(S_h \times \overline{R}_+; \overline{R}_+)$ is the family of all nonnegative functions $V(x,t)$ defined on $S_h \times \overline{R}_+$ such that they are continuously twice differentiable in x and once in t, and

$$V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_d}), V_{xx} = (\frac{\partial^2 V}{\partial x_k \partial x_j})_{d\times d}.$$

2. Asymptotic pathwise estimation

In this section, we will investigate pathwise behaviour of the solution of system (1.2), which is one of desired population dynamical properties of population system.

Theorem 2.1. *For any given initial value $x_0 \in R_+^n$, the solution of system* (1.2) *has the following property:*

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log t} \le 1 \quad a.s.$$

Proof. Define a Lyapunov function for any $p \in (0,1)$

$$V(x) = \sum_{i=1}^{n} {x_i}^p.$$

Then

$$dV(x) = p\sum_{i=1}^{n} {x_i}^p \left(r_i + \sum_{j=1}^{n} a_{ij}x_j - \frac{1-p}{2}{\sigma_i}^2 {x_i}^2 \right) dt + p\sum_{i=1}^{n} \sigma_i {x_i}^{(p+1)} dB_i(t).$$

Let

$$z(x) = \frac{p\sum\limits_{i=1}^{n} \sigma_i {x_i}^{(p+1)}}{V(x)}, \quad K(x) = p\sum_{i=1}^{n} {x_i}^p \left(r_i + \sum_{j=1}^{n} a_{ij}x_j - \frac{1-p}{2}{\sigma_i}^2 {x_i}^2 \right).$$

By Itô's formula, we get

$$d\log V(x) = \left(\frac{K(x)}{V(x)} - \frac{z^2(x)}{2} \right) dt + z(x)dB(t)$$

and

$$de^t \log V(x) = \left(\log V(x) + \frac{K(x)}{V(x)} - \frac{z^2(x)}{2} \right) dt + z(x)dB(t).$$

Integrating both sides from 0 to t yields

$$e^t \log V(x(t)) = \log V(x(0)) + \int_0^t e^s \left(\log V(x(s)) + \frac{K(x(s))}{V(x(s))} - \frac{z^2(x(s))}{2} \right) ds + \int_0^t e^s z(x(s))dB(s).$$

Using the Exponential Martingale Inequality, for any $\alpha, \beta, T > 0$, we have

$$P\left\{ \omega : \sup_{0\le t\le T} \left[\int_0^t e^s z(x(s))dB(s) - \frac{\alpha}{2}\int_0^t e^{2s} z^2(x(s))ds \right] \ge \beta \right\} \le e^{-\alpha\beta}.$$

Choose $T = K\delta, \alpha = \epsilon e^{-K\delta}, \beta = \frac{(1+\delta)e^{K\delta}\log(K\delta)}{\epsilon}$, where $0 < \delta < 1, 0 < \epsilon < 1$. Note that

$$\sum_{K=1}^{\infty} \frac{1}{(K)^{1+\delta}} < \infty,$$

an application of the Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega) > 0$ such that

$$\begin{aligned} \int_0^t e^s z(x(s))dB(s) &\le \frac{(1+\delta)e^{K\delta}\log(K\delta)}{\epsilon} + \frac{\epsilon e^{-K\delta}}{2}\int_0^t e^{2s} z^2(x(s))ds \\ &\le \frac{(1+\delta)e^{K\delta}\log(K\delta)}{\epsilon} + \frac{\epsilon}{2}\int_0^t e^s z^2(x(s))ds, \quad 0 \le t \le K\delta, \quad n \ge n_0. \end{aligned} \tag{2.1}$$

Hence

$$\begin{aligned}\log V(x(t)) \le e^{-t}\log V(x(0)) + \frac{(1+\delta)e^{K\delta-t}\log(K\delta)}{\epsilon} \\ + \int_0^t e^{s-t}\left(\log V(x(s)) + \frac{K(x(s))}{V(x(s))} - \frac{(1-\epsilon)z^2(x(s))}{2}\right)ds.\end{aligned}$$

