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The existence of solution for a stochastic fourth-order parabolic equation

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Abstract

The authors consider stochastic equations of the prototype

$$du + (\gamma D^4 u - \gamma D^2 f'(u) + D^2 u - f'(u))dt - dw = 0,$$

where $\gamma > 0$ is a constant and w is a Q-Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We establish the global existence and uniqueness of the solution for this prototype in one dimension space. The random attractor is also discussed.

Keywords: Random term, stochastic fourth-order equation, global existence.
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1. Introduction

In this paper, we shall consider the existence and uniqueness of the solution to the fourth-order equation with a random term

$$du + (\gamma D^4 u - \gamma D^2 f'(u) - D^2 u + f'(u))dt - dw = 0, \tag{1.1}$$

where $\gamma > 0$ is a constant, $f'(u) = u^3 - u$ and w is a Q-Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will be mainly interested in the case that the wiener process w takes value in a Hilbert space. The noise term dw represents the thermal fluctuation.

The equation (1.1) is supplemented by the boundary value conditions

$$Du|_{x=0,1} = D^3 u|_{x=0,1} = 0, \ t > 0, \tag{1.2}$$

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and the initial value condition

$$u(x,0) = u_0(x). \tag{1.3}$$

The usual equation without random term,

$$\frac{\partial u}{\partial t} + \gamma D^4 u - \gamma D^2 f'(u) - D^2 u + f'(u) = 0 \tag{1.4}$$

is introduced as a simplification of multiple microscopic mechanisms model [10] in cluster interface evolution. Karali and Katsoulakis [10] discuss microscopic models describing pattern formation mechanisms for a prototypical model of surface processes that involves multiple microscopic mechanisms. G. Karali and Y. Nagase [11] proved the existence of the solution of this problem. By using the semigroups and the classical existence theorem of global attractors, Tang, Liu and Zhao [13] gave the existence of the global attractor in H^k $(0 \le k < 5)$ space of the equation (1.4), and it attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

During the past years, many authors have paid much attention to the stochastic partial differential equations. However, only a few works have been devoted to stochastic higher order parabolic equations. Elezović and Mikelić [8] studied the stochastic Cahn-Hilliard equation. Authors prove the existence of weak statistical solutions and it is shown that under additional conditions, the Cahn-Hilliard equation has a unique strong solution with some additional regularity properties. Da Prato and Debussche [4] proved the existence of an invariant measure for stochastic Cahn-Hilliard equation. Antonopoulou, Blömker and Karali [1] proved stochastic stability of the approximate slow manifold of solutions over a very long time scale and evaluated the noise effect (see also [2, 9, 12]). Duan and Ervin [7] studied the stochastic Kuramoto-Sivashinsky equation under the influence of white noise (see also [15]).

We study the problem (1.1)–(1.3). We first show the existence and uniqueness of local solution to the equations for $0 < t < T$ via an application of a transformation which can change the stochastic equation to a deterministic equation, and then we use the general methods to deal with the deterministic one. Secondly, we show the local solution remains bounded for any $T > 0$, that is, the existence of global solution of the equation.

The plan of the paper is as follows. In section 2, we give some lemmas, notions. The proof of the existence and uniqueness of local solution is finished in section 3. In section 4, we give the existence of global solution, the random attractor is also discussed in section 5.

2. Some Lemmas

In order to study the problem (1.1)–(1.3), we shall need the following Lemmas.

Lemma 2.1 ([5]). *Let $\mathscr{F}$ be a transformation from a Banach space $\mathscr{E}$ into $\mathscr{E}$, a is an element of $\mathscr{E}$ and $\alpha > 0$ is a positive number. If $\mathscr{F}(0) = 0$, $\|a\|_{\mathscr{E}} \le \frac{1}{2}\alpha$ and*

$$\|\mathscr{F}(z_1) - \mathscr{F}(z_2)\|_{\mathscr{E}} \le \frac{1}{2}\|z_1 - z_2\|_{\mathscr{E}} \quad \textit{for} \quad \|z_1\|_{\mathscr{E}} \le \alpha, \quad \|z_2\|_{\mathscr{E}} \le \alpha, \tag{2.1}$$

then the equation

$$z = a + \mathscr{F}(z), \quad z \in \mathscr{E} \tag{2.2}$$

has a unique solution $z \in \mathscr{E}$ satisfying $\|z\|_{\mathscr{E}} \le \alpha$.

Lemma 2.2 ([17]). *Assume that A is a negative self-adjoint operator on H and*

$$V = D((-A)^{\frac{1}{2}}) \subset H \subset V'.$$

Then A and $S(t) = e^{tA}$ have a continuous extension from V to V'. If

$$y(t) = y(t;g) = S(t)y_0 + \int_0^t e^{(t-s)A} g(s)ds, \quad t \in [0,T]$$

for $y_0 \in H$*, and* $g \in L^2(0,T;V')$*, then*

$$y \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

and for some constant $L > 0$*, independent of* $T > 0$*,*

$$\|y\|_{L^\infty(0,T;H)} + \|y\|_{L^2(0,T;V)} \leq \; L\left(\|y_0\|_H + \|g\|_{L^2(0,T;V')}\right). \tag{2.3}$$

In this article, we let

$$H := H_0^1(I), \;\; V := H_0^2(I), \;\; I = (0,1), \; E := L^6\left(0,T;W^{1,6}(I)\right).$$

Then, we have the following lemma

Lemma 2.3. *For each* $T > 0$*, we obtain*

$$L^\infty(0,T;H) \cap L^2(0,T;V) \subset E$$

and there exists a constant K*, independent of* $T > 0$*, such that*

$$\|u\|_E \leq K\left(\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)}\right), \quad u \in E. \tag{2.4}$$

