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Finite time blow up of solutions to an inverse problem for a quasilinear parabolic equation with power nonlinearity

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Abstract

We consider an inverse problem for quasilinear parabolic equations with type power nonlinearity. Sufficient conditions on initial data for blow up result are obtained with positive initial energy. Overdetermination condition is given as an integral form. To get the blow up result for considered nonlinear inverse parabolic equation, we use the concavity of a special positive function. The life span of the solution is also computed.

Keywords: Blow-up, inverse problem, quasilinear parabolic equation.
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1. Introduction

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium, from the observation of a response of this object, or medium, to a probing signal. Thus, the theory of inverse problems yields a theoretical basis for remote sensing and nondestructive evaluation. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes, missiles, etc.), objects immersed in water (submarines, paces of fish, etc.), and in many other situations.

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In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in the material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor, or some abnormality in a human body.

We now consider the following inverse problem for a quasilinear parabolic equation

$$u_t - \nabla . \left[\left(k_1 + k_2 |\nabla u|^{m-2})\nabla u\right)\right] + h\left(u, \nabla u\right) - |u|^{p-2} u = F(t) w(x), \tag{1.1}$$

$$u\left(x,t\right) = 0, x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u\left(x,0\right) = u_0, x \in \Omega, \tag{1.3}$$

$$\int\limits_{\Omega} u\left(x,t\right) w\left(x\right) dx = 1, t > 0, \tag{1.4}$$

where $\Omega \subset R^n$, $n \geq 1$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. ρ, k_1 and k_2 are positive constants and $p > m \geq 2$. Also assume that $w(x)$ is a given function satisfying

$$\int\limits_{\Omega} w^2\left(x\right) dx = 1,\ w \in H^m\left(\Omega\right) \cap H_0^1\left(\Omega\right) \cap L^p\left(\Omega\right), m \geq 2. \tag{1.5}$$

The inverse problem consists of finding a pair of functions $\{u\left(x,t\right), F\left(t\right)\}$ satisfying (1.1)–(1.4) when

$$\int\limits_{\Omega} u_0 w dx = 1,\ u_0 \in H_0^1\left(\Omega\right) \cap L^p\left(\Omega\right) \tag{1.6}$$

and $h\left(u,\ \nabla u\right)$ is a continuous function which satisfies the relation

$$\left|h\left(u, \nabla u\right)\right| \leq K\left(|u|^{\frac{p}{2}} + |\nabla u|^{\frac{m}{2}}\right), K > 0. \tag{1.7}$$

Additional information about the solution to the inverse problem is given in the form of the integral over-determination condition (1.4). Temperature $u(x,t)$ is averaged by function w over the domain Ω [10].

Existence and uniqueness of solutions to inverse problems for parabolic equations are studied by several authors [4, 5, 7, 8, 9]. Asymptotic stability of solutions to such problems are investigated in [2, 9, 10, 11].

Global nonexistence and blow up results for nonlinear parabolic equations are discussed in [1, 3]. But less is known about inverse problem for nonlinear parabolic equations. Eden and Kalantarov [2] studied the following problem

$$u_t - \Delta u - |u|^p u + b\left(x,t,u,\nabla u\right) = F\left(t\right) w\left(x\right),\ p > 0.$$

In this work, we consider blow up results in finite time for solutions to inverse problems for nonlinear parabolic equation (1.1)–(1.4) with weight function $w(x)$. The technique of our proofs is similar to the one in [1]. In this paper, we use the following notations: $\|u\| = \|u\|_{L_2(\Omega)}$, $\|u\|_p = \|u\|_{L_p(\Omega)}$ where $L_2(\Omega)$ and $L_p(\Omega)$ are usual Lebesgue spaces, $(u,v) = \int_\Omega uvdx$ is the inner product,

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,\ \ \varepsilon > 0$$

is a form of the weighted arithmetic–geometric inequality for $a, b > 0$ and

$$ab \leq \beta a^q + C(q,\beta) b^{q'}$$

is the Young's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$, $C\left(q,\beta\right) = \frac{1}{q'(q\beta)^{q'/q}}$. The following lemma is a useful tool for obtaining blow up results for dynamical problems.