Applying inequalities $\log x \le x-1$ and $n^{(1-\frac{p}{2})\wedge 0}|x|^p \le \sum_{i=1}^{n} {x_i}^p \le n^{(1-\frac{p}{2})\vee 0}|x|^p$, one see that

$$\begin{aligned}\log V(x(t)) + \frac{K(x(t))}{V(x(t))} - \frac{(1-\epsilon)z^2(x(t))}{2} &\le V(x(t)) - 1 + \frac{K(x(t))}{V(x(t))} \\ &\le \sum_{i=1}^{n} {x_i}^p - 1 + p\check{r} + p\check{a}_{ij}\sum_{i=1}^{n} x_i - \frac{1}{n}|x|^2 \\ &\le C,\end{aligned}$$

where $\check{r} = \max\{r_1, r_2, \cdots, r_n\}$, $\check{a}_{ij} = \max\{a_{ij}\}, i, j = 1, 2, \cdots n$ and C is a constant.
Therefore

$$\log V(x(t)) \le e^{-t}\log V(x(0)) + \frac{(1+\delta)e^{K\delta-t}\log(K\delta)}{\epsilon} + C(1-e^{-t}).$$

It then follows that for almost all $\omega \in \Omega$, if $n \ge n_0$, $(K-1)\delta \le t \le K\delta$,

$$\frac{\log|x(t)|^p}{\log t} \le \frac{\log V(x(t))}{\log t} \le \frac{1}{\log(K-1)\delta}[e^{-t}\log V(x(0)) + C] + \frac{(1+\delta)e^{\delta}\log(K\delta)}{\epsilon\log(K-1)\delta}.$$

This implies

$$\limsup_{t\to\infty}\frac{\log|x(t)|^p}{\log t} \le \frac{(1+\delta)e^{\delta}}{\epsilon} \quad a.s.$$

Letting $\delta \to 0, \epsilon \to 1$ gives

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log t} \le \frac{1}{p} \quad a.s.$$

Letting $p \to 1$ we obtain

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log t} \le 1 \quad a.s.$$

This completes the proof. □

3. Existence of stationary distribution

In this section, we prove the existence of stationary distribution of system (1.2). The following theorem gives a criterion for the existence of stationary distribution in terms of Lyapunov function (see [3], Chapter 3, p.103, [8], [17], p.1163).

Let $X(t)$ be a homogeneous Markov Process in E_l (E_l denotes l dimensional Euclidean space) and is described by the following stochastic equation,

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t). \tag{3.1}$$

The diffusion matrix is defined as follows,

$$\Lambda(x) = (\lambda_{ij}(x)), \quad \lambda_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Theorem 3.1. *The Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $U \in E_l$ with regular boundary Γ and*

A_1 *: there is a positive number M such that $\sum_{i,j=1}^{l} \lambda_{ij}(x)\xi_i\xi_j \geq M|\xi|^2, \quad x \in U,\ \xi \in R^l,$*

A_2 *: there exist a nonnegative C^2-function V such that LV is negative for any $E_l \setminus U$.*

Then

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_l} f(x)\mu(dx)\right\} = 1$$

for all $x \in E_l$, where $f(\cdot)$ is a function integrable with respect to the measure μ.

Theorem 3.2. *Assume that $r_i > \frac{{\sigma_i}^2}{2}$, $i = 1, 2, \cdots, n$, then for any initial value $x_0 \in R^n_+$, there is a stationary distribution $\mu(.)$ for system (1.2) and it has ergodic property.*

Proof. For any $p \in (0,1)$, $\theta \in (0,1)$, define a nonnegative C^2- function V by

$$V(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n}\left({x_i}^p + \frac{1}{{x_i}^\theta}\right).$$

Denote

$$V_1 = \sum_{i=1}^{n} {x_i}^p, \quad V_2 = \sum_{i=1}^{n}\frac{1}{{x_i}^\theta}.$$

