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Generalized mixed equilibrium and fixed point problems in a Banach space

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Abstract

In this paper, a quasi-ϕ-nonexpansive mapping and a generalized mixed equilibrium problem are investigated. A strong convergence theorem of common solutions is established in a non-uniformly convex Banach space. The results presented in the paper improve and extend some recent results.

Keywords: Algorithm, equilibrium problem, quasi-ϕ-nonexpansive mapping, nonexpansive mapping, fixed point.
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1. Introduction and Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let S_E be the unit sphere of E. Recall that E is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in S_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2\delta.$$

E is said to be a strictly convex space if and only if $\|x + y\| < 2$ for all $x, y \in S_E$ and $x \neq y$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that E is said to have a Gâteaux differentiable norm if and only if

$$\lim_{t \to 0} \frac{\|x\| - \|x + ty\|}{t}$$

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exists for each $x, y \in S_E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S_E$, the limit is attained uniformly for all $x \in S_E$. E is also said to have a uniformly Fréchet differentiable norm if and only if the above limit is attained uniformly for $x, y \in S_E$. In this case, we say that E is uniformly smooth. It is known that a uniformly smooth Banach space is reflexive and smooth.

Recall that E is said to have the KKP if $\lim_{m\to\infty} \|x_m - x\| = 0$, for any sequence $\{x_m\} \subset E$, and $x \in E$ with $\{x_m\}$ converges weakly to x, and $\{\|x_m\|\}$ converges strongly to $\|x\|$. It is known that every uniformly convex Banach space has the KKP; see [11] and the references therein.

Recall that normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x, y\rangle = \|y\|^2\}.$$

It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a smooth Banach space, then J is single-valued and demi-continuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Let C be nonempty convex and closed subset of E. Let $B : C \times C \to \mathbb{R}$ be a bifunction, $Y : C \to \mathbb{R}$ be a real valued function and $S : C \to E^*$ be a nonlinear mapping. Consider that the following generalized mixed equilibrium problem is to find $\bar{x} \in C$ such that

$$B(\bar{x}, x) + \langle S\bar{x}, x - \bar{x}\rangle + Yx - Y\bar{x} \geq 0, \forall x \in C. \tag{1.1}$$

The solution set of the generalized mixed equilibrium problem is denoted by $Sol(B, S, Y)$.

The generalized mixed equilibrium problem, which finds a lot of applications in physics, economics, finance, transportation, network and structural analysis, elasticity and optimization, provides a natural, novel and unified framework to study fixed point problems, variational inequality, complementarity problems, and optimization problems; see [2], [12], [13], [19], [18], [20] and the references therein.

If $S = 0$, then the generalized mixed equilibrium problem is reduced to the following mixed equilibrium problem: find $\bar{x} \in C$ such that

$$B(\bar{x}, x) + Yx - Y\bar{x} \geq 0, \forall x \in C. \tag{1.2}$$

The solution set of the mixed equilibrium problem is denoted by $Sol(B, Y)$.

If $B = 0$, then the generalized mixed equilibrium problem is reduced to the following mixed variational inequality of Browder type: find $\bar{x} \in C$ such that

$$\langle S\bar{x}, x - \bar{x}\rangle + Yx - Y\bar{x} \geq 0, \forall x \in C. \tag{1.3}$$

The solution set of the mixed equilibrium problem is denoted by $VI(C, B, Y)$.

If $Y = 0$, then the generalized mixed equilibrium problem is reduced to the following generalized equilibrium problem: find $\bar{x} \in C$ such that

$$B(\bar{x}, x) + \langle S\bar{x}, x - \bar{x}\rangle \geq 0, \forall x \in C. \tag{1.4}$$

The solution set of the generalized equilibrium problem is denoted by $Sol(B, S)$.

If $S = 0$ and $Y = 0$, then the generalized mixed equilibrium problem is reduced to the following equilibrium problem in the terminology of Blum and Oettli [4]: find $\bar{x} \in C$ such that

$$B(\bar{x}, x) \geq 0, \forall x \in C. \tag{1.5}$$

The solution set of the equilibrium problem is denoted by $Sol(B)$.

The following restrictions on bifunction B are essential in this paper.

(R-1) $B(a, a) \equiv 0, \forall a \in C$;
(R-2) $B(b, a) + B(a, b) \le 0, \forall a, b \in C$;
(R-3) $B(a, b) \ge \limsup_{t\downarrow 0} B(tc + (1 - t)a, b)$, $\forall a, b, c \in C$;
(R-4) $b \mapsto B(a, b)$ is convex and weakly lower semi-continuous, $\forall a \in C$.