Proof. From the Nirenberg inequality, we know that there exists a constant C_1 such that

$$\|u\|_{W^{1,6}(D)} \leq C_1 \|u(t)\|_{L^2(D)}^{\frac{2}{3}} \|u(t)\|_{H^2(D)}^{\frac{1}{3}}.$$

Integrating the above equation over $[0,T]$, by Young's inequality and the definitions of the norms, we deduce the result with $K = C_1/2$. □

Lemma 2.4.

$$\|S(t)y_0\|_E \leq 2KT^{\frac{1}{6}}\left(\|S(t)y_0\|^6_{L^\infty(0,T;H)} + \|S(t)y_0\|^6_{L^\infty(0,T;V)}\right)^{\frac{1}{6}}. \tag{2.5}$$

Moreover, the right-hand of (2.5) is finite.

Proof. From the Lemma 2.3, we have

$$\begin{aligned}
\int_0^T \|u\|^6_{W^{1,6}} dt &\leq C_1^6 \int_0^T \|u\|_H^4 \|u\|_V^2 dt \\
&\leq \frac{C_1^6}{2}\left(\int_0^T \|u\|_H^6 dt + \int_0^T \|u\|_V^6 dt\right) \\
&\leq \frac{C_1^6 T}{2}\left(\|u\|^6_{L^\infty(0,T;H)} + \|u\|^6_{L^\infty(0,T;V)}\right).
\end{aligned}$$

Let $u = S(t)y_0$, we deduce the result. □

3. Local existence

In this section, we are going to prove the local existence.

Theorem 3.1. *For* $u_0 \in H$*, there exists a random variable* τ *taking values* $\mathbb{P}$*-a.s.in* $(0,T]$ *such that equations* (1.1)–(1.3) *have a unique solution* u *on the interval* $[0,\tau]$*.*

In order to prove the Theorem 3.1, we need to introduce a transformation to simplify (1.1) to a deterministic equation.

We consider the self-adjoint operator A corresponding to the form

$$Au := -\gamma u_{xxxx} + u_{xx}, \tag{3.1}$$

which is a strictly negative operator on $H_0^4(I)$. $(-A)^\alpha$ with the domain $D((-A)^\alpha) = H_0^{4\alpha}$ is an operator defined via Fourier analysis (see [14]). Thus, the equation (1.1) can be written in the form

$$du = (Au + \gamma D^2 f'(u) - u^3 + u) + dw, \tag{3.2}$$

where w which takes value in the separable space $H := H_0^1(I)$ is a Wiener process with the covariance operator Q. We define

$$w_A(t) := \int_0^T S(t-s)dw(s), \tag{3.3}$$

where $S(t) := e^{tA}$, $t \geq 0$. With the substitution

$$y(t,x) := u(t,x) - w_A(t,x), \quad t \in [0,T] \ P \ a.s.. \tag{3.4}$$

Then the equation (3.2) is changed to a deterministic equation

$$y_t = Ay + \gamma D^2 f'(y + w_A) - (y + w_A)^3 + (y + w_A), \tag{3.5}$$

together with

$$y(0,x) = u_0(x) \quad \text{and} \quad Dy(t,x)|_{x=0,1} = D^3 y|_{x=0,1} = 0. \tag{3.6}$$

Definition 3.2. The function $u \in E$ is called the (mild) solution to the equation (1.1), if

$$y(t,x) := u(t,x) - w_A(t,x), \quad t \in [0,T] \ P \ a.s.,$$

and y satisfies

$$y(t) = S(t)u_0 + T(y + w_A)(t), \quad t \in [0,T], \tag{3.7}$$

where

$$T : E \to E$$

is given by

$$T(u)(t) = \int_0^T S(t-s)(\gamma D^2 f'(u(s)) - u^3(s) + u(s))ds. \tag{3.8}$$

It is necessary for us to give the rationality of the definition.

Firstly, we should notice that, when $(-A)^\beta$ takes value in $D((-A)^\beta)$ for $0 \leq \beta < \frac{1}{2}$ and Hölder exponent smaller than $\frac{1}{2} - \beta$, there exists a version of $w_A(t)$ is Hölder continuous (see [5]). By Lemma 2.3, we know that w_A has a continuous version in $W^{1,6}(I)$. Secondly, we define

$$T_0 : C^1([0,T]; V) \to E$$

by

$$T_0(u)(t) = \int_0^T S(t-s)(K_0 u)(s)ds, \quad t \in [0,T], \tag{3.9}$$

where

$$K_0 : C^1([0,T]; V) \to E$$

is given by

$$K_0(u)(t) = \gamma D^2 f'(u(t)) - u^3(t) + u(t), \quad t \in [0,T]. \tag{3.10}$$

Thus, it is easy to know that the solution of (3.5) can written as follows

$$y(t) = S(t)u_0 + T_0(y + w_A)(t), \quad t \in [0,T]. \tag{3.11}$$

We say that the Definition 3.2 is a extension of (3.11). In order to explain it, we need next lemma.

