Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2347–2363
Research Article

Journal of Nonlinear Science and Applications

Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point theorems for generalized contraction mappings in multiplicative metric spaces

Afrah A. N. Abdou

Departement of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

Communicated by Y. J. Cho

Abstract

The purpose of this paper is to study and discuss the existence of common fixed points for weakly compatible mappings satisfying the generalized contractiveness and the (CLR)-property. Our results improve the corresponding results given in He et al. [X. He, M. Song, D. Chen, Fixed Point Theory Appl., **2014** (2014), 9 pages]. Moreover, we give some examples to illustrate for the main results.

Keywords: Multiplicative metric space, common fixed point, compatible mappings, weakly compatible mappings, the (CLR)-property.
2010 MSC: 47H10, 54H25, 54E50.

1. Introduction

In 1922, Banach [5] proved the theorem which is well known as "Banach's Fixed Point Theorem" to establish the existence of solutions for nonlinear operator equations and integral equations. It is widely considered as a source of metric fixed point theory and also its significance lies in its vast applications. The study on the existence of fixed points of some mappings satisfying certain contractions has many applications and has been the center various research activities. In the past years, many authors generalized Banach's Fixed Point Theorem in various spaces such as quasi-metric spaces, fuzzy metric spaces, 2-metric spaces, cone metric spaces, partial metric spaces, probabilistic metric spaces and generalized metric spaces (see, for instance, [2, 3, 4, 7, 9, 15, 16, 17, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30]).

On the other hand, in 2008, Bashirov et al. [6] defined a new distance so called a multiplicative distance by using the concepts of multiplicative absolute value. After then, in 2012, by using the same idea of

Email address: aabdou@kau.edu.sa (Afrah A. N. Abdou)

Received 2015-12-22

multiplicative distance, Özavşar and Çevikel [18] investigate multiplicative metric spaces by remarking its topological properties and introduced the concept of a multiplicative contraction mapping and proved some fixed point theorems for multiplicative contraction mappings on multiplicative spaces. In 2012, He et al. [8] proved a common fixed point theorem for four self-mappings in multiplicative metric space.

Recently, motivated by the concepts of compatible mappings and compatible mappings of types (A), (B) in metric spaces given by Jungck [10], [14], Jungck et al. [12] and Pathak and Khan [19], in 2015, Kang et al. [11] introduced the concepts of compatible mappings and its variants in multiplicative metric spaces, that is compatible mappings of types (A), (B) and others, and prove some common fixed point theorems for these mappings.

Especially, in 2002, Aamri and Moutawakil [1] introduced the concept of the (E.A)-property. Afterward, in 2011, Sintunaravat and Kumam [27] obtained that the notion of the (E.A)-property always requires a completeness of underlying subspaces for the existence of common fixed points for single-valued mappings and hence they coined the idea of *common limit in the range* (shortly, the (CLR) property), which relaxes the completeness of the underlying of the subspaces.

Motivated by the above results, in this paper, we prove some common fixed point theorems for weakly compatible mappings satisfying some generalized contractions and the common limit range with respect to the value of given mappings in multiplicative metric spaces. Also, we give some examples to illustrate for our main results.

2. Preliminaries

Now, we present some necessary definitions and results in multiplicative metric spaces, which will be needed in the sequel.

Definition 2.1 ([6]). Let X be a nonempty set. A *multiplicative metric* is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(M1) $d(x,y) \geq 1$ for all $x,y \in X$ and $d(x,y) = 1 \iff x = y$;
(M2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(M3) $d(x,y) \leq d(x,z) \cdot d(z,y)$ for all $x,y,z \in X$ (: multiplicative triangle inequality).

The pair (X,d) is called a *multiplicative metric space*.

Proposition 2.2 ([18]). *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and let $x \in X$. Then*

$$x_n \to x \ (n \to \infty) \ \text{ if and only if } \ d(x_n,x) \to 1 \ (n \to \infty).$$

Definition 2.3 ([18]). Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a *multiplicative Cauchy sequence* if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n,x_m) < \epsilon$ for all $n,m \geq N$.

Proposition 2.4 ([18]). *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n,x_m) \to 1$ as $n,m \to \infty$.*

Definition 2.5 ([18]). A multiplicative metric space (X,d) is said to be *multiplicative complete* if every multiplicative Cauchy sequence in (X,d) is multiplicative convergent in X.

Note that $\mathbb{R}^+$ is not complete under the ordinary metric, of course, under the multiplicative metric, $\mathbb{R}^+$ is a complete multiplicative and the convergence of a sequence in $\mathbb{R}^+$ in both multiplicative and ordinary metric space are equivalent. But they may be different in more general cases.

Proposition 2.6 ([18]). *Let (X,d_X) and (Y,d_Y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be a sequence in X. Then f is multiplicative continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$.*

Proposition 2.7 ([18]). *Let (X, d_X) be a multiplicative metric spaces, $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to \infty$. Then*

$$d(x_n, y_n) \to d(x, y) \ \ (n \to \infty).$$

Definition 2.8. The self-mappings f and g of a set X are said to be:

(1) *commutative* or *commuting* on X [10] if $fgx = gfx$ for all $x \in X$;

(2) *weakly commutative* or *weakly commuting* on X [21] if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$;

(3) *compatible* on X [11] if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$;

(4) *weakly compatible* on X [13] if $fx = gx$ for all $x \in X$ implies $fgx = gfx$, that is, $d(fx, gx) = 1 \Longrightarrow d(fgx, gfx) = 1$.

Remark 2.9. Weakly commutative mappings are compatible and compatible mappings are be weakly compatible, but the converses are not true (see [11, 13]).