Lemma 1.1 ([6], Lemma 1). *Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies, for $t > 0$, the following inequality*

$$\psi(t)\psi''(t) - (1+\gamma)\left(\psi'(t)\right)^2 \geq -2M_1\psi(t)\psi'(t) - M_2\psi^2(t),$$

where $\gamma > 0, M_1, M_2 \geq 0$. If $\psi(0) > 0$, $\psi'(0) > -\gamma_2\gamma^{-1}\psi(0)$ and $M_1 + M_2 > 0$, then $\psi(t)$ tends to infinity as

$$t \to t_1 \leq t_2 \leq \frac{1}{2\sqrt{M_1^2 + \gamma M_2}} \ln\left(\frac{\gamma_1\psi(0) + \gamma\psi'(0)}{\gamma_2\psi(0) + \gamma\psi'(0)}\right),$$

where $\gamma_1 = -M_1 + \sqrt{M_1^2 + \gamma M_2}$, $\gamma_2 = -M_1 - \sqrt{M_1^2 + \gamma M_2}$.

2. Blow-up Result

Theorem 2.1. *Suppose that conditions* (1.3) *and* (1.4) *are satisfied. Let $u(x,t), F(t)$ be a solution to inverse problem* (1.1)–(1.4)*. Assume that the following conditions are satisfied*

$$\gamma = \sqrt{1+\beta} - 1, \beta \in (0,\alpha), \alpha = \frac{p+m-4}{8}, \lambda = \frac{K^2(m+pk_2)(1+\alpha)}{k_2(p-m)(\alpha-\beta)} \tag{2.1}$$

$$E(0) = -\frac{\lambda}{2}\|u_0\|^2 - \frac{k_1}{2}\|\nabla u_0\|^2 - \frac{k_2}{m}\|\nabla u_0\|_m^m + \frac{1}{p}\|u_0\|_p^p > 0 \tag{2.2}$$

$$4(1+2\alpha)E(0) - \frac{2\lambda(1+\gamma)^2}{\gamma}\|u_0\|^2 > D_3 \tag{2.3}$$

where

$$\begin{aligned} D_3 = &\frac{8K^2(m+pk_2)}{k_2(p-m)}\|w\|^2 + \frac{4k_2}{m}\left(\frac{8(m-1)}{(p-m)}\right)^{m-1}\|\nabla \mathrm{w}\|_m^m \\ &+ \frac{4}{p}\left(\frac{8(p-1)}{(p-m)}\right)^{p-1}\|w\|_p^p + \frac{4k_1}{p+m-4}\|\nabla w\|^2. \end{aligned} \tag{2.4}$$

Then there exists a finite time t_1 such that

$$\|u\|^2 \to +\infty \quad as \quad t \to t_1^-.$$

Proof For $\lambda > 0$, we apply the transformation $(x,t) = e^{\lambda t}v(x,t)$ in (1.1) and obtain the equation

$$v_t - \nabla.\left[(k_1 + k_2e^{\lambda(m-2)t}|\nabla v|^{m-2})\nabla v\right] + \lambda v + e^{-\lambda t}h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right) - e^{\lambda(p-2)t}|v|^{p-2}v = e^{-\lambda t}F(t)w(x) \tag{2.5}$$

with the boundary condition

$$v(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2.6}$$

the initial condition

$$v(x,0) = u_0, \quad x \in \Omega, \tag{2.7}$$

and the integral over-determination condition

$$\int_\Omega v(x,t)\,w(x)dx = e^{-\lambda t}, \quad t > 0. \tag{2.8}$$

Multiplying equation (2.5) by v_t in $L^2(\Omega)$, we get the relation

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\lambda}{2}\|v\|^2 + \frac{k_1}{2}\|\nabla v\|^2 + \frac{k_2}{m}e^{\lambda(m-2)t}\|\nabla v\|_m^m - \frac{1}{p}e^{\lambda(p-2)t}\|v\|_p^p\right] \\ &\quad - \frac{k_2\lambda(m-2)}{m}e^{\lambda(m-2)t}\|\nabla v\|_m^m + \|v_t\|^2 + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p \\ &\quad + e^{-\lambda t}\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), v_t\right) = -\lambda e^{-2\lambda t}F(t). \end{aligned} \tag{2.9}$$