Lemma 3.3. *The operator K_0 defined by (3.10) can be continuously extended to*

$$K := E \to L^2(0,T;V')$$

and satisfies

$$\|K(u)-K(v)\|_{L^2(0,T;V')} \leq M_1(\|u\|_E^2 + \|v\|_E^2 + T^{\frac{1}{3}})\|u-v\|_E, \quad u,v \in E.$$

Proof. Let $u, v, \psi \in L^2(0,T;V)$, and $\langle\cdot,\cdot\rangle$ is a duality mapping between $L^2(0,T;V)$ and $L^2(0,T;V')$, we obtain

$$\begin{aligned}
\langle K_0(u)-K_0(v),\psi\rangle &= \gamma\int_0^T\int_I[(u^3-v^3)-(u-v)]D^2\psi dxdt + \int_0^T\int_I((-u^3+v^3)\psi+(u-v)\psi)dxdt \\
&= \int_0^T\int_I((u-v)(u^2+uv+v^2)D^2\psi-(u-v)D^2\psi)dxdt \\
&\quad + \int_0^T\int_I((v-u)(u^2+uv+v^2)\psi+(u-v)\psi)dxdt \\
&\leq \left(\int_0^T\int_I(\frac{3}{2}(|u|^2+|v|^2)+1)^2(u-v)^2)dxdt\right)^{\frac{1}{2}} \times \left(\int_0^T\int_I|D^2\psi|^2dxdt\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^T\int_I\left(\frac{3}{2}\left(|u|^2+|v|^2\right)+1\right)^2(u-v)^2\right)dxdt\Bigg)^{\frac{1}{2}} \times \left(\int_0^T\int_I(|\psi|)^2dxdt\right)^{\frac{1}{2}} \\
&\leq 2\left(\int_0^T\int_I\left(\frac{3}{2}|u|^2+\frac{3}{2}|v|^2+1\right)^3dxdt\right)^{\frac{1}{3}} \times \left(\int_0^T\int_I(u-v)^6dxdt\right)^{\frac{1}{6}}\|\psi\|_{L^2(0,T;V)} \\
&\leq M_1\left(\|u\|_E^2+\|v\|_E^2+T^{\frac{1}{3}}\right)\times\|u-v\|_E\|\psi\|_{L^2(0,T;V)}.
\end{aligned}$$

□

It is clearly that we can obtain next lemma by Lemma 3.3.

Lemma 3.4.

$$\|K(u)-K(v)\|_{V'} \leq M_1\left(\|u\|^2_{W^{1,6}(I)}+\|v\|^2_{W^{1,6}(I)}+1\right)\|u-v\|_{L^6(I)},\ u,v\in E. \tag{3.12}$$

Lemma 3.5. *The operator T_0 can be continuously extended to*

$$T := E \to E.$$

Furthermore, there exists a constant $M_2 > 0$, independent of $T > 0$, such that

$$\|T(u)-T(v)\|_E \leq M_2\left(\|u\|_E^2+\|v\|_E^2+T^{\frac{1}{3}}\right)\|u-v\|_E, \quad u,v\in E. \tag{3.13}$$

Proof. By Lemma 2.2, we have

$$T(u)(t) = y(t;K(u)) \in L^\infty(0,T;H)\cap L^2(0,T;V).$$

Moreover from Lemma 2.3, we obtain

$$\begin{aligned}
\|T(u)-T(v)\|_E &\leq K(\|y(\cdot;K(u))-y(\cdot;K(v))\|_{L^\infty(0,T;H)} \\
&\quad + \|y(\cdot;K(u))-y(\cdot;K(v))\|_{L^2(0,T;V)}) \\
&\leq KL\|K(u)-K(v)\|_{L^2(0,T;V')} \\
&\leq M_2\left(\|u\|_E^2+\|v\|_E^2+T^{\frac{1}{3}}\right)\|u-v\|_E,
\end{aligned}$$

for $M_2 = M_1KL$. □

Thus the T is 'good'–non-trivial point– enough to be used on the Lemma 2.1.

Next, we establish the solutions defined as Definition 3.2 are continuous with respect to the initial data.

Lemma 3.6. *The solution defined as* Definition 3.2 *depends continuously on the initial data* $u_0 \in H$*, and* $w_A \in E$*.*

Proof. Let y_i be the solutions generated by u_0^i and $w_A^i(t)$, $i = 1, 2$, then

$$y_1 - y_2 = S(t)(u_0^1 - u_0^2) + \int_0^T [S(t-s)K(y_1 + w_A^1)(s) - S(t-s)K(y_2 + w_A^2)(s)]ds.$$

Since $y_1, y_2 \in L^2(0,T;V)$, then $(y_1 - y_2)(t) \in V \subset H$, μ a. e. for almost all $t \in (0,T)$. From Lemma 2.2, the continuity of $S(t)$ and Lemma 3.4, we have

$$\begin{aligned}\|y_1 - y_2\|_V &\le L_1\|u_0^1 - u_0^2\|_H + L_2\int_0^T \|K(y_1 + w_A^1)(s) - K(y_2 + w_A^2)(s)\|_{V'}ds \\ &\le L_1\|u_0^1 - u_0^2\|_H + L_2\int_0^T M_2\Big(\|y_1 + w_A^1\|^2_{W^{1,6}(I)} \\ &\quad +\|y_2 + w_A^2\|^2_{W^{1,6}(I)} + 1\Big)\|(y_1 + w_A^1) - (y_2 + w_A^2)\|_{W^{1,6}(I)}ds.\end{aligned}$$

Using the Sobolev embedding theorem, we have

$$\|y_1 - y_2\|_{W^{1,6}(I)} \le C_3\left(\|u_0^1 - u_0^2\|_H + \|w_A^1 - w_A^2\|_E\right) + C_4\int_0^T \|y_1 - y_2\|_{W^{1,6}(I)}, \quad \mu \ a.e.$$

From Gronwall's inequality, we have

$$\|y_1 - y_2\|_{W^{1,6}(I)} \le C_3\left(\|u_0^1 - u_0^2\|_H + \|w_A^1 - w_A^2\|_E\right)e^{C_4 t}, \quad \mu \ a.e.$$

□

Now, we give the proof of the Theorem 3.1. From G. D. Prato and J. Zabczyk [6], we know that the solution u has a measurable modification, and the solution u is about the measurable modification in the proof.