Example 2.10. Let $X = [0, +\infty)$ and define a mapping as follows: for all $x, y \in X$,

$$d(x, y) = e^{|x-y|}.$$

Then d satisfies all the conditions of a multiplicative metric and so (X, d) is a multiplicative metric space. Let f and g be two self-mappings of X defined by $fx = x^3$ and $gx = 2 - x$ for all $x \in X$. Then we have

$$d(fx_n, gx_n) = e^{|x_n-1|\cdot|x_n^2+x_n+2|} \to 1 \text{ if and only if } x_n \to 1$$

and

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \lim_{n\to\infty} e^{6|x_n-1|^2} = 1 \text{ if } x_n \to 1$$

as $n \to \infty$. Thus f and g are compatible. Note that

$$d(fg(0), gf(0)) = d(8, 2) = e^6 > e^2 = d(0, 2) = d(f(0), g(0))$$

and so the pair (f, g) is not weakly commuting.

Example 2.11. Let $X = [0, +\infty)$, (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Let f and g be two self-mappings of X defined by

$$fx = \begin{cases} x, & \text{if} \quad 0 \leq x < 2, \\ 2, & \text{if} \quad x = 2, \\ 4, & \text{if} \quad 2 < x < +\infty, \end{cases} \qquad gx = \begin{cases} 4 - x, & \text{if} \quad 0 \leq x < 2, \\ 2, & \text{if} \quad x = 2, \\ 7, & \text{if} \quad 2 < x < +\infty. \end{cases}$$

By the definition of the mappings f and g, only for $x = 2$, $fx = gx = 2$ and so $fgx = gfx = 2$. Thus the pair (f, g) is weakly compatible.

For $x_n = 2 - \frac{1}{n} \in (0, 2)$, from the definition of the mappings f and g, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2,$$

but

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \lim_{n\to\infty} e^{x_n} = e^2 \neq 1$$

and so the pair (f, g) is not compatible.

Definition 2.12. Two pairs S, A and T, B of self-mappings of a multiplicative metric spaces (X, d) are said to have the *common limit range* with respect to the value of the mapping A (or B) (shortly, the (CLR)-property) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Az$$

for some $z \in X$.

Example 2.13. Let $X = [0, \infty)$ be the usual metric space and define a mapping $d : X \times X \longrightarrow R$ by $d(x, d) = e^{|x-y|}$ for all $x, y \in X$. Then (X, d) is a complete multiplicative metric space. Define mappings $S, T, A, B : X \longrightarrow X$ by

$$Sx = \frac{1}{64}x, \quad Tx = \frac{1}{32}x, \quad Ax = x, \quad Bx = 2x$$

for all $x \in X$. Then, for the sequences $\{x_n\}$ and $\{y_n\}$ in X defined by $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$ for each $n \geq 1$, clearly, we can see that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = A(0) = 0$$

or $B(0) = 0$. This show that the pairs (S, A) and (T, B) have the common limit range with respect to the value of the mapping A or the mapping B.

Definition 2.14 ([18])**.** Let (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a *multiplicative contraction* if there exists a real constant $\lambda \in (0, 1]$ such that $d(fx, fy) \leq [d(x, y)]^\lambda$ for all $x, y \in X$.

The following is Banach's Fixed Point Theorem in multiplicative metric spaces, which was proved by Özavşar and Çevikel.

Theorem 2.15 ([18])**.** *Let (X, d) be a multiplicative metric space and $f : X \to X$ be a multiplicative contraction. If (X, d) is complete, then f has a unique fixed point in X.*

Recently, He et al. [8] proved the following.

Theorem 2.16 ([8])**.** *Let S, T, A and B be four self-mappings of a multiplicative metric space X satisfying the following conditions:*

(a) *$S(X) \subset B(X)$ and $T(X) \subset A(X)$;*
(b) *the pairs (A, S) and (B, T) are weak commuting on X;*
(c) *one of S, T, A and B is continuous;*
(d) *there exists a number $\lambda \in (0, \frac{1}{2})$ such that, for all $x, y \in X$,*

$$d(Sx, Ty) \leq [\max\{d(Ax, By), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}]^\lambda.$$

Then S, T, A and B have a unique common fixed point in X.

3. Common fixed points for weakly compatible mappings

In this section we prove some common fixed point theorems for weakly compatible mappings in multiplicative metric spaces.

Theorem 3.1. *Let (X, d) be a complete multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d^p(Sx, Ty) \leq \Big[\varphi\Big(\max\Big\{d^p(Ax, By), \frac{d^p(Ax, Sx)d^p(By, Ty)}{1 + d^p(Ax, By)}, \frac{d^p(Ax, Ty)d^p(By, Ax)}{1 + d^p(Ax, By)}\Big\}\Big)\Big]^\lambda \tag{3.1}$$

for all $x, y \in X$ and $p \geq 1$, where $\varphi : [0,\infty) \to [0,\infty)$ is a monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

Suppose that one of the following conditions is satisfied:

(a) *either A or S is continuous, the pair (S, A) is compatible and the pair (T, B) is weakly compatible;*

(b) *either B or T is continuous, the pair (T, B) is compatible and the pair (S, A) is weakly compatible.*

Then S, T, A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $S(X) \subset B(X)$ and $T(X) \subset A(X)$, there exist $x_1, x_2 \in X$ such that $y_0 = Sx_0 = Bx_1$ and $y_1 = Tx_1 = Ax_2$. By induction, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \tag{3.2}$$

for all $n \geq 0$.

Now, we prove that $\{y_n\}$ is a multiplicative Cauchy sequence in X. From (3.1) and (3.2), it follows that, for all $n \geq 1$,

$$\begin{aligned}
d^p(y_{2n}, y_{2n+1}) &= d^p(Sx_{2n}, Tx_{2n+1}) \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(Ax_{2n}, Bx_{2n+1})), \frac{d^p(Ax_{2n}, Sx_{2n})d^p(Bx_{2n+1}, Tx_{2n+1})}{1 + d^p(Ax_{2n}, Bx_{2n+1})}, \\
&\qquad \frac{d^p(Ax_{2n}, Tx_{2n+1})d^p(Bx_{2n+1}, Ax_{2n})}{1 + d^p(Ax_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda} \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(y_{2n-1}, y_{2n})), d^p(y_{2n}, y_{2n+1}), d^p(y_{2n-1}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\
&\leq \Big[\varphi\Big(d^p(y_{2n-1}, y_{2n}) \cdot d^p(y_{2n}, y_{2n+1})\Big)\Big]^{\lambda} \\
&\leq [d(y_{2n-1}, y_{2n})]^{p\lambda} \cdot [d(y_{2n}, y_{2n+1})]^{p\lambda},
\end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \leq [d(y_{2n-1}, y_{2n})]^{\frac{p\lambda}{p-p\lambda}} = [d(y_{2n-1}, y_{2n})]^{\frac{\lambda}{1-\lambda}} = [d(y_{2n-1}, y_{2n})]^{h}, \tag{3.3}$$