Applying Itô's formula, we get

$$\begin{aligned}
LV_1 &= p\sum_{i=1}^{n}{x_i}^p\left(r_i + \sum_{j=1}^{n}a_{ij}x_j - \frac{1-p}{2}(\sigma_i x_i)^2\right)\\
&\leq p\sum_{i=1}^{n}{x_i}^p\left(r_i + \sum_{j=1}^{n}a_{ij}x_j\right) - \frac{p(1-p)}{2}\sum_{i=1}^{n}\sigma_i^2{x_i}^{2+p}\\
&\leq p\sum_{i=1}^{n}\left(r_i{x_i}^p + \sum_{j=1}^{n}\frac{a_{ij}}{2}({x_i}^{2p} + {x_j}^2)\right) - \frac{p(1-p)}{2}\sum_{i=1}^{n}\sigma_i^2{x_i}^{2+p}\\
&\leq p\sum_{i=1}^{n}\left(r_i{x_i}^p + \sum_{j=1}^{n}\frac{a_{ij}}{2}{x_i}^{2p} + \sum_{j=1}^{n}\frac{a_{ji}}{2}{x_i}^2\right) - \frac{p(1-p)}{2}\sum_{i=1}^{n}\sigma_i^2{x_i}^{2+p}\\
&\leq M - \frac{p(1-p)}{4}\sum_{i=1}^{n}\sigma_i^2{x_i}^{2+p},
\end{aligned} \tag{3.2}$$

where $M = \sup_{x\in R^n_+}\{p\sum_{i=1}^{n}\left(r_i{x_i}^p + \sum_{j=1}^{n}\frac{a_{ij}}{2}{x_i}^{2p} + \sum_{j=1}^{n}\frac{a_{ji}}{2}{x_i}^2\right) - \frac{p(1-p)}{4}\sum_{i=1}^{n}\sigma_i^2{x_i}^{2+p}\} < \infty$.

$$\begin{aligned}
LV_2 &= -\theta\sum_{i=1}^{n}{x_i}^{-\theta}\left(r_i + \sum_{j=1}^{n}a_{ij}x_j - \frac{\theta+1}{2}\sigma_i^2{x_i}^2\right)\\
&\leq -\theta\sum_{i=1}^{n}{x_i}^{-\theta}\left(r_i + a_{ii}x_i - \frac{\theta+1}{2}\sigma_i^2{x_i}^2\right).
\end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned} LV &= LV_1 + LV_2 \\ &\leq M - \frac{p(1-p)}{4} \sum_{i=1}^{n} \sigma_i^2 {x_i}^{2+p} - \theta \sum_{i=1}^{n} {x_i}^{-\theta} \left(r_i + a_{ii} x_i - \frac{\theta+1}{2} \sigma_i^2 {x_i}^2 \right). \end{aligned} \tag{3.4}$$

Define

$$U = \{(x_1, x_2, , \cdots, x_n) \in R_+^n, \epsilon \leq x_i \leq \frac{1}{\epsilon}\},$$

where ϵ is sufficiently small number.
Now, for any fixed $m(1 \leq m \leq n)$, if $0 < x_m < \epsilon$, we have

$$LV \leq M - \frac{p(1-p)}{4} \sum_{i=1}^{n} \sigma_i^2 {x_i}^{2+p} - \theta r_m {x_m}^{-\theta} - \theta \sum_{i=1}^{n} {x_i}^{-\theta} \left(a_{ii} x_i - \frac{\theta+1}{2} \sigma_i^2 {x_i}^2 \right).$$