Proof of Theorem 3.1. Let $z(t) = y(t) + w_A(t) - S(t)u_0$, the equation (3.7) is changed to

$$z = a + \mathscr{F}(z), \tag{3.14}$$

where

$$a = w_A(t), \quad \mathscr{F}(z) = T(z + S(t)u_0).$$

Next, we verify the conditions of Lemma 2.1.
Firstly,

$$\mathscr{F}(0) = T(S(t)u_0) - T(S(t)u_0) = 0.$$

Secondly, let $\alpha = \sqrt{\frac{1}{12M_2}}$, and

$$\tau_1 = \left(6M_2[1 + 16K^2(\|S(t)(u_0)\|^6_{L^\infty(0,T;H)} + \|S(t)(u_0)\|^6_{L^\infty(0,T;V)})^{\frac{1}{3}}]\right)^{-3}, \tag{3.15}$$

where K and M_2 are defined before.
Since $w_A(t)$ is continuous and $w_A(0) = 0$, there exists τ_2 such that

$$\left(\int_0^t \|w_A(s)\|^6_{L^6}ds\right)^{\frac{1}{6}} \le \frac{\alpha}{4} \quad \text{for} \quad 0 \le t \le \tau_2.$$

Let

$$\mathcal{E} := L^6(0,\tau; W^{1,6}(I)),$$

and $\tau := \min\{\tau_1, \tau_2\}$, with z_1 and z_2 satisfying $\|z_i\|_{\mathcal{E}} \leq \alpha, i = 1, 2$, from Lemma 3.5 and Lemma 2.4, we have

$$\begin{aligned}
&\|\mathscr{F}(z_1) - \mathscr{F}(z_2)\|_{\mathcal{E}} = \|T(z_1 + S(t)u_0) - T(z_2 + S(t)u_0)\|_{\mathcal{E}} \\
&\leq M_2 \left(\|z_1 + S(t)u_0\|^2_{\mathcal{E}} + \|z_2 + S(t)u_0\|^2_{\mathcal{E}} + \tau^{\frac{1}{3}}\right) \|z_1 - z_2\|_{\mathcal{E}} \\
&\leq M_2 \left(2\|z_1\|^2_{\mathcal{E}} + 2\|z_2\|^2_{\mathcal{E}} + 4\|S(t)u_0\|^2_{\mathcal{E}} + \tau^{\frac{1}{3}}\right) \|z_1 - z_2\|_{\mathcal{E}} \leq \frac{1}{2}\|z_1 - z_2\|_{\mathcal{E}}.
\end{aligned}$$

By applying Lemma 2.1, we obtain the existence and uniqueness of the solution of $z(t)$ when $t \in [0, \tau]$, and then consequently $y(t)$. The proof is completed. □

4. Global existence

To prove the global existence of solution, we will show that the E-norm of u obtained by Theorem 3.1 is finite for any T, then u still lies in E.

Lemma 4.1. *Let $u_0 \in H$, w_A is given by (3.3), and y is the solution of*

$$y(t) = S(t)u_0 + T(y + w_A)(t), \quad t \in [0, T]. \tag{4.1}$$

Then, y satisfies

$$\sup_{t\in[0,T]} \|y(t)\|^2_H \leq C, \tag{4.2}$$

and

$$\int_0^T \|y\|^2_V dt \leq C. \tag{4.3}$$

Proof. As we know, $D(A)$ and $C(0, T; H_0^2)$ are dense in H and E , and the solutions are continuous with respect to the initial data from Lemma 3.6. We fix our attention to deal with the (strong) solution of the equation

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + \gamma D^2 f'(y(t) + w_A(t)) - (y(t) + w_A(t))^3 + y(t) + w_A(t), \\ y_0 = u_0. \end{cases} \tag{4.4}$$

Multiplying the equation (4.4) by $y(t)$ and integrating over I, then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y\|^2_{L^2(I)} + \gamma\|D^2y\|^2_{L^2(I)} + \|Dy\|^2_{L^2(I)} &= \gamma\int_0^1 D^2\left[(y + w_A)^3 - (y + w_A)\right] ydx \\
&\quad - \int_0^1 y(y + w_A)^3 dx + \int_0^1 yw_A dx + \int_0^1 y^2 dx.
\end{aligned} \tag{4.5}$$

Now we deal with the first term of righthand of (4.5),

$$\begin{aligned}
\gamma\int_0^1 D^2(y + w_A)^3 ydx &= -\gamma\int_0^1 D(y + w_A)^3 Dydx = -3\gamma\int_0^1 (y + w_A)^2(Dy + Dw_A)Dydx \\
&\leq -3\gamma\int_0^1 (y + w_A)^2(Dy)^2dx + \frac{3\gamma}{4}\int_0^1 (y + w_A)^2(Dy)^2dx \\
&\quad + C\int_0^1 (y^2 + (w_A)^2)(Dw_A)^2dx
\end{aligned}$$

$$\begin{aligned}&\leq -\frac{9\gamma}{4}\int_0^1 (y+w_A)^2(Dy)^2dx + \frac{1}{4}\int_0^1 y^4dx\\&\quad + C\int_0^1 (w_A)^4dx + C\int_0^1 (Dw_A)^4dx.\end{aligned} \tag{4.6}$$