where $h = \frac{\lambda}{1-\lambda} \in (0, 1)$. Similarly, we have

$$\begin{aligned}
d^p(y_{2n+2}, y_{2n+1}) &= d^p(Sx_{2n+2}, Tx_{2n+1}) \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(Ax_{2n+2}, Bx_{2n+1})), \frac{d^p(Ax_{2n+2}, Sx_{2n+2})d^p(Bx_{2n+1}, Tx_{2n+1})}{1 + d^p(Ax_{2n+2}, Bx_{2n+1})}, \\
&\qquad \frac{d^p(Ax_{2n+2}, Tx_{2n+1})d^p(Bx_{2n+1}, Ax_{2n+2})}{1 + d^p(A_{2n+2}, Bx_{2n+1})}\Big\}\Big)\Big]^{\lambda} \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(y_{2n}, y_{2n+1}), d^p(y_{2n+2}, y_{2n+1}), d^p(y_{2n}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(y_{2n}, y_{2n+1}) \cdot d^p(y_{2n+2}, y_{2n+1})\Big\}\Big)\Big]^{\lambda} \\
&\leq [d(y_{2n}, y_{2n+1})]^{p\lambda} \cdot [d(y_{2n+2}, y_{2n+1})]^{p\lambda},
\end{aligned}$$

which implies that

$$d(y_{2n+1}, y_{2n+2}) \leq [d(y_{2n}, y_{2n+1})]^{\frac{\lambda}{1-\lambda}} = [d(y_{2n}, y_{2n+1})]^{h}. \tag{3.4}$$

It follows from (3.3) and (3.4) that, for all $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) \leq [d(y_{n-1}, y_n)]^{h} \leq [d(y_{n-2}, y_{n-1})]^{h^2} \leq \cdots \leq [d(y_0, y_1)]^{h^n}.$$

Therefore, for all $n, m \in \mathbb{N}$ with $n < m$, by the multiplicative triangle inequality, we obtain

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) \cdot d(y_{n+1}, y_{n+2}) \cdots d(y_{m-1}, y_m) \\ &\leq [d(y_0, y_1)]^{h^n} \cdot [d(y_0, y_1)]^{h^{n+1}} \cdots [d(y_0, y_1)]^{h^{m-1}} \\ &\leq [d(y_0, y_1)]^{\frac{h^n}{1-h}}. \end{aligned}$$

This means that $d(y_n, y_m) \to 1$ as $n, m \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in X. By the completeness of X, there exists $z \in X$ such that $y_n \to z$ as $n \to \infty$. Moreover, since $\{y_{2n}\} = \{Sx_{2n}\} = \{Bx_{2n+1}\}$ and $\{y_{2n+1}\} = \{Tx_{2n+1}\} = \{Ax_{2n+2}\}$ are subsequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z. \tag{3.5}$$

Next, we show that z is a common fixed point of S, T, A and B under the condition (a).

Case 1. Suppose that A is continuous. Then it follows that $\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n} = Az$. Since the pair (S, A) is compatible, it follows from (3.5) that

$$\lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = \lim_{n\to\infty} d(SAx_{2n}, Az) = 1,$$

this is, $\lim_{n\to\infty} SAx_{2n} = Az$. Using (3.1), we have

$$\begin{aligned} d^p(SAx_{2n}, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d^p(A^2x_{2n}, Bx_{2n+1}), \frac{d^p(A^2x_{2n}, SAx_{2n})d^p(Bx_{2n+1}, Tx_{2n+1})}{1 + d^p(A^2x_{2n}, Bx_{2n+1})}, \\ &\frac{d^p(Ax^2x_{2n}, Tx_{2n+1})d^p(Bx_{2n+1}, A^2x_{2n})}{1 + d^p(A^2_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^\lambda. \end{aligned}$$

Taking $n \to \infty$ on the two sides of the above inequality and using (3.5), we can obtain

$$\begin{aligned} d^p(Az, z) &\leq \Big[\varphi\Big(\max\Big\{d^p(Az, z), \frac{d^p(Az, Az)d^p(z, z)}{1 + d^p(Az, z)}, \frac{d^p(Az, z)d^p(z, Az)}{1 + d^p(Az, z)}\Big\}\Big)\Big]^\lambda \\ &\leq \Big[\varphi\Big(\max\Big\{d^p(Az, z), \frac{1}{d^p(Az, z)}, d^p(Az, z)\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(Az, z)\Big)\Big]^\lambda \\ &\leq [d(Az, z)]^{p\lambda}. \end{aligned}$$

This means that $d(Az, z) = 1$, this is, $Az = z$. Again, applying (3.1), we obtain

$$\begin{aligned} d^p(Sz, Tx_{2n+1}) \leq \Big[\varphi\Big(\max\Big\{&d^p(Az, Bx_{2n+1})), \frac{d^p(Az, Sz)d^p(Bx_{2n+1}, Tx_{2n+1})}{1 + d^p(Az, Bx_{2n+1})}, \\ &\frac{d^p(Az, Tx_{2n+1})d^p(Bx_{2n+1}, Az)}{1 + d^p(Az, Bx_{2n+1})}\Big\}\Big)\Big]^\lambda. \end{aligned}$$

Letting $n \to \infty$ on both sides in the above inequality and using $Az = z$ and (3.4), we can obtain

$$\begin{aligned} d^p(Sz, z) &\leq \Big[\varphi\Big(\max\Big\{d^p(Az, z), \frac{d^p(z, Sz)d^p(z, z)}{1 + d^p(Az, z)}, \frac{d^p(Az, z)d^p(z, Az)}{1 + d^p(z, z)}\Big\}\Big)\Big]^\lambda \\ &\leq \Big[\varphi\Big(d^p(Sz, z)\Big)\Big]^\lambda \\ &\leq [d(Sz, z)]^{p\lambda}. \end{aligned}$$