Now, multiplying equation (2.5) by w in $L^2(\Omega)$ and using over-determination condition (2.8), then we obtain

$$\begin{aligned} F(t) =& k_1 e^{\lambda t}(\nabla v, \nabla w) + k_2 e^{\lambda(m-1)t}\left(|\nabla v|^{m-2}\nabla v, \nabla w\right) \\ &+ \left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), w\right) - e^{\lambda(p-1)t}\left(|v|^{p-2}v, w\right). \end{aligned} \tag{2.10}$$

Substituting equation (2.10) into equation (2.9) we get the relation

$$\begin{aligned} &-\frac{d}{dt}E(t) - \frac{\lambda k_2(m-2)}{m}e^{\lambda(m-2)t}\|\nabla v\|_m^m + \|v_t\|^2 + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p \\ &= e^{-\lambda t}\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), v_t\right) - \lambda e^{-2\lambda t}\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), w\right) \\ &\quad - \lambda k_2 e^{\lambda(m-3)t}\left(|\nabla v|^{m-2}\nabla v, \nabla w\right) + \lambda e^{\lambda(p-3)t}\left(|v|^{p-2}v, w\right) - \lambda k_1 e^{-\lambda t}(\nabla v, \nabla w), \end{aligned} \tag{2.11}$$

where

$$E(t) = \frac{1}{p}e^{\lambda(p-2)t}\|v\|_p^p - \frac{k_2}{m}e^{\lambda(m-2)t}\|\nabla v\|_m^m - \frac{\lambda}{2}\|v\|^2 - \frac{k_1}{2}\|\nabla v\|^2.$$

Use the property of the function $h(e^{\lambda t}v, e^{\lambda t}\nabla v)$ given by (1.7) and then apply the weighted arithmetic–geometric inequality to the first term on the right-hand side of (2.11) with $a = e^{\lambda(p-2)t/2}\|v\|_p^{p/2}$, $b = K\|v_t\|$, $\varepsilon = \frac{\lambda(p-m)}{4p}$ and $a = e^{\lambda(m-2)t/2}\|\nabla v\|_m^{m/2}$, $b = K\|v_t\|$, $\varepsilon = \frac{\lambda k_2(p-m)}{4m}$ to get the estimate

$$\begin{aligned} e^{-\lambda t}\left|\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), v_t\right)\right| \leq& \frac{\lambda(p-m)}{4p}e^{\lambda(p-2)t}\|v\|_p^p + \frac{\lambda k_2(p-m)}{4m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ &+ \frac{K^2(m+pk_2)}{\lambda k_2(p-m)}\|v_t\|^2. \end{aligned} \tag{2.12}$$

We can obtain a similar result for the second term on the right-hand side of (2.11) with $a = e^{\lambda(p-2)t/2}\|v\|_p^{p/2}$, $b = \lambda K\|w\|$, $\varepsilon = \frac{\lambda(p-m)}{8p}$ and $a = e^{\lambda(m-2)t/2}\|\nabla v\|_m^{m/2}$, $b = \lambda K\|w\|$, $\varepsilon = \frac{\lambda k_2(p-m)}{4m}$

$$\begin{aligned} \lambda e^{-2\lambda t}\left|\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), w\right)\right| \leq& \frac{\lambda(p-m)}{8p}e^{\lambda(p-2)t}\|v\|_p^p + \frac{\lambda k_2(p-m)}{8m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ &+ \frac{2\lambda K^2(m+pk_2)}{k_2(p-m)}e^{-2\lambda t}\|w\|^2. \end{aligned} \tag{2.13}$$