Recently, the above nonlinear problems have been extensively studied based on iterative techniques; see [3], [6]-[10], [14]–[17], [19], [22]-[26] and the references therein. In this paper, we study generalized mixed equilibrium problem (1.1) based on a monotone projection technique without any compactness assumption. Let T be a mapping on C. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} Tx_n = y'$, then $Tx' = y'$. From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively. Recall that a point p is said to be a fixed point of T if and only if $p = Tp$. p is said to be an asymptotic fixed point of T if and only if C contains a sequence $\{x_n\}$, where $x_n \rightharpoonup p$ such that $x_n - Tx_n \to 0$. From now on, We use $Fix(T)$ to stand for the fixed point set and $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set.

Next, we assume that E is a smooth Banach space which means mapping J is single-valued. Study the functional

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists an unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [1], Alber studied a new mapping $Proj_C$ in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$.

Recall that T is said to be relatively nonexpansive [5] if $Fix(T) = \widetilde{Fix}(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

T is said to be quasi-ϕ-nonexpansive [17] if $Fix(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

Remark 1.1. The class of quasi-ϕ-nonexpansive mappings is more desirable than the class of relatively nonexpansive mappings because of strong restriction $Fix(T) = \widetilde{Fix}(T)$.

Remark 1.2. The class of quasi-ϕ-nonexpansive mappings is reduced to the class of quasi-nonexpansive mappings in the framework of Hilbert spaces.

The following lemmas also play an important role in this paper.

Lemma 1.3 ([21])**.** *Let r be a positive real number and let E be uniformly convex. Then there exists a convex, strictly increasing and continuous function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|(1 - t)b + ta\|^2 + t(1 - t)g(\|b - a\|) \le t\|a\|^2 + (1 - t)\|b\|^2$$

for all $a, b \in B^r := \{a \in E : \|a\| \le r\}$ and $t \in [0, 1]$.

Lemma 1.4 ([1])**.** *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \le \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

and $x_0 = \Pi_C x$ if and only if

$$\langle y - x_0, Jx - Jx_0\rangle \le 0, \forall y \in C.$$

Lemma 1.5 ([18])**.** *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let T be a closed quasi-ϕ-nonexpansive mappings on C. Then $F(T)$ is closed and convex.*

Lemma 1.6 ([4], [17])**.** *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a function with restrictions (R-1), (R-2), (R-3) and (R-4), from $C \times C$ to $\mathbb{R}$. Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that*

$$rB(z, y) + \langle z - y, Jz - Jx\rangle \leq 0, \forall y \in C.$$

Define a mapping $C^{B,r}$ by

$$C^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$C^{B,r}$ is single-valued quasi-ϕ-nonexpansive;*

(2) *$Sol(B) = Fix(C^{B,r})$ is closed and convex.*

2. Main results

We are now in a position to state our main results.

Theorem 2.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $S : C \to E^*$ be a continuous and monotone mapping and let $Y : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let T be a quasi-ϕ-nonexpansive mappings on C. Assume that $Sol(B, S, Y) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj_{C_1} x_0, \\ r_n B(z_n, z) + r_n(Yz - Yz_n) + r_n\langle Sz_n, z - z_n\rangle \geq \langle z_n - z, Jz_n - Jx_n\rangle, \forall z \in C_n, \\ Jy_n = \alpha_n JTx_n + (1 - \alpha_n)Jz_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to a special common solution $\bar{x}$, where $\bar{x} = Proj_{Sol(B,S,Y)\cap Fix(T)} x_1$.

Proof. Define

$$G(a, b) = B(a, b) + \langle Sa, b - a\rangle + Yb - Ya, \forall a, b \in C.$$

Next, we prove that bifunction G satisfies (R-1), (R-2), (R-3) and (R-4). Therefore, the generalized mixed equilibrium problem is equivalent to the following equilibrium problem: find $a \in C$ such that $G(a, b) \geq 0$, $\forall b \in C$. First, we prove G is monotone. Since S is a continuous and monotone operator, we find from the definition of G that

$$\begin{aligned} G(b, c) + G(c, b) &= B(b, c) + \langle Sb, c - b\rangle + Yc - Yb + B(c, b) \\ &\quad + \langle Sc, b - c\rangle + Yb - Yc \\ &= B(c, b) + \langle Sc, b - c\rangle + B(b, c) + \langle Sb, c - b\rangle \\ &\leq \langle Sc - Sb, b - c\rangle \leq 0. \end{aligned}$$