This implies that $d(Sz, z) = 1$, that is, $Sz = z$. On the other hand, since $z = Sz \in S(X) \subset B(X)$, there exist $z^* \in X$ such that $z = Sz = Bz^*$. By using (3.1) and $z = Sz = Az = Bz^*$, we can obtain

$$\begin{aligned} d^p(z, Tz^*) &= d^p(Sz, Tz^*) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Az, Bz^*), \frac{d^p(Az, Sz)d^p(Bz^*, Tz^*)}{1 + d^p(Az, Bz^*)}, \frac{d^p(Az, Tz^*)d^p(Bz^*, Az)}{1 + d^p(Az, Bz^*)}\Big\}\Big)\Big]^\lambda \\ &\le \Big[\varphi\Big(d^p(z, Tz^*)\Big)\Big]^\lambda \\ &\le [d(z, Tz^*)]^{p\lambda}. \end{aligned}$$

This implies that $d(z, Tz^*) = 1$ and so $Tz^* = z = Bz^*$. Since the pair T, B is weakly compatible, we have

$$Tz = TBz^* = BTz^* = Bz.$$

Now, we prove that $Tz = z$. From (3.1), we have

$$\begin{aligned} d^p(z, Tz) &= d^p(Sz, Tz) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Az, Bz), \frac{d^p(Az, Sz)d^p(Bz, Tz)}{1 + d^p(Az, Bz)}, \frac{d^p(Az, Tz)d^p(Bz, Az)}{1 + d^p(Az, Bz)}\Big\}\Big)\Big]^\lambda \\ &\le \Big[\varphi\Big(\max\Big\{d^p(z, Tz), \frac{1}{d^p(z, Tz)}, d^p(z, Tz)\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(z, Tz)\Big)\Big]^\lambda \\ &\le [d(z, Tz)]^{p\lambda}. \end{aligned}$$

This implies that $d(z, Tz) = 1$ and so $z = Tz$. Therefore, we obtain $z = Sz = Az = Tz = Bz$ and so z is a common fixed point of S, T, A and B.

Case 2. Suppose that S is continuous. Then $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} S^2x_{2n} = Sz$. Since the pair (S, A) is compatible, it follows from (3.5) that

$$\lim_{n\to\infty} d^p(SAx_{2n}, ASx_{2n}) = \lim_{n\to\infty} d^p(Sz, ASx_{2n}) = 1,$$

this is, $\lim_{n\to\infty} ASx_{2n} = Sz$. From (3.1), we obtain

$$\begin{aligned} d^p(S^2x_{2n}, Tx_{2n+1}) \le \Big[\varphi\Big(\max\Big\{&d^p(ASx_{2n}, Bx_{2n+1})), \frac{d^p(ASx_{2n}, S^2x_{2n})d^p(Bx_{2n+1}, Tx_{2n+1})}{1 + d^p(ASx_{2n}, Bx_{2n+1})}, \\ &\frac{d^p(ASx_{2n}, Tx_{2n+1})d^p(Bx_{2n+1}, ASx_{2n})}{1 + d^p(ASx_{2n}, Bx_{2n+1})}\Big\}\Big)\Big]^\lambda. \end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality and using (3.4), we can obtain

$$\begin{aligned} d^p(Sz, z) &\le \Big[\varphi\Big(\max\Big\{d^p(Sz, z), \frac{d^p(Sz, Sz)d^p(z, z)}{1 + d^p(Sz, z)}, \frac{d^p(Sz, z)d^p(z, Sz)}{1 + d^p(Sz, z)}\Big\}\Big)\Big]^\lambda \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Sz, z), \frac{1}{d^p(Sz, z)}, d^p(z, Sz)\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(z, Tz)\Big)\Big]^\lambda \\ &\le [d(Sz, z)]^{p\lambda}. \end{aligned}$$

This means that $d(Sz, z) = 1$, this is, $Sz = z$. Since $z = Sz \in S(X) \subset B(X)$, there exist $z^* \in X$ such that $z = Sz = Bz^*$. From (3.1), we have

$$\begin{aligned} d^p(S^2x_{2n}, Tz^*) \le \Big[\varphi\Big(\max\Big\{&d^p(ASx_{2n}, Bz^*), \frac{d^p(ASx_{2n}, S^2x_{2n})d^p(Bz^*, Tz^*)}{1 + d^p(ASx_{2n}, Bz^*)}, \\ &\frac{d^p(ASx_{2n}, Tz^*)d^p(Bz^*, ASx)}{1 + d^p(ASx_{2n}, Bz^*)}\Big\}\Big)\Big]^\lambda. \end{aligned}$$

Letting $n \to \infty$ on both sides in the above inequality and using $Az = z$ and (3.5), we can obtain

$$\begin{aligned} d^p(Sz, Tz^*) &\le \Big[\varphi\Big(\max\Big\{d^p(Sz, z), \frac{d^p(Sz, Sz)d^p(z, Tz^*)}{1 + d^p(Sz, z)}, \frac{d^p(Sz, Tz^*)d^p(z, Sz)}{1 + d^p(Sz, z)}\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(z, Tz^*)\Big)\Big]^\lambda \\ &\le [d(z, Tz^*)]^{p\lambda}, \end{aligned}$$

which implies that $d(z, Tz^*) = 1$ and so $Tz^* = z = Bz^*$. Since the pair (T, B) is weakly compatible, we obtain

$$Tz = TBz^* = BTz^* = Bz$$

and so $Tz = Bz$. By (3.1), we have

$$\begin{aligned} d^p(Sx_{2n}, Tz) \le \Big[\varphi\Big(\max\Big\{&d^p(Ax_{2n}, Bz)), \frac{d^p(Ax_{2n}, Sx_{2n})d^p(Bz, Tz)}{1 + d^p(Ax_{2n}, Bz)}, \\ &\frac{d^p(Ax_{2n}, Tz)d^p(Bz, Ax_{2n})}{1 + d^p(Ax_{2n}, Bz)}\Big\}\Big)\Big]^\lambda. \end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality and using $Bz = Tz$, we can obtain