Noting that $M - \frac{p(1-p)}{4} \sum\limits_{i=1}^{n} \sigma_i^2 {x_i}^{2+p} - \theta \sum\limits_{i=1}^{n} {x_i}^{-\theta} \left(a_{ii} x_i - \frac{\theta+1}{2} \sigma_i^2 {x_i}^2 \right)$ is bounded, we obtain

$$LV \leq M_1 - \theta r_m \epsilon^{-\theta}. \tag{3.5}$$

If $x_m > \frac{1}{\epsilon}$, we get from (3.4) that

$$\begin{aligned} LV &\leq M - \frac{p(1-p)}{8} \sum_{i=1}^{n} \sigma_i^2 {x_i}^{2+p} - \theta \sum_{i=1}^{n} {x_i}^{-\theta} \left(r_i + a_{ii} x_i - \frac{\theta+1}{2} \sigma_i^2 {x_i}^2 \right) - \frac{p(1-p)}{8} \sigma_m^2 {x_m}^{2+p} \\ &\leq M_2 - \frac{p(1-p)}{8} \sigma_m^2 \frac{1}{\epsilon^{2+p}}, \end{aligned} \tag{3.6}$$

where $M_2 = \sup\limits_{x \in R_+^n} \{M - \frac{p(1-p)}{8} \sum\limits_{i=1}^{n} \sigma_i^2 {x_i}^{2+p} - \theta \sum\limits_{i=1}^{n} {x_i}^{-\theta} \left(r_i + a_{ii} x_i - \frac{\theta+1}{2} \sigma_i^2 {x_i}^2 \right)\} < \infty$.

Now, we can choose ϵ sufficiently small such that

$$M_1 - \theta r_m \epsilon^{-\theta} < -1$$

and

$$M_2 - \frac{p(1-p)}{8} \sigma_m^2 \frac{1}{\epsilon^{2+p}} < -1,$$

which together with (3.5) and (3.6) yields that

$$LV < -1$$

for any $x \in R_+^n \backslash U$. Hence condition A_2 in Theorem 3.1 is satisfied. Besides, the diffusion matrix of Equation (1.2) is

$$\Lambda = \operatorname{diag}\{\sigma_1^2 {x_1}^4, \sigma_2^2 {x_2}^4, \cdots, \sigma_n^2 {x_n}^4\}.$$

Choosing $M = \min\{\sigma_1^2 {x_1}^4, \sigma_2^2 {x_2}^4, \cdots, \sigma_n^2 {x_n}^4, (x_1, x_2, \cdots, x_n) \in \overline{U}\}$, we get

$$\sum_{i,j=1}^{n} \lambda_{ij}(x) \xi_i \xi_j = \sum_{i=1}^{n} \sigma_i^2 {x_i}^4 {\xi_i}^2 \geq M |\xi|^2.$$

Hence condition A_1 is satisfied. According to Theorem 3.1, the desired results can be obtained. □

4. Simulations

In this section, we will consider 2-species mutualism system with stochastic perturbation which can be represented by

$$\begin{cases} dx_1(t) = x_1(t)[(r_1 - a_{11}x_1(t) + a_{12}x_2(t))dt + \sigma_1 x_1(t)dB_1(t)], \\ dx_2(t) = x_2(t)[(r_2 + a_{21}x_1(t) - a_{22}x_2(t))dt + \sigma_2 x_2(t)dB_2(t)]. \end{cases} \tag{4.1}$$

Using the Milstein method mentioned in [5], we get the corresponding discretization equation:

$$\begin{cases} x_{1,k+1} = x_{1,k} + x_{1,k}[(r_1 - a_{11}x_{1,k} + a_{12}x_{2,k})\Delta t + \sigma_1 x_{1,k}\epsilon_{1,k}\sqrt{\Delta t} + \sigma_1^2 x_{1,k}^2(\epsilon_{1,k}^2\Delta t - \Delta t)], \\ x_{2,k+1} = x_{2,k} + x_{2,k}[(r_2 + a_{21}x_{1,k} - a_{22}x_{2,k})\Delta t + \sigma_2 x_{2,k}\epsilon_{2,k}\sqrt{\Delta t} + \sigma_2^2 x_{2,k}^2(\epsilon_{2,k}^2\Delta t - \Delta t)]. \end{cases}$$

Let $r_1 = 0.7, r_2 = 0.7, a_{11} = 0.6, a_{12} = 0.2, a_{21}(t) = 0.3, a_{22}(t) = 0.8, x_1(0) = 1.0, x_2(0) = 1.5$.