It is clear that G satisfies (R-2). Next, we show that for each $a \in C$, $b \longmapsto G(a, b)$ is a convex and lower semicontinuous. For each $a \in C$, for all $t \in (0, 1)$ and for all $b, c \in C$, since Y is convex, we have

$$
\begin{aligned}
&G(a, tb + (1-t)c) \\
&= B(a, tb + (1-t)c) + \langle Sa, tb + (1-t)c - a\rangle + Y(tb + (1-t)c) - Ya \\
&\leq t\big(B(a,b) + Yb - Ya + \langle Sa, b - a\rangle\big) \\
&\quad + (1-t)\big(B(a,c) + Yc - Ya + \langle Sa, c - a\rangle\big) \\
&= (1-t)G(a,c) + tG(a,b).
\end{aligned}
$$

So, $b \longmapsto G(a,b)$ is convex. Similarly, we find that $b \longmapsto G(a,b)$ is also lower semicontinuous. Since S is continuous and Y is lower semicontinuous, we have

$$
\begin{aligned}
\limsup_{t\downarrow 0} G(tc + (1-t)a, b) &= \limsup_{t\downarrow 0} B(tc + (1-t)a, b) \\
&\quad + \limsup_{t\downarrow 0} \big(Yb - Y(tc + (1-t)a)\big) \\
&\quad + \limsup_{t\downarrow 0} \langle S\big(tc + (1-t)a\big), b - \big(tc + (1-t)a\big)\rangle \\
&\leq B(a,b) + Yb - Ya + \langle Sa, b - a\rangle \\
&= G(a,b).
\end{aligned}
$$

Using Lemma 1.6, one sees that $Sol(G) = Sol(B, S, Y)$ is closed and convex. Using Lemma 1.5, one sees that $Fix(T)$ is also convex and closed. Hence, $Sol(B, S, Y) \cap Fix(T)$ is convex and closed.

We are now in a position to show that C_n is convex and closed. It is obvious that $C_1 = C$ is convex and closed. Assume that C_i is convex and closed for some $i \geq 1$. Let $p_1, p_2 \in C_{i+1}$. It follows that $p = sp_1 + (1-s)p_2 \in C_i$, where $s \in (0,1)$. Since

$$\phi(p_1, y_i) \leq \phi(p_1, x_i),$$

and

$$\phi(p_2, y_i) \leq \phi(p_2, x_i),$$

one has

$$2\langle p_1, Jx_i - Jy_i\rangle \leq \|x_i\|^2 - \|y_i\|^2$$

and

$$2\langle p_2, Jx_i - Jy_i\rangle \leq \|x_i\|^2 - \|y_i\|^2.$$

Using the above two inequalities, one has $\phi(p, y_i) \leq \phi(p, x_i)$. This shows that C_{i+1} is closed and convex. Hence, C_n is a convex and closed set.

Next, one proves $Fix(T) \cap Sol(B, S, Y) \subset C_n$. It is obvious $Fix(T) \cap Sol(B, S, Y) \subset C_1 = C$. Suppose that $Fix(T) \cap Sol(B, S, Y) \subset C_i$ for some positive integer i. For any $z \in Fix(T) \cap Sol(B) \subset C_i$, we see that

$$
\begin{aligned}
\phi(z, y_i) &= \|z\|^2 + \|\alpha_i JTx_i + (1-\alpha_i)Jz_i\|^2 \\
&\quad - 2\langle z, \alpha_i JTx_i + (1-\alpha_i)Jz_i\rangle \\
&\leq \|z\|^2 + \alpha_i\|Tx_i\|^2 + (1-\alpha_i)\|Jz_i\|^2 \\
&\quad - 2(1-\alpha_i)\langle z, Jz_i\rangle - 2\alpha_i\langle z, JTx_i\rangle \\
&\leq \alpha_i\phi(z, Tx_i) + (1-\alpha_i)\phi(z, C^{G,r_i}x_i) \\
&\leq \phi(z, x_i),
\end{aligned}
$$

where

$$C^{G,r_i}x = \{z \in C : r_iG(z,y) + \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C_i\}.$$

This shows that $z \in C_{i+1}$. This implies that $Fix(T) \cap Sol(B, S, Y) \subset C_n$. Using Lemma 1.4, we find

$$\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0, \forall z \in C_n.$$

It follows that

$$\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0, \quad \forall z \in Fix(T) \cap Sol(B, S, Y) \subset C_n.$$