Similarly, we have

$$-\gamma\int_0^1 D^2(y+w_A)ydx = \gamma\int_0^1 (Dy+Dw_A)Dydx \leq C\int_0^1 (Dy)^2dx + C\int_0^1 (Dw_A)^2dx. \tag{4.7}$$

On the other hand, we obtain

$$-\int_0^1 y(y+w_A)^3dx = -\int_0^1 \left(y^4+3y^3w_A+3y^2w_A+yw_A^3\right)dx = I_1+I_2+I_3+I_4. \tag{4.8}$$

By Hölder's inequality and Young's inequality,

$$I_2 \leq \left|\int_0^1 3y^3w_Adx\right| \leq \frac{1}{4}\int_0^1 y^4dx + C\int_0^1 w_A^4dx,$$

and

$$I_3 \leq \left|\int_0^1 3y^2w_Adx\right| \leq \frac{1}{4}\int_0^1 y^4dx + C\int_0^1 w_A^2dx. \tag{4.9}$$

Similarly,

$$I_4 \leq \frac{1}{2}\|y\|_{L^2(I)}^2 + \frac{1}{2}\|w_A\|_{W^{1,6}(I)}^6, \tag{4.10}$$

and

$$\int_0^1 yw_Adx \leq \frac{1}{2}\|y\|_{L^2}^2 + \frac{1}{2}\|w_A\|_{L^2}^2. \tag{4.11}$$

From (4.6)–(4.11), we obtain

$$\frac{d}{dt}\|y\|_{L^2}^2 + 2\gamma\|D^2y\|_{L^2}^2 \leq C_1\|Dy\|_{L^2}^2 + C_0\|y\|_{L^2}^2 + f(t), \tag{4.12}$$

where

$$f(t) = C\|w_A\|_{L^2}^2 + C\|w_A\|_{W^{1,6}(I)}^6 + C\|w_A\|_{W^{1,4}(I)}^4.$$

Note that

$$\begin{aligned}C_1\|Dy\|_{L^2}^2 &= C_1\int_0^1 DyDydx = -C_1\int_0^1 D^2yydx\\&\leq \frac{\gamma}{2}\|D^2y\|_{L^2}^2 + \frac{C_1^2}{2\gamma}\|y\|_{L^2}^2.\end{aligned}$$

Hence

$$\frac{d}{dt}\|y\|_{L^2}^2 + \gamma\|D^2y\|_{L^2}^2 \leq \left(C_0 + \frac{C_1^2}{2\gamma}\right)\|y\|_{L^2}^2 + f(t). \tag{4.13}$$

By Gronwall's inequality, we get (4.3) and

$$\sup_{t\in[0,T]}\|y(t)\|_{L^2}^2 \leq C. \tag{4.14}$$

Multiplying the equation (4.4) by $D^2y(t)$ and integrating over I, then

$$\frac{1}{2}\frac{d}{dt}\|Dy\|^2_{L^2(I)}+\gamma\|D^3y\|^2_{L^2(I)}+\|D^2y\|^2_{L^2(I)}=-\gamma\int_0^1 D^2[(y+w_A)^3-(y+w_A)]D^2ydx$$
$$+\int_0^1 D^2y(y+w_A)^3dx-\int_0^1 D^2y(y+w_A)dx. \tag{4.15}$$

Using the Hölder inequality, we get

$$\begin{aligned}
-\gamma\int_0^1 D^2(y+w_A)^3D^2ydx=&-3\gamma\int_0^1(y+w_A)^2(D^2y+D^2w_A)D^2ydx\\
&-6\gamma\int_0^1(y+w_A)(Dy+Dw_A)^2D^2ydx\\
\le&-3\gamma\int_0^1(y+w_A)^2(D^2y)^2+\frac{\gamma}{4}\int_0^1(D^3y)^2dx\\
&+72\gamma\int_0^1(y^4+(w_A)^4)(Dw_A)^2dx\\
&+\frac{3\gamma}{2}\int_0^1(y+w_A)^2(D^2y)^2dx+24\gamma\int_0^1((Dy)^2+(Dw_A)^2)(Dw_A)^2dx\\
&+\frac{3\gamma}{2}\int_0^1(y+w_A)^2(D^2y)^2dx+96\gamma\int_0^1((Dy)^4+(Dw_A)^4)dx\\
\le&\frac{\gamma}{4}\int_0^1(D^3y)^2dx+72\gamma\int_0^1y^8dx+72\gamma\int_0^1w_A^8dx\\
&+36\gamma\int_0^1(Dw_A)^4dx+12\gamma\int_0^1(Dy)^4dx+12\gamma\int_0^1(Dw_A)^4dx\\
&+96\gamma\int_0^1((Dy)^4+(Dw_A)^4)dx.
\end{aligned}$$

Similarly, we have

$$\gamma\int_0^1 D^2(y+w_A)D^2ydx\le 2\gamma\int_0^1(Dy)^2dx+\frac{\gamma}{4}\int_0^1(D^3y)^2dx+2\gamma\int_0^1(Dw_A)^2dx,$$
$$\int_0^1 D^2y(y+w_A)^3dx\le\frac{1}{2}\int_0^1(D^2y)^2dx+32\int_0^1(y^6+w_A^6)dx,$$

and

$$\int_0^1 D^2y(y+w_A)dx\le\frac{1}{2}\int_0^1(D^2y)^2dx+\int_0^1(y^2+w_A^2)dx.$$