$$\begin{aligned} d^p(z, Tz) &\le \Big[\varphi\Big(\max\Big\{d^p(z, Tz), \frac{d^p(z, z)d^p(Tz, Tz)}{1 + d^p(z, Tz)}, \frac{d^p(z, Tz)d^p(Tz, z)}{1 + d^p(z, Tz)}\Big\}\Big)\Big]^\lambda \\ &\le \Big[\varphi\Big(\max\Big\{d^p(z, Tz), \frac{1}{d^p(z, Tz)}, d^p(Tz, z)\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(z, Tz)\Big)\Big]^\lambda \\ &\le [d(z, Tz)]^{p\lambda}. \end{aligned}$$

This implies that $d(z, Tz) = 1$ and so $z = Tz = Bz$. On the other hand, since $z = Tz \in T(X) \subset A(X)$, there exist $z^{**} \in X$ such that $z = Tz = Az^{**}$. By (3.1), using $Tz = Bz = z$, we can obtain

$$\begin{aligned} d^p(Sz^{**}, z) &= d^p(Sz^{**}, Tz) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Az^{**}, Bz), \frac{d^p(Az^{**}, Sz^{**})d^p(Bz, Tz)}{1 + d^p(Az^{**}, Bz)}, \frac{d(Az^{**}, Tz)d^p(Bz, Az^{**})}{1 + d^p(Az^{**}, Bz)}\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(Sz^{**}, z)\Big)\Big]^\lambda \\ &\le [d(Sz^{**}, z)]^{p\lambda}. \end{aligned}$$

This implies that $d(Sz^{**}, z) = 1$ and so $Sz^{**} = z = Az^{**}$. Since the pair S, A is compatible, we have

$$d(Az, Sz) = d(SAz^{**}, ASz^{**}) = d(z, z) = 1$$

and so $Az = Sz$. Hence $z = Sz = Az = Tz = Bz$.

Next, we prove that S, T, A and B have a unique common fixed point. Suppose that $w \in X$ is another common fixed point of S, T, A and B. Then we have

$$\begin{aligned} d^p(z, w) &= d^p(Sz, Tw) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Az, Bw), \frac{d^p(Az, Sz)d^p(Bw, Tw)}{1 + d^p(Az, Bw)}, \frac{d^p(Az, Tw)d^p(Bz, Az)}{1 + d^p(Az, Bw)}\Big\}\Big)\Big]^\lambda \\ &= \Big[\varphi\Big(d^p(z, w)\Big)\Big]^\lambda \\ &\le [d(z, w)]^{p\lambda}, \end{aligned}$$

which implies that $d(z, w) = 1$ and so $w = z$. Therefore, z is a unique common fixed point of S, T, A and B.

Finally, if the condition (b) holds, then we obtain the same result. This completes the proof. □

Example 3.2. Let $X = [0, 2]$ and (X, d) be a complete multiplicative metric space, where d is defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Let S, T, A and B be four self-mappings of X defined by

$$Sx = \frac{5}{4}, \quad \text{if } x \in [0, 2]; \qquad Tx = \begin{cases} \frac{7}{4}, & \text{if } x \in [0, 1], \\ \frac{5}{4}, & \text{if } x \in (1, 2], \end{cases}$$

$$Ax = \begin{cases} 1, & \text{if } x \in [0, 1], \\ \frac{5}{4}, & \text{if } x \in (1, 2), \\ \frac{7}{4}, & \text{if } x = 2, \end{cases} \qquad Bx = \begin{cases} \frac{1}{4}, & \text{if } x \in [0, 1], \\ \frac{5}{4}, & \text{if } x \in (1, 2), \\ 1, & \text{if } x = 2. \end{cases}$$

Note that S is multiplicative continuous in X, but T, A and B are not multiplicative continuous mappings in X. Also, we have the following:

(1) Clearly, $S(X) \subset B(X)$ and $T(X) \subset A(X)$.

(2) If $\{x_n\} \subset (1, 2)$, then we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t = \frac{5}{4}$$

and so

$$\lim_{n\to\infty} d(SAx_n, AS_nx) = d\left(\frac{5}{4}, \frac{5}{4}\right) = 1,$$

which means that the pair (S, A) is compatible. By the definition of the mappings T, B, for any $x \in (1, 2)$, we have $Tx = Bx = \frac{5}{4}$ and so

$$TBx = T(\frac{5}{4}) = \frac{5}{4} = B(\frac{5}{4}) = BTx.$$

Thus $TBx = BTx$, which implies that the pair T, B is weakly compatible.

(3) Now, we prove that the mappings S, T, A and B satisfy the condition (3.1) of Theorem 3.1 with $\lambda = \frac{2}{3}$ and $p = 1$. For this, we consider the following cases:

Case 1. If $x, y \in [0, 1]$, then we have

$$d(Sx, Ty) = d\left(\frac{5}{4}, \frac{7}{4}\right) = e^{\frac{1}{2}}$$

and, since $\varphi(t) < t$ for all $t > 0$, we have

$$\begin{aligned}&\Big[\varphi\Big(\max\Big\{d^p(Ax, By), \frac{d^p(Ax, Sx)d^p(By, Ty)}{1 + d^p(Ax, By)}, \frac{d^p(Ax, Ty)d^p(By, Ax)}{1 + d^p(Ax, By)}\Big\}\Big)\Big]^\lambda \\ &\le \max\Big\{d^{\frac{2}{3}}\Big(1, \frac{1}{4}\Big), d^{\frac{2}{3}}\Big(1, \frac{5}{4}\Big)d^{\frac{2}{3}}\Big(\frac{1}{4}, \frac{7}{4}\Big), d^{\frac{2}{3}}\Big(1, \frac{7}{4}\Big)d^{\frac{2}{3}}\Big(\frac{1}{4}, 1\Big)\Big\} \\ &= \max\{e^{\frac{1}{2}}, e^{\frac{1}{6}}e, e^{\frac{3}{4}}e^{\frac{3}{4}}\} \le e.\end{aligned}$$