Apply Young's inequality to the third and fourth terms on the right-hand side of (2.11) with

$$e^{\frac{\lambda(m-2)(m-1)}{m}t}\|\nabla v\|_m^{m-1}, b = \lambda k_2 e^{\frac{-2\lambda t}{m}}\|\nabla w_m\|, \varepsilon = \frac{\lambda k_2(p-m)}{8m}$$

and

$$a = e^{\frac{\lambda(p-2)(p-1)t}{p}}\|v\|_p^{p-1}, \lambda e^{\frac{-2\lambda t}{p}}\|w\|_p, \varepsilon = \frac{\lambda(p-m)}{8p}$$

to get the estimates, respectively

$$\begin{aligned} \lambda k_2 e^{\lambda(m-3)t}\left|(|\nabla v|^{m-2}\nabla v, \nabla w)\right| \leq& \frac{\lambda k_2(p-m)}{8m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ &+ \frac{\lambda k_2}{m}\left(\frac{8(m-1)}{p-m}\right)^{m-1}e^{-2\lambda t}\|\nabla w\|_m^m, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \lambda e^{\lambda(p-3)t}\left|\left(|v|^{p-2}v, w\right)\right| \leq& \frac{\lambda(p-m)}{8p}e^{\lambda(p-2)t}\|v\|_p^p \\ &+ \frac{\lambda}{p}\left(\frac{8(p-1)}{p-m}\right)^{p-1}e^{-2\lambda t}\|w\|_p^p. \end{aligned} \tag{2.15}$$

The last term on the right-hand side of (2.11) can be estimated by weighted arithmetic–geometric inequality with $a = \|\nabla v\|$, $b = \lambda k_1 e^{-\lambda t} \|\nabla w\|$, $\varepsilon = \frac{\lambda k_1 (p+m-4)}{4}$

$$\lambda k_1 e^{-\lambda t} \left|(\nabla v, \nabla w)\right| \leq \frac{\lambda k_1 \left(p+m-4\right)}{4} \|\nabla v\|^2 + \frac{\lambda k_1}{p+m-4} e^{-2\lambda t} \|\nabla w\|^2. \tag{2.16}$$

Substituting (2.12)–(2.16) into equation (2.11), we get

$$\begin{aligned} \frac{d}{dt} E\left(t\right) \geq & \frac{\lambda}{2}\left(p+m-4\right) E\left(t\right) + \frac{\lambda^2}{2}\left(p+m-4\right)\|v\|^2 \\ & + \frac{k_1}{4}\left(p+m-4\right)\|\nabla v\|^2 + \left\{1 - \frac{K^2\left(m+pk_2\right)}{\lambda k_2\left(p-m\right)}\right\}\|v_t\|^2 - D_0 e^{-2\lambda t}, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} D_0 = & \frac{2\lambda K^2 \left(m+pk_2\right)}{k_2\left(p-m\right)}\|w\|^2 + \frac{\lambda k_2}{m}\left(\frac{8\left(m-1\right)}{p-m}\right)^{m-1} \|\nabla \mathrm{w}\|_m^m \\ & + \frac{\lambda}{p}\left(\frac{8\left(p-1\right)}{p-m}\right)^{p-1} \|w\|_p^p + \frac{\lambda k_1}{p+m-4}\|\nabla w\|^2. \end{aligned}$$

Using (2.1), we rewrite the inequality (2.17) as follows

$$\begin{aligned} \frac{d}{dt} E\left(t\right) \geq & \frac{\lambda}{2}\left(p+m-4\right) E\left(t\right) + \frac{\lambda^2}{2}\left(p+m-4\right)\|v\|^2 \\ & + \frac{\lambda k_1}{4}\left(p+m-4\right)\|\nabla v\|^2 + \left(\frac{1+\beta}{1+\alpha}\right)\|v_t\|^2 - D_0 e^{-2\lambda t}. \end{aligned} \tag{2.18}$$