Therefore, we have

$$\frac{d}{dt}\|Dy\|^2_{L^2}+\gamma\|D^3y\|^2_{L^2}\le 2\gamma\int_0^1(Dy)^2dx+108\gamma\int_0^1(Dy)^4dx+72\gamma\int_0^1y^8dx+32\int_0^1y^6dx+g_1(t),$$

where

$$g_1(t)=C_2\|w_A\|^2_{W^{1,2}}+C_3\|w_A\|^6_{W^{1,6}(I)}+C_4\|w_A\|^4_{W^{1,4}(I)}.$$

On the other hand, by the Nirenberg inequality and (4.14), we know that

$$\int_0^1y^6dx\le C\left(\int_0^1(D^3y)^2dx\right)^{1/3}\left(\int_0^1y^2dx\right)^{8/3}\le C\left(\int_0^1(D^3y)^2dx\right)^{1/3},$$
$$\int_0^1y^8dx\le C\left(\int_0^1(D^3y)^2dx\right)^{1/2}\left(\int_0^1y^2dx\right)^{7/2}\le C\left(\int_0^1(D^3y)^2dx\right)^{1/2},$$

$$\int_0^1 (Dy)^4 dx \le C \left(\int_0^1 (D^3y)^2 dx\right)^{5/6} \left(\int_0^1 y^2 dx\right)^{7/6} \le C \left(\int_0^1 (D^3y)^2 dx\right)^{5/6}.$$

Hence, we obtain

$$\frac{d}{dt}\|Dy\|_{L^2}^2 + \gamma\|D^3y\|_{L^2}^2 \le 2\gamma\|Dy\|_{L^2}^2 + g_2(t), \tag{4.16}$$

where

$$g_2(t) = C_2\|w_A\|_{W^{1,2}}^2 + C_3\|w_A\|_{W^{1,6}(I)}^6 + C_4\|w_A\|_{W^{1,4}(I)}^4 + C_5.$$

By Gronwall's inequality, we complete the proof. □

Theorem 4.2. *Let $u_0 \in H^1(I)$, then the equations (1.1)–(1.3) have a unique $\mathbb{P}$ a.s. solution $u(\cdot, x)$ concentrated on the space E.*

Proof. From Theorem 3.1, the equation (3.4) and Lemma 4.1, we deduce the result. □

5. The random attractor

In this section, we prove the equation (1.1) possesses a random attractor. The existence result of random attractors can be stated as follows

Lemma 5.1 ([3, 16])**.** *If there exists a random compact set absorbing every bounded nonrandom set $B \subset X$, the random dynamical system φ possesses a random attractor $\mathcal{A}(\omega)$:*

$$\mathcal{A}(\varphi) = \overline{\bigcup_{B\subset X} \Lambda_B(\omega)},$$

where $\Lambda_B(\omega) := \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} \varphi(t, \theta_t\omega)B}$ is the omega-limit set of B.

Let $R = \gamma D^4 - D^2 - 1$, the linear equation

$$dz + (\gamma D^4 z - D^2 z - z)dt = dw(t)$$

has a unique stationary solution given by

$$z(t) = \int_{-\infty}^t e^{-R(t-s)} dw(s).$$

Consider the set of continuous function with value 0 at 0

$$\Omega = \{\omega \in C(R, R) : \omega(0) = 0\}.$$

Let $\mathcal{F}$ be the Borel sigma-algebra induced by the compact-open topology of Ω, and P a Wiener measure on $(\Omega\mathcal{F})$. Writing $w(t, \omega) = \omega(t)$, we define

$$\theta_t\omega(s) = \omega(t + s) - \omega(t), t \in R,$$

which satisfies $\theta_t \circ \theta_s = \theta_{t+s}$. Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R})$ is an ergodic metric dynamical system which models white noise.

Lemma 5.2. *Given any ball of H, $B(0, \rho)$ centered at 0 of radius ρ, for any $-1 \le t \le 1$, there exist random variables $R_t(\omega)$ and $t(\rho, \omega) < -1$ such that for any $s \le t(\rho, \omega)$, $u_s \in B(0, \rho)$,*

$$\|S(t, s; \omega)u_s\|_{H_1} \le R_t(\omega),$$

holds P-a.s.

Proof. Let $v(t) = u(t) - z(t)$. We have

$$\frac{dv(t)}{dt} = -Rv(t) + \gamma D^2 f'(v(t) + z(t)) - (v(t) + z(t))^3 + v(t). \tag{5.1}$$

Multiplying the equation (5.1) by $v(t)$ and integrating over I, similar to (4.13), we have

$$\frac{d}{dt}\|v\|_{L^2}^2 + \gamma\|D^2 v\|_{L^2}^2 \le \left(C_0 + \frac{C_1^2}{2\gamma}\right)\|v\|_{L^2}^2 + f_1(t), \tag{5.2}$$

where

$$f_1(t) = C\|z\|_{L^2}^2 + C\|z\|_{W^{1,6}(I)}^6 + C\|z\|_{W^{1,4}(I)}^4.$$

On the other hand, using the Poincaré inequality, we know

$$\|v\|_{L^2} \le \|Dv\|_{L^2} \le \|D^2 y\|_{L^2}.$$

Hence, we get

$$\frac{d}{dt}\|v\|_{L^2}^2 + \left(\gamma - C_0 - \frac{C_1^2}{2\gamma}\right)\|v\|_{L^2}^2 \le f_1(t).$$