Thus we have

$$\begin{aligned}d(Sx, Ty) &= e^{\frac{1}{2}} \\ &< \max\Big\{d^{\frac{2}{3}}(Ax, By), \frac{d^{\frac{2}{3}}(Ax, Sx)d^{\frac{2}{3}}(By, Ty)}{1 + d^{\frac{2}{3}}(Ax, By)}, \frac{d^{\frac{2}{3}}(Ax, Ty)d^{\frac{2}{3}}(By, Ax)}{1 + d^{\frac{2}{3}}(Ax, By)}\Big\}.\end{aligned}$$

Case 2. If $x \in [0, 1]$ and $y \in (1, 2]$, then we obtain

$$d(Sx, Ty) = d\left(\frac{5}{4}, \frac{5}{4}\right) = 1 \le \max\Big\{d^{\frac{2}{3}}(Ax, By), \frac{d^{\frac{2}{3}}(Ax, Sx)d^{\frac{2}{3}}(By, Ty)}{1 + d^{\frac{2}{3}}(Ax, By)}, \frac{d^{\frac{2}{3}}(Ax, Ty)d^{\frac{2}{3}}(By, Ax)}{1 + d^{\frac{2}{3}}(Ax, By)}\Big\}.$$

Case 3. If $x \in (1,2)$ and $y \in [0,1]$, then we obtain

$$d(Sx,Ty) = d\left(\frac{5}{4},\frac{7}{4}\right) = e^{\frac{1}{2}}$$

and

$$\begin{aligned}
&\Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^\lambda \\
&\le \max\Big\{d^{\frac{2}{3}}(\frac{5}{4},\frac{1}{4}), \frac{d^{\frac{2}{3}}(\frac{5}{4},\frac{5}{4})d^{\frac{2}{3}}(\frac{1}{4},\frac{7}{4})}{1+d^{\frac{2}{3}}(\frac{5}{4},\frac{1}{4})}, \frac{d^{\frac{2}{3}}(\frac{5}{4},\frac{7}{4})d^{\frac{2}{3}}(\frac{1}{4},\frac{5}{4})}{1+d^{\frac{2}{3}}(\frac{5}{4},\frac{1}{4})}\Big\} \\
&= \max\Big(e^{\frac{2}{3}}, e, e\Big) < e.
\end{aligned}$$

Hence we have

$$d(Sx,Ty) = e^{\frac{1}{2}} < \Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^\lambda.$$

Case 4. If $x = 2$ and $y \in [0,1]$, then we have

$$d(Sx,Ty) = d\left(\frac{5}{4},\frac{7}{4}\right) = e^{\frac{1}{2}}.$$

Hence we have

$$d(Sx,Ty) = e^{\frac{1}{2}} < \Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^\lambda.$$

Case 5. If $x, y \in (1,2]$, then we have

$$d(Sx,Ty) = d\left(\frac{5}{4},\frac{5}{4}\right) = 1 \le \Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^\lambda.$$

Then, as in all the above cases, the mappings S, T, A and B satisfy the condition (3.1) of Theorem 3.1. So, all the conditions of Theorem 3.1 are satisfied. Moreover, $\frac{5}{4}$ is the unique common fixed point for all of the mappings S, T, A and B.

Theorem 3.3. *Let (X,d) be a complete multiplicative metric space. Let $S,T,A,B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $TX \subset AX$ and there exists $\lambda \in (0,\frac{1}{2})$ such that*

$$d^p(S^m x, T^q y) \le \Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,S^m x)d^p(By,T^q y)}{1+d^p(Ax,By)}, \frac{d^p(Ax,T^q y)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^\lambda \tag{3.6}$$

for all $x, y \in X$, $p \ge 1$ and $m, q \in Z^+$, where $\varphi : [0,\infty) \to [0,\infty)$ is a monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

Assume the following conditions are satisfied:

(a) *the pairs (S,A) and (T,B) are commutative mappings;*

(b) *one of S, T, A and B is continuous.*

Then S,T,A and B have a unique common fixed point in X.

Proof. From $S(X) \subset B(X)$ and $T(X) \subset A(X)$ we have

$$S^m(X) \subset S^{m-1}(X) \subset \cdots \subset S^2(X) \subset S(X) \subset B(X)$$

and

$$T^q(X) \subset T^{q-1}(X) \subset \cdots \subset T^2(X) \subset T(X) \subset A(X).$$

Since the pairs (S, A) and (T, B) are commutative mappings, we have

$$S^m A = S^{p-1}SA = S^{m-1}AS = S^{m-2}(SA)S = S^{m-2}AS^2 = \cdots = AS^m$$

and

$$T^q B = T^{q-1}TB = T^{q-1}BT = T^{q-2}(TB)T = T^{q-2}BT^2 = \cdots = BT^q,$$

that is, $S^m A = AS^m$ and $T^q B = BT^q$. It follows from Remark 2.9 that the pairs (S^p, A) and (T^q, B) are compatible and also weakly compatible. Therefore, by Theorem 3.1, we can obtain that S^m, T^q, A and B have a unique common fixed point $z \in X$.