Since $p+m-4>0$, the second and third terms on the right-hand side of (2.18) can be omitted to get the inequality

$$\frac{d}{dt} E\left(t\right) \geq \frac{\lambda}{2}\left(p+m-4\right) E\left(t\right) + \left(\frac{1+\beta}{1+\alpha}\right)\|v_t\|^2 - D_0 e^{-2\lambda t}. \tag{2.19}$$

Solving the differential inequality (2.19) with the estimation $1 - e^{-\frac{\lambda}{2}(p+m)t}$ by 1, we get

$$E\left(t\right) \geq \left(E\left(0\right) - D_1\right) e^{\frac{\lambda}{2}(p+m-4)t} + \left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau, \tag{2.20}$$

where $D_1 = \frac{2D_0}{\lambda(p+m)}$. It is easy to see that

$$E\left(t\right) \geq e^{\frac{\lambda}{2}(p+m-4)t}\left(E\left(0\right) - D_1\right) \geq E\left(0\right) - D_1$$

by assumption (2.4). Thus we obtain a lower bound for $E(t)$:

$$E\left(t\right) \geq \left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau + E\left(0\right) - D_1. \tag{2.21}$$

Multiplying the equation (2.5) by v in $L^2\left(\Omega\right)$ and using (2.10), we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + k_1\|\nabla v\|^2 + k_2 e^{\lambda(m-2)t}\|\nabla v\|_m^m - e^{\lambda(p-2)t}\|v\|_p^p \\ & = -e^{-\lambda t}\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), v\right) + k_2 e^{\lambda(m-3)t}(|\nabla v|^{m-2}\nabla v, \nabla w) + e^{-2\lambda t}\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), w\right) \\ & \quad - e^{(p-3)\lambda t}\left(|v|^{p-2}v, w\right) + k_1 e^{-\lambda t}\left(\nabla v, \nabla w\right). \end{aligned} \tag{2.22}$$

Recall condition (1.7) and apply the weighted arithmetic–geometric inequality to the first term on the right-hand side of the equation (2.22) with $a = e^{\frac{\lambda(p-2)}{2}t}\|v\|_p^{p/2}$, $b = K\|v\|$, $\varepsilon = \frac{p-m}{4p}$ and $a = e^{\frac{\lambda(m-2)}{2}t}\|\nabla v\|_m^{m/2}$, $b = K\|v\|$, $\varepsilon = \frac{k_2(p-m)}{4m}$ to get the estimate

$$\begin{aligned} e^{-\lambda t}\left|\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), v\right)\right| \leq & \frac{p-m}{4p}e^{\lambda(p-2)t}\|v\|_p^p + \frac{k_2(p-m)}{4m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ & + \frac{K^2(m+pk_2)}{k_2(p-m)}\|v\|^2. \end{aligned} \tag{2.23}$$

We can find a similar result for the third term on the right-hand side of (2.22) with $a = e^{\frac{\lambda(p-2)}{2}t}\|v\|_p^{p/2}$, $b = K\|w\|$, $\varepsilon = \frac{p-m}{8p}$ and $a = e^{\frac{\lambda(m-2)}{2}t}\|\nabla v\|_m^{m/2}$, $b = K\|w\|$, $\varepsilon = \frac{k_2(p-m)}{8m}$

$$\begin{aligned} e^{-2\lambda t}\left|\left(h\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right), w\right)\right| \leq & \frac{p-m}{8p}e^{\lambda(p-2)t}\|v\|_p^p + \frac{k_2(p-m)}{8m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ & + \frac{2K^2(m+pk_2)}{k_2(p-m)}e^{-2\lambda t}\|w\|^2. \end{aligned} \tag{2.24}$$