If $M_\gamma = \gamma - C_0 - \frac{C_1^2}{2\gamma} > 0$. The Gronwall lemma leads to

$$\|v(t)\|_{L^2}^2 \le \|v(s)\|_{L^2}^2 e^{-\int_s^t M_\gamma d\tau} + \int_s^t e^{-\int_\tau^t M_\gamma d\eta} f_1(\tau)d\tau.$$

On the other hand, multiplying the equation (5.1) by $D^2 v(t)$ and integrating over I, similar to (4.16), we obtain

$$\frac{d}{dt}\|Dv\|_{L^2}^2 + \gamma\|D^3 v\|_{L^2}^2 \le 2\gamma\|Dv\|_{L^2}^2 + g_3(t),$$

where

$$g_3(t) = C_2\|z\|_{W^{1,2}}^2 + C_3\|z\|_{W^{1,6}(I)}^6 + C_4\|z\|_{W^{1,4}(I)}^4 + C_5.$$

By the Nirenberg inequality, we see that

$$\int_0^1 (Dv)^2 dx \le C\left(\int_0^1 (D^3 v)^2 dx\right)^{1/3}\left(\int_0^1 v^2 dx\right)^{2/3} \le C_6\left(\int_0^1 (D^3 v)^2 dx\right)^{1/3}.$$

Thus,

$$\frac{d}{dt}\|Dv\|_{L^2}^2 + \frac{\gamma}{2}\|Dv\|_{L^2}^2 \le h(t),$$

where

$$h(t) = g_3(t) + 3\sqrt{2}C_6^{\frac{3}{2}}\gamma.$$

By the Gronwall inequality, we have

$$\|Dv(t)\|_{L^2}^2 \le \|Dv(s)\|_{L^2}^2 e^{-\int_s^t \frac{\gamma}{2} d\tau} + \int_s^t e^{-\int_\tau^t \frac{\gamma}{2} d\eta} h(\tau)d\tau.$$

Hence

$$\begin{aligned}\|v(t)\|_{H^1}^2 \le& \|v(s)\|_{L^2}^2 e^{-\int_s^t M_\gamma d\tau} + \|Dv(s)\|_{L^2}^2 e^{-\int_s^t \frac{\gamma}{2} d\tau} \\ &+ \int_s^t e^{-\int_\tau^t M_\gamma d\eta} f_1(\tau)d\tau + \int_s^t e^{-\int_\tau^t \frac{\gamma}{2} d\eta} h(\tau)d\tau.\end{aligned}$$

Thus, for any ball $B(0,\rho) \subset H$, we conclude that there exists a random variable $t(\rho;\omega) < -1$ such that for all $s < t(\rho;\omega)$, $t \in [-1,1]$, and all $u_s \in B(0,\rho)$,

$$\|v(t,\omega;s,v_s)\|_{L^2}^2 \leq 1 + \int_{-\infty}^{t} e^{-\int_\tau^t M_\gamma d\eta} f_1(\tau)d\tau + \int_{-\infty}^{t} e^{-\int_\tau^t \frac{\gamma}{2}d\eta} h(\tau)d\tau, \tag{5.3}$$

holds P-a.s. Indeed, it is enough to choose $t(\rho;\omega) < -1$ such that for $s < t(\rho;\omega)$, $t \in [-1,1]$,

$$\begin{aligned}\|v(s)\|_{L^2}^2 e^{-\int_s^t M_\gamma d\tau} + \|Dv(s)\|_{L^2}^2 e^{-\int_s^t \frac{\gamma}{2}d\tau} &= \|u(s)-z(s)\|_{L^2}^2 e^{-\int_s^t M_\gamma d\tau} + \|Du(s)-Dz(s)\|_{L^2}^2 e^{-\int_s^t \frac{\gamma}{2}d\tau}\\ &\leq 2(\rho^2 + \|z\|_{L^2}^2)e^{-\int_s^t M_\gamma d\tau} + 2(\rho^2 + \|Dz\|_{L^2}^2)e^{-\int_s^t \frac{\gamma}{2}d\tau} < 1,\end{aligned}$$

holds P-a.s. Denote by $r_t^2(\omega)$ the right side of the inequality (5.3). Writing $R_t^2 = 2(r_t^2(\omega) + |z(t)|^2)$, we complete the proof of the lemma. □

Lemma 5.2 shows that for any deterministic bounded set $B \subset B(0,\rho)$ in H, there exists a random time $t(\rho;\omega) < -1$ such that for any $s < t(\rho;\omega)$,

$$S(-1,s;\omega)B \subset B(0, R_{-1}(\omega)).$$

Noticing that

$$\varphi(-s,\theta_s\omega) = S(-s,0;\theta_s\omega) = S(0,s;\omega) = S(0,-1;\omega)S(-1,s;\omega),$$

we find that $\mathcal{B}(\omega) := S(0,-1;\omega)B(0,R_{-1}(\omega))$ is a random absorbing set in H. Moreover, $\mathcal{B}(\omega)$ satisfies the following property.

Lemma 5.3. *The random set $\mathcal{B}(\omega)$ described above is a compact absorbing set, that is, it is compact and absorbs any nonrandom bounded set: for every bounded deterministic set $B \subset B(0,\rho)$, we have*

$$\varphi(-s,\theta_s\omega)B \subset \mathcal{B}(\omega),$$

holds P-a.s., for any $s \leq t(\rho,\omega)$.