In addition, we prove that S, T, A and B have a unique common fixed point. From (3.6), we have

$$\begin{aligned}
d^p(Sz, z) &= d(S^m(Sz), T^q z) \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(ASz, Bz), \frac{d^p(ASz, S^m Sz)d^p(Bz, T^q z)}{1 + d^p(ASz, Bz)}, \frac{d^p(ASz, T^q z)d^p(Bz, ASz)}{1 + d^p(ASz, Bz)}\Big\}\Big)\Big]^\lambda \\
&= \Big[\varphi\Big(\max\Big\{d^p(Sz, z), \frac{d^p(Sz, Sz)d^p(z, z)}{1 + d^p(Sz, z)}, \frac{d^p(Sz, z)d^p(z, Sz)}{1 + d^p(Sz, z)}\Big\}\Big)\Big]^\lambda \\
&\leq \Big[\varphi\left(d^p(Sz, z)\right)\Big]^\lambda \\
&\leq [d(Sz, z)]^{p\lambda}.
\end{aligned}$$

This implies that $d(Sz, z) = 1$ and so $Sz = z$. On the other hand, we have

$$\begin{aligned}
d(z, Tz) &= d(S^m(z), T^q(Tz)) \\
&\leq \Big[\varphi\Big(\max\Big\{d^p(Az, BTz), \frac{d^p(Az, S^m z)d^p(BTz, T^q z)}{1 + d^p(Az, BTz)}, \frac{d^p(Az, T^q(Tz))d^p(BTz, Az)}{1 + d^p(Az, BTz)}\Big\}\Big)\Big]^\lambda \\
&= \Big[\varphi\Big(\max\Big\{d^p(z, z), \frac{d^p(z, z)d^p(Tz, z)}{1 + d^p(z, Tz)}, \frac{d^p(z, Tz)d^p(z, Tz)}{1 + d^p(z, Tz)}\Big\}\Big)\Big]^\lambda \\
&\leq \Big[\varphi\left(d^p(z, Tz)\right)\Big]^\lambda \\
&\leq [d(z, Tz)]^{p\lambda}.
\end{aligned}$$

This implies that $d(z, Tz) = 1$, i.e., $Tz = z$. Therefore, we obtain $Sz = Tz = Az = Bz = z$ and so z is a common fixed point of S, T, A and B.

Finally, we prove that S, T, A and B have a unique common fixed point z. Suppose that $w \in X$ is also a common fixed point of S, T, A and B. Then we have

$$\begin{aligned}
d(z, w) &= d(S^m(z), T^q(w)) \\
&\leq \Big[\varphi\Big(\max\Big(d^p(Az, Bz), \frac{d^p(Az, S^m z)d^p(Bw, T^q w)}{1 + d^p(Az, Bz)}, \frac{d^p(Az, T^q(w))d^p(Bw, Az)}{1 + d^p(Az, Bz)}\Big)\Big)\Big]^\lambda \\
&= \varphi\Big(\max\Big(d^p(z, Tz), \frac{d^p(z, z)d^p(Tz, Tz)}{1 + d^p(z, Tz)}, \frac{d^p(z, Tz)d^p(Tz, z)}{1 + d^p(z, Tz)}\Big)\Big]^\lambda \\
&\leq \Big[\varphi\left(d^p(z, Tz)\right)\Big]^\lambda \\
&\leq [d(z, Tz)]^{p\lambda}.
\end{aligned}$$

This implies that $d(z, w) = 1$ and so $w = z$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proof. $\square$

Now, if we take $\varphi(t) = t$ and $p = 1$ in Theorem 3.1, then we have the following result.

Corollary 3.4. *Let (X, d) be a complete multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that, for all $x, y \in X$,*

$$d(Sx, Ty) \quad \leq \max\Big\{d^{\lambda}(Ax, By), \frac{d^{\lambda}(Ax, Sx)d^{\lambda}(By, Ty)}{1 + d^{\lambda}(Ax, By)}, \frac{d^{\lambda}(Ax, Ty)d^{\lambda}(By, Ax)}{1 + d^{\lambda}(Ax, By)}\Big\}. \tag{3.7}$$

Suppose that one of the following conditions is satisfied:

(a) *either A or S is continuous, the pair (S, A) is compatible and the pair (T, B) is weakly compatible;*

(b) *either B or T is continuous, the pair (T, B) is compatible and the pair (S, A) is weakly compatible.*

Then S, T, A and B have a unique common fixed point in X.

Now, if we take $\varphi(t) = t$ and $p = 1$ in Theorem 3.3, then we have the following result.

Corollary 3.5. *Let (X, d) be a complete multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d(S^m x, T^q y) \leq \max\Big\{d^{\lambda}(Ax, By), \frac{d^{\lambda}(Ax, S^m x)d^{\lambda}(By, T^q y)}{1 + d^{\lambda}(Ax, By)}, \frac{d^{\lambda}(Ax, T^q y)d^{\lambda}(By, Ax)}{1 + d^{\lambda}(Ax, By)}\Big\} \tag{3.8}$$

for all $x, y \in X$ and $m, q \in Z^+$. Assume that the following conditions are satisfied:

(a) *the pairs (S, A) and (T, B) are commutative mappings;*

(b) *one of S, T, A and B is continuous.*

Then S, T, A and B have a unique common fixed point in X.

Now, we give an example to illustrate Corollary 3.4.

Example 3.6. Let $X = [0, 2]$ and (X, d) be a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$ for all x, y in X. Let S, T, A and B be four self mappings defined by

$$Sx = \frac{7}{6}, \quad \text{if } x \in [0, 2], \qquad Tx = \begin{cases} \frac{3}{2}, & \text{if } x \in [0, 1], \\ \frac{7}{6}, & \text{if } x \in (1, 2], \end{cases}$$

$$Ax = \begin{cases} 1, & \text{if } x \in [0, 1], \\ \frac{7}{6}, & \text{if } x \in (1, 2), \\ \frac{3}{2}, & \text{if } x = 2, \end{cases} \qquad Bx = \begin{cases} \frac{1}{6}, & \text{if } x \in [0, 1], \\ \frac{7}{6}, & \text{if } x \in (1, 2), \\ 1, & \text{if } x = 2. \end{cases}$$

Clearly, we can get $S(X) \subset B(X)$ and $T(X) \subset A(X)$. Note that T, A and B are not multiplicative continuous mappings and S is multiplicative continuous in X. By the definition of the mappings S and A, we have

$$d(SAx, ASx) = d\left(\frac{7}{6}, \frac{7}{6}\right) = 1 \leq d(Sx, Ax),$$

which implies that the pair S, A is weak commuting. Therefore, the pair (S, A) must be compatible.

Clearly, only for $x \in (1, 2)$, $Tx = Bx = \frac{7}{6}$ and $TBx = T(\frac{7}{6}) = \frac{7}{6} = B(\frac{7}{6}) = BTx$ and so $TBx = BTx$. Thus the pair (T, B) is also weakly compatible.