Using Lemma 1.4, one has

$$\begin{aligned}\phi(x_n, x_1) &\leq \phi(Proj_{Fix(T)\cap Sol(B,S,Y)}x_1, x_1) - \phi(Proj_{Fix(T)\cap Sol(B,S,Y)}x_1, x_n)\\ &\leq \phi(Proj_{Fix(T)\cap Sol(B)}x_1, x_1),\end{aligned}$$

which shows that $\{\phi(x_n, x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Without loss of generality, we assume $x_n \rightharpoonup \bar{x}$. Since every C_n is convex and closed. So $\bar{x} \in C_n$. Since $\bar{x} \in C_n$, one has $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. This implies that

$$\begin{aligned}\phi(\bar{x}, x_1) &\leq \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle)\\ &= \liminf_{n\to\infty} \phi(x_n, x_1)\\ &\leq \limsup_{n\to\infty} \phi(x_n, x_1)\\ &\leq \phi(\bar{x}, x_1).\end{aligned}$$

Hence, one has $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. It follows that $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. Using the KKP, one obtains that $\{x_n\}$ converges strongly to $\bar{x}$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we find that $\phi(x_{n+1}, x_1) \geq \phi(x_n, x_1)$, which shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows that $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Since

$$\phi(x_{n+1}, x_1) - \phi(x_n, x_1) \geq \phi(x_{n+1}, x_n) \geq 0,$$

one has $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Using the fact $x_{n+1} \in C_{n+1}$, one sees

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Therefore, one has $\lim_{n\to\infty}(\|y_n\| - \|x_{n+1}\|) = 0$. This implies that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Jy_n\}$ is bounded. Without loss of generality, we assume that $\{Jy_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n\rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2\\ &= \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy\rangle\\ &= \phi(\bar{x}, y)\\ &\geq 0.\end{aligned}$$

That is, $\bar{x} = y$, which in turn implies that $J\bar{x} = y^*$. Hence, $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E is uniformly smooth, hence, E^* is uniformly convex and it has the KKP, we obtain $\lim_{n\to\infty} Jy_n = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demi-continuous and E has the KKP, one gets that $y_n \to \bar{x}$, as $n \to \infty$.

On the other hand, we find from Lemma 1.3 that

$$\begin{aligned}\phi(z, y_n) &\leq \|z\|^2 + \alpha_n\|Tx_n\|^2 + (1 - \alpha_n)\|Jz_n\|^2\\ &\quad - 2(1 - \alpha_n)\langle z, Jz_n\rangle - 2\alpha_n\langle z, JTx_n\rangle\\ &\quad - \alpha_n(1 - \alpha_n)g(\|JTx_n - Jz_n\|)\\ &\leq \alpha_n\phi(z, Tx_n) + (1 - \alpha_n)\phi(z, C^{G,r_n}x_n)\\ &\quad - \alpha_n(1 - \alpha_n)g(\|JTx_n - Jz_n\|)\\ &\leq \phi(z, x_n) - \alpha_n(1 - \alpha_n)g(\|JTx_n - Jz_n\|).\end{aligned}$$

Since

$$\phi(z,x_n)-\phi(z,y_n)\le(\|x_n\|+\|y_n\|)\|y_n-x_n\|+2\langle z,Jy_n-Jx_n\rangle,$$

we find

$$\lim_{n\to\infty}\big(\phi(z,x_n)-\phi(z,y_n)\big)=0,\quad \forall z\in Fix(T)\cap Sol(B).$$

This implies $\lim_{n\to\infty}\|Jz_n-JTx_n\|=0$. Hence, one has $JTx_n\to J\bar{x}$ as $n\to\infty$. Since $J^{-1}:E^*\to E$ is demi-continuous, one has $Tx_n\rightharpoonup\bar{x}$. Using the fact

$$|\|Tx_n\|-\|\bar{x}\||=|\|JTx_n\|-\|J\bar{x}\||\le\|JTx_n-J\bar{x}\|,$$

one has $\|Tx_n\|\to\|\bar{x}\|$ as $n\to\infty$. Since E has the KKP, one has $\lim_{n\to\infty}\|\bar{x}-Tx_n\|=0$. Using the closedness of T, we find $T\bar{x}=\bar{x}$. This proves $\bar{x}\in Fix(T)$. Since $\{z_n\}$ converges strongly to $\bar{x}$ and G is a monotone bifunction, one has $r_nG(z,z_n)\le\|z-z_n\|\|Jz_n-Jx_n\|$. Since $\liminf_{n\to\infty}r_n>0$, we may assume there exists $\mu>0$ such that $r_n\ge\mu$. It follows that

$$G(z,z_n)\le\|z-z_n\|\frac{\|Jz_n-Jx_n\|}{\mu}.$$

Hence, one has $G(z,\bar{x})\le 0$. For $0<s<1$, define $z^s=(1-s)\bar{x}+sz$. This implies that $0\ge G(z^s,\bar{x})$. Hence, we have

$$0=G(z^s,z^s)\le sB(z^s,z).$$

It follows that $G(\bar{x},z)\ge 0$, $\forall z\in C$. This implies that $\bar{x}\in Sol(G)=Sol(B,S,Y)$. Using Lemma 1.4, we find

$$\langle x_n-z,Jx_1-Jx_n\rangle\ge 0,\forall z\in Fix(T)\cap Sol(B,S,Y).$$

Let $n\to\infty$, one has $\langle\bar{x}-z,Jx_1-J\bar{x}\rangle\ge 0$. It follows that $\bar{x}=Proj_{Fix(T)\cap Sol(B,S,Y)}x_1$. This completes the proof. □