Apply Young's inequality to the second and fourth terms on the right-hand side of equation (2.22) with $a = e^{\frac{\lambda(m-2)(m-1)}{m}t}\|\nabla v\|_m^{m-1}$, $b = k_2 e^{\frac{-2\lambda}{m}t}\|\nabla w\|_m$, $\varepsilon = \frac{k_2(p-m)}{8m}$ and $a = e^{\frac{\lambda(p-2)(p-1)}{p}t}\|v\|_p^{p-1}$, $b = e^{\frac{-2\lambda}{p}t}\|w\|_p$, $\varepsilon = \frac{(p-m)}{8p}$ to get the estimates, respectively,

$$\begin{aligned} k_2 e^{\lambda(m-3)t}\left|\left(|\nabla v|^{m-2}\nabla v, \nabla w\right)\right| \leq & \frac{k_2(p-m)}{8m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ & + \frac{k_2}{m}\left(\frac{8(m-1)}{(p-m)}\right)^{m-1}e^{-2\lambda t}\|\nabla w\|_m^m, \end{aligned} \tag{2.25}$$

$$\begin{aligned} e^{\lambda(p-3)t}\left|\left(|v|^{p-2}v, w\right)\right| \leq & \frac{p-m}{8p}e^{\lambda(p-2)t}\|v\|_p^p \\ & + \frac{1}{p}\left(\frac{8(p-1)}{(p-m)}\right)^{p-1}e^{-2\lambda t}\|w\|_p^p. \end{aligned} \tag{2.26}$$

The last term on the right-hand side of equation (2.22) can be estimated by using the weighted arithmetic–geometric inequality with $a = \|\nabla v\|$, $b = k_1 e^{-\lambda t}\|\nabla w\|$, $\varepsilon = \frac{k_1(p+m-4)}{4}$

$$k_1 e^{-\lambda t}|(\nabla v, \nabla w)| \leq \frac{k_1(p+m-4)}{4}\|\nabla v\|^2 + \frac{k_1}{p+m-4}e^{-2\lambda t}\|\nabla w\|^2. \tag{2.27}$$

Substitute estimates (2.23)–(2.27) into (2.22), we obtain the following differential inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}v^2 \geq & \frac{p+m}{2p}e^{\lambda(p-2)t}\|v\|_p^p - \frac{k_2(p+m)}{2m}e^{\lambda(m-2)t}\|\nabla v\|_m^m \\ & - \left\{\lambda + \frac{K^2(m+pk_2)}{k_2(p-m)}\right\}\|v\|^2 - \frac{k_1(p+m)}{4}\|\nabla v\|^2 - D_2 e^{-2\lambda t}, \end{aligned} \tag{2.28}$$

where

$$\begin{aligned} D_2 = & \frac{2K^2(m+pk_2)}{k_2(p-m)}\|w\|^2 + \frac{k_2}{m}\left(\frac{8(m-1)}{p-m}\right)^{m-1}\|\nabla \mathrm{w}\|_m^m \\ & + \frac{1}{p}\left(\frac{8(p-1)}{p-m}\right)^{p-1}\|w\|_p^p + \frac{k_1}{p+m-4}\|\nabla w\|^2. \end{aligned}$$

Rewrite inequality (2.28) as follows

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq \frac{p+m}{2}E(t) + \left[\frac{\lambda(p+m-4)}{4} - \frac{K^2(m+pk_2)}{k_2(p-m)}\right]\|v\|^2 - D_2e^{-2\lambda t}. \tag{2.29}$$

Since $-D_2e^{-2\lambda t} \geq -D_2$ and $p+m-4>0$, we can write (2.29) as

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq \frac{p+m}{2}E(t) - \frac{K^2(m+pk_2)}{k_2(p-m)}\|v\|^2 - D_2. \tag{2.30}$$

Substituting the estimate (2.21) and $p+m=4(1+2\alpha)$ into (2.30), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \geq & 2(1+2\alpha)\left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau \\ & + 2(1+2\alpha)(E(0)-D_1) - D_2 - \frac{K^2(m+pk_2)}{k_2(p-m)}\|v\|^2.\end{aligned} \tag{2.31}$$

Since $\lambda > \frac{K^2(m+pk_2)}{k_2(p-m)}$, by assumption (2.1), then it follows from (2.31)

$$\frac{d}{dt}\|v\|^2 \geq 4(1+2\alpha)\left(\frac{1+\beta}{1+\alpha}\right)\int_0^t \|v_\tau\|^2 d\tau - 2\lambda\|v\|^2 + 4(1+2\alpha)E(0) - D_3, \tag{2.32}$$

where $D_3 = 4(1+2\alpha)D_1 + 2D_2$.