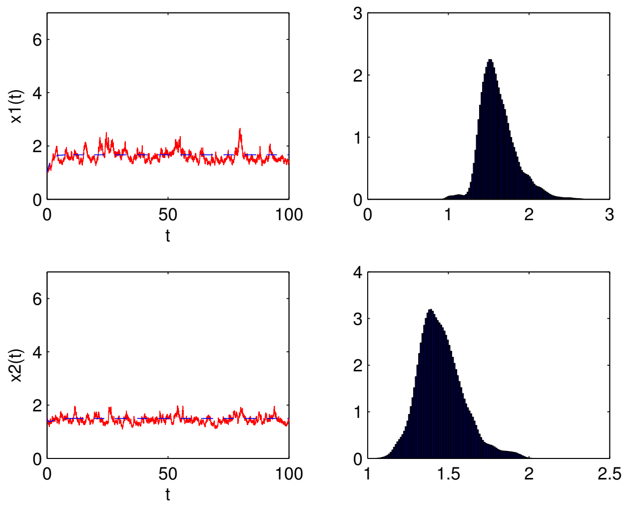

Figure 1: Solution $(x_1(t), x_2(t))$ for system (4.1) compared to the deterministic system with $\sigma_1 = 0.1, \sigma_2 = 0.1$ and its corresponding histogram.

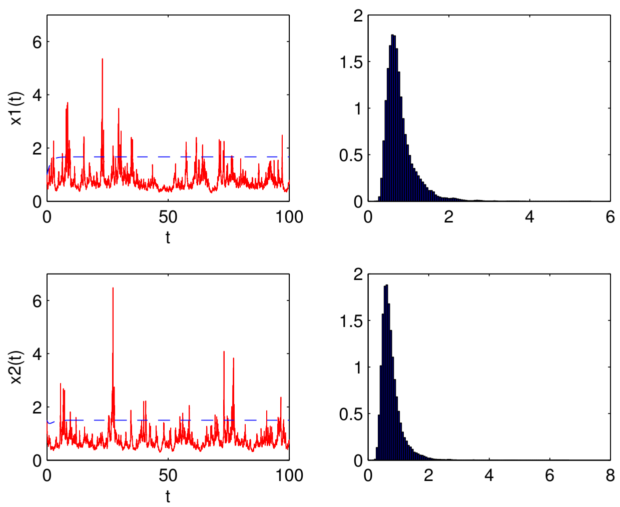

Figure 2: Solution $(x_1(t), x_2(t))$ for system (4.1) compared to the deterministic system with $\sigma_1 = 0.85, \sigma_2 = 0.9$ and its corresponding histogram.

In Figure 1, we choose $\sigma_1 = 0.1, \sigma_2 = 0.1$, According to Theorem 3.2, we can conclude that there is a stationary distribution for system (4.1). Figure 1 shows the histogram of the approximate stationary distribution of system (4.1).

In Figure 2, we choose the same parameters as in Figure 1, but increase the intensities of the white noise($\sigma_1 = 0.85, \sigma_2 = 0.9$), Theorem 3.2 is still satisfied. We can see that with the increasing of the white noise, the zone which the solution is fluctuating in is getting large compared to the earlier case. However, there is still a stationary distribution for system (4.1) (see the histogram on the right in Figure 2).

5. Conclusion

In this paper, we have considered a stochastic mutualism system. By constructing suitable Lyapunov functions, we have shown that the system has a stationary distribution with no parametric restriction. The result is very interesting, since the existence of a stationary distribution is independent both of the system parameters and of the initial value. It is also independent of the noise intensity as long as the noise intensity $\sigma_i^2 > 0$. Usually, for the stochastic system, the strong white noise may make the system to be extinct, however, the new stochastic system does not. On the other hand, the stability of positive equilibrium of the corresponding deterministic system (1.1) need some restrictions on the parameters (see [2]), but the stochastic system (1.2) has a stationary distribution with no parametric restriction if $\sigma_i^2 > 0$. The existence of a stationary distribution means stochastic stability to some extent, namely that noise is helpful for the stability of the population system.

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