Remark 2.2. Theorem 2.1 mainly improve the corresponding results in [14], [15], [17] and [18]. The framework of the space is weak which do not require the uniform convexness.

In the framework of Hilbert spaces, we have the following result.

Theorem 2.3. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $S:C\to E$ be a continuous and monotone mapping and let $Y:C\to\mathbb{R}$ be a lower semi-continuous and convex function. Let T be a quasi-nonexpansive mappings on C. Assume that $Sol(B,S,Y)\cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in (0,1) such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0\in E \text{ chosen arbitrarily},\\ C_1=C,\\ x_1=P_{C_1}x_0,\\ r_nB(z_n,z)+r_n(Yz-Yz_n)+r_n\langle Sz_n,z-z_n\rangle\ge\langle z_n-z,z_n-x_n\rangle,\forall z\in C_n,\\ y_n=\alpha_nTx_n+(1-\alpha_n)z_n,\\ C_{n+1}=\{z\in C_n:\|z-x_n\|\ge\|z-y_n\|\},\\ x_{n+1}=P_{C_{n+1}}x_1,\end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty}r_n>0$. Then $\{x_n\}$ converges strongly to a special common solution $\bar{x}$, where $\bar{x}=Proj_{Sol(B,S,Y)\cap Fix(T)}x_1$.

Proof. The generalized projection is reduced to the metric projection and the class of quasi-ϕ-nonexpansive mappings is reduced to the class of quasi-nonexpansive mappings. Using Theorem 2.1, we find the following results. □

From Theorem 2.1, we also have the following result on generalized equilibrium problem (1.4).

Corollary 2.4. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $S : C \to E^*$ be a continuous and monotone mapping and let T be a quasi-ϕ-nonexpansive mappings on C. Assume that $Sol(B,S) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in (0,1) such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj_{C_1} x_0, \\ r_n B(z_n, z) + r_n\langle Sz_n, z - z_n\rangle \geq \langle z_n - z, Jz_n - Jx_n\rangle, \forall z \in C_n, \\ Jy_n = \alpha_n JTx_n + (1-\alpha_n)Jz_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. *Then $\{x_n\}$ converges strongly to a special common solution $\bar{x}$, where $\bar{x} = Proj_{Sol(B,S)\cap Fix(T)} x_1$.*

From Theorem 2.1, we also have the following result on mixed equilibrium problem (1.2).

Corollary 2.5. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $Y : C \to \mathbb{R}$ be a lower semi-continuous and convex function and let T be a quasi-ϕ-nonexpansive mappings on C. Assume that $Sol(B,Y) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in (0,1) such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj_{C_1} x_0, \\ r_n B(z_n, z) + r_n(Yz - Yz_n) \geq \langle z_n - z, Jz_n - Jx_n\rangle, \forall z \in C_n, \\ Jy_n = \alpha_n JTx_n + (1-\alpha_n)Jz_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. *Then $\{x_n\}$ converges strongly to a special common solution $\bar{x}$, where $\bar{x} = Proj_{Sol(B,Y)\cap Fix(T)} x_1$.*

Finally, we give a result on equilibrium problem (1.5).

Corollary 2.6. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be a quasi-ϕ-nonexpansive mappings on C. Assume that $Sol(B) \cap Fix(T)$ is nonempty and T is closed. Let $\{\alpha_n\}$ be real sequence in (0,1) such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj_{C_1} x_0, \\ r_n B(z_n, z) \geq \langle z_n - z, Jz_n - Jx_n\rangle, \forall z \in C_n, \\ Jy_n = \alpha_n JTx_n + (1-\alpha_n)Jz_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ *is a real sequence such that* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to a special common solution* $\bar{x}$, *where* $\bar{x} = Proj_{Sol(B)\cap Fix(T)}x_1$.

Remark 2.7. Corollary 2.5 and Corollary 2.6 mainly improve the corresponding results in [22]. We relax the uniform convexness and the class of relatively nonexpansive mappings is also improved to the class of quasi-ϕ-nonexpansive mappings.

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