Now let us introduce the positive function

$$\psi(t) = \int_0^t \|v\|^2 d\tau + C_0, \tag{2.33}$$

where C_0 is a positive constant that will be chosen later. The first and second derivatives of (2.33) are given by

$$\psi'(t) = \|v\|^2 = 2\int_0^t (v, v_\tau)\, d\tau + \|u_0\|^2, \tag{2.34}$$

$$\psi''(t) = \frac{d}{dt}\|v\|^2. \tag{2.35}$$

Apply the Cauchy–Schwarz inequality and the weighted arithmetic–geometric inequality to get an upper bound for $\psi'(t)$:

$$\begin{aligned}\left[\psi'(t)\right]^2 &= 4\left[\int_0^t (v, v_\tau)\, d\tau + \frac{1}{2}\|u_0\|^2\right]^2 \\ &\leq 4\left[\sqrt{\int_0^t \|v\|^2 d\tau}\sqrt{\int_0^t \|v_\tau\|^2 d\tau} + \frac{1}{2}\|u_0\|^2\right]^2 \\ &\leq 4\left[(1+4\varepsilon)\left(\int_0^t \|v\|^2 d\tau\right)\left(\int_0^t \|v_\tau\|^2 d\tau\right) + \frac{1}{4}\left(1+\frac{1}{4\varepsilon}\right)\|u_0\|^4\right].\end{aligned} \tag{2.36}$$

Recalling the relations (2.33)–(2.36), we can estimate the term $\psi\psi'' - (1+\gamma)(\psi')^2$:

$$\begin{aligned}\psi\psi'' - (1+\gamma)\left(\psi'\right)^2 \geq & 4(1+\beta)\left(\int_0^t \|v_\tau\|^2 d\tau\right)\psi + \left[(p+m)E(0) - D_3\right]\psi - 2\lambda\|v\|^2\psi \\ & - 4(1+\gamma)\left[(1+4\varepsilon)\left(\int_0^t \|v\|^2 d\tau\right)\left(\int_0^t \|v_\tau\|^2 d\tau\right)\right] \\ & - (1+\gamma)\left(1+\frac{1}{4\varepsilon}\right)\|u_0\|^4. \end{aligned} \tag{2.37}$$

It also should be noted that, since β (and so γ) can be neglected, $\varepsilon > 0$ can be taken in this way followed by assumption $\max\left\{1+4\varepsilon, 1+\frac{1}{4\varepsilon}\right\} = \frac{1+\beta}{1+\gamma}$.

By assumption (2.1) and inequality (2.37), we get the estimation

$$\psi\psi'' - (1+\gamma)\left(\psi'\right)^2 \geq -2\lambda\psi\psi' + \left((p+m)E(0) - D_3\right)C_0 - (1+\gamma)^2\|u_0\|^4.$$

Now Lemma 1.1 can be applied if

$$C_0 = \frac{(1+\gamma)^2}{(p+m)E(0) - D_3}\|u_0\|^4. \tag{2.38}$$

3. Conclusion

We get the relation $\psi\psi'' - (1+\gamma)(\psi')^2 \geq -2\lambda\psi\psi'$, with $M_1 = \lambda$, $M_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = -2\lambda$. The conditions of Lemma 1, positivity of $\psi(0)$ and the condition $\psi'(0) > -\gamma_2\gamma^{-1}\psi(0)$ are satisfied by the constant (2.38) and the assumption (2.3) respectively. Thus solutions to the inverse problem for nonlinear parabolic equation (1.1)–(1.4) blow up as

$$t \to t_1 \leq \frac{1}{2\lambda} ln \frac{\gamma((p+m)E(0) - D_3)}{\gamma((p+m)E(0) - D_3) - 2\lambda(1+\gamma)^2\|u_0\|^2}.$$

As a result, we find conditions on data, guaranteeing global nonexistence of solution to an inverse source problem for a class of nonlinear parabolic equations.

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