Proof. We only verify that the random absorbing set $B(\omega)$ is compact. Let $\{u_0^n : n \in N\}$ be a sequence in $B(\omega)$ and v_n a solution of the equation (5.1) such that $v^n(0) = u_0^n - z(0)$. Multiplying the equation (5.1) by $v(t)$ and integrating over I, we have the inequality (5.2)

$$\frac{d}{dt}\|v\|_{L^2}^2 + 2\gamma\|D^2 v\|_{L^2}^2 \leq M\|v\|_{L^2}^2 + f_1(t), \tag{5.4}$$

where

$$M = C_0 + \frac{C_1^2}{2\gamma}, \quad f_1(t) = C\|z\|_{L^2}^2 + C\|z\|_{W^{1,6}(I)}^6 + C\|z\|_{W^{1,4}(I)}^4.$$

Applying the Gronwall inequality to (5.4), we get that for $t \in [-1,0]$,

$$\begin{aligned}\|v(t)\|_{L^2}^2 \leq& \|v(-1)\|_{L^2}^2 e^{-\int_{-1}^t M d\tau} + \int_{-1}^t e^{-\int_\tau^t M d\eta} f_1(\tau)d\tau\\ \leq& e^{-\int_{-1}^0 M d\tau}\left\{\|v(-1)\|_{L^2}^2 + \int_{-1}^0 f_1(\tau)d\tau\right\}.\end{aligned}$$

Denoting by M_2 the right side of the above inequality and integrating (5.4) over $[-1,t], t \in [-1,0]$, we have

$$\|v(t)\|_{L^2}^2 + \frac{\gamma}{2}\int_{-1}^t \|D^2 v\|_{L^2}^2 ds \leq \|v(-1)\|_{L^2}^2 + M_2\int_{-1}^0 M ds + \int_{-1}^0 f_1(\eta)d\eta.$$

Similarly, multiplying the equation (5.1) by $D^2 v(t)$ and integrating over I, we obtain

$$\|Dv(t)\|_{L^2}^2 + 2\gamma\int_{-1}^t \|D^3 v\|_{L^2}^2 ds \leq \|Dv(-1)\|_{L^2}^2 + 2\gamma M_3 + \int_{-1}^0 g_3(\eta)d\eta.$$

Since $\|v^n(-1)\|_{H^1} \leq R_{-1}(\omega) + \|z(-1)\|_{H^1}$, we obtain that $\{v^n : n \in N\}$ is bounded in $L^\infty(-1,0;H) \cap L^2(-1,0;V)$, and so, it is compact in $L^2(-1,0;H)$. Hence, there exists a subsequence $\{v^{n_k} : n \in N\}$ convergent to a function v in $L^2(-1,0;H)$. Moreover, v is a solution of the equation (5.1). Let $u_0 = v(0) + z(0)$, it is easy to yield

$$u_0 - u_0^{n_k} = v(0) - v^{n_k}(0).$$

We now prove the subsequence $\{u_0^{n_k}\}$ converges in H to the function u_0. Writing $X(t) = v(t) - v^{n_k}(t)$, then $X(t)$ satisfies the following equation

$$\begin{aligned}\frac{dX}{dt} &= -RX - D^2X + \gamma D^2[X((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)] \\ &\quad - X((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2) + X, \qquad (5.5)\\ X(0) &= v(0) - v^{n_k}(0). \qquad (5.6)\end{aligned}$$

Multiplying the equation (5.5) by X and integrating over I, we have

$$\begin{aligned}\frac{d}{dt}\|X\|_{L^2}^2 + 2\gamma\|D^2X\|_{L^2}^2 &= \gamma\int_0^1 D^2[X((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)]Xdx \\ &\quad - \int_0^1 X^2((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)dx.\end{aligned}$$

Now we deal with the first term of right-hand of above equation,

$$\begin{aligned}&\gamma\int_0^1 D^2[X((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)]Xdx \\ &= -\gamma\int_0^1 D[X((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)]DXdx \\ &\leq 8(|v|_{L^\infty} + |v^{n_k}|_{L^\infty} + |z|_{L^\infty})\|DX\|_{L^2}^2 \leq \frac{\gamma}{2}\|D^2X\|_{L^2}^2 + C\|X\|_{L^2}^2.\end{aligned}$$

Similarly, we have

$$-\int_0^1 X^2((v+z)^2 + (v+z)(v^{n_k}+z) + (v^{n_k}+z)^2)dx \leq C\|X\|_{L^2}^2.$$

Therefore, we get

$$\frac{d}{dt}\|X\|_{L^2}^2 + \gamma\|D^2X\|_{L^2}^2 \leq C\|X\|_{L^2}^2. \qquad (5.7)$$

From (5.7), we conclude

$$\|u_0 - u_0^{n_k}\|_{L^2}^2 = \|X(0)\|_{L^2}^2 \leq \|X(t)\|_{L^2}^2 e^{\int_t^0 Cds}.$$

Integrating the above inequality on $[-1,0]$, we obtain

$$\|u_0 - u_0^{n_k}\|_{L^2}^2 \leq \|X(t)\|_{L^2(-1,0:L^2)}^2 e^{\int_{-1}^0 Cds}.$$

Similarly, multiplying the equation (5.5) by D^2X and integrating over I, we have

$$\|Du_0 - Du_0^{n_k}\|_{L^2}^2 \leq \|DX(t)\|_{L^2(-1,0:L^2)}^2 e^{\int_{-1}^0 C_1 ds},$$

which implies that the sequence $u_0^{n_k}$ converges to u_0, and thus, $\mathcal{B}(\omega)$ is compact. □

Applying Lemma 5.1, we conclude

Theorem 5.4. *The random dynamical system associated with the stochastic fourth-order equation* (1.1) *possesses a random attractor* $\mathcal{A}(\omega)$.

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