Now, we prove that the mappings S, T, A and B satisfy the Condition (3.7) of Corollary 3.4 with $\lambda = \frac{2}{3}$. Let

$$M(x, y) = \Big(\max\Big\{d(Ax, By), \frac{d(Ax, Sx)d(By, Ty)}{1 + d(Ax, By)}, \frac{d(Ax, Ty)d(By, Ax)}{1 + d(Ax, By)}\Big\}\Big)^{\lambda}.$$

Now, we have the following 4 cases:

Case 1. If $x, y \in [0, 1]$, then we have

$$d(Sx, Ty) = d\left(\frac{7}{6}, \frac{3}{2}\right) = e^{\frac{1}{3}} < e^{\frac{5}{6} \cdot \frac{2}{3}} = d^{\frac{2}{3}}\left(1, \frac{1}{6}\right) = d^{\frac{2}{3}}(Ax, By) \leq M(x, y).$$

Case 2. If $x \in [0,1]$ and $y \in (1,2]$, then we have

$$d(Sx,Ty) = d\left(\frac{7}{6},\frac{7}{6}\right) = 1 \leq M(x,y).$$

Case 3. If $x \in (1,2]$ and $y \in [0,1]$, then we have

$$d(Sx,Ty) = d\left(\frac{7}{6},\frac{3}{2}\right) = e^{\frac{1}{3}} < 1 \leq M(x,y).$$

Case 4. If $x, y \in (1,2]$, then we have

$$d(Sx,Ty) = d\left(\frac{7}{6},\frac{7}{6}\right) = 1 \leq M(x,y).$$

Then, in all the above cases, the mappings S, T, A and B satisfy the Condition (3.7) of Corollary 3.4 with $\lambda = \frac{2}{3}$. So, all the conditions of Corollary 3.4 are satisfied. Moreover, $\frac{7}{6}$ is the unique common fixed point for the mappings S, T, A and B.

4. Common fixed points for mappings with the (CLR)-property

In this section, we prove some common fixed point theorems for weakly compatible mappings satisfying the (CLR) property without completeness of multiplicative metric space.

Theorem 4.1. *Let (X,d) be a multiplicative metric space. Let $S,T,A,B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d^p(Sx,Ty) \leq \Big[\varphi\Big(\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big)\Big]^{\lambda} \tag{4.1}$$

for all $x,y \in X$ and $p \geq 1$, where $\varphi : [0,\infty) \to [0,\infty)$ is a monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Assume the following conditions are satisfied:

(a) *the pairs (S,A) and (T,B) are weakly compatible;*

(b) *the pairs (B,T) and (T,B) have the common limit with respect to the value of the mapping A (or B).*

Then S,T,A and B have a unique common fixed point in X.

Proof. Since two pairs (S,A) and (T,B) have the common limit with respect to the value of A, there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Az$$

for some $z \in X$.

Now, we show that $Sz = Az$. By (4.1), we have

$$d^p(Sz,Ty_n) \leq \Big[\varphi\Big(\max\Big\{d^p(Az,By_n), \frac{d^p(Az,Sz)d^p(By_n,Ty_n)}{1+d^p(Az,By_n)}, \frac{d^p(Az,Ty_n)d^p(By_n,Az)}{1+d^p(Az,By_n)}\Big\}\Big)\Big]^{\lambda}.$$

Taking $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} d^p(Sz,Az) &\leq \Big[\varphi\Big(\max\Big\{d^p(Az,Az), \frac{d^p(Az,Sz)d^p(Az,Az)}{1+d^p(Az,Az)}, \frac{d^p(Az,Az)d^p(Az,Az)}{1+d^p(Az,Az)}\Big\}\Big)\Big]^{\lambda} \\ &\leq [d(Sz,Az)]^{p\lambda}, \end{aligned}$$

which implies that $Sz = Az$. Since $S(X) \subset B(X)$, there exists $v \in X$ such that $Sz = Bz$.

Next, we show $Tv = Bv$. By (4.1), we have

$$d^p(Bv,Tv) = d(Sz,Tv) \le \Big[\varphi\Big(\max\Big\{d^p(Az,Bv), \frac{d^p(Az,Sz)d^p(Bv,Tv)}{1+d^p(Az,Bv)}, \frac{d^p(Az,Tv)d^p(Bv,Az)}{1+d^p(Az,Bv)}\Big\}\Big)\Big]^\lambda$$
$$\le [d(Bv,Tv)]^{p\lambda}$$

and so $Bv = Tv$. Therefore, we have

$$Sz = Az = Bv = Tv.$$

Since the pairs A, S and B, T are weakly compatible, $Sz = Az$, $Tv = Bv$ and

$$SAz = ASz = AAz = SSz, \quad TBv = BTv = TTv = BBv. \tag{4.2}$$

Now, we show that Sz is a common fixed point of S, T, A and B. By (4.1), we obtain

$$\begin{aligned} d^p(S^2z,Sz) &= d(S^2z,Tv) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(ASz,Bv), \frac{d^p(ASz,S^2z)d^p(Bv,Tv)}{1+d^p(ASz,Bv)}, \frac{d^p(ASz,Tv)d^p(Bv,Az)}{1+d^p(ASz,Bv)}\Big\}\Big)\Big]^\lambda \\ &\le [d(S^2z,Sz)]^{p\lambda}. \end{aligned}$$

This implies that $S^2z = Sz$. Therefore, $SSz = ASz = Sz$. By (4.1), we obtain

$$\begin{aligned} d^p(Tv,T^2v) &= d(Sz,T^2v) \\ &\le \Big[\varphi\Big(\max\Big\{d^p(Az,BTv), \frac{d^p(Az,Sz)d^p(BTv,T^2v)}{1+d^p(Az,BTv)}, \frac{d^p(Az,T^2v)d^p(BTv,Az)}{1+d^p(ASz,BTv)}\Big\}\Big)\Big]^\lambda \\ &\le [d(Tv,T^2v)]^{p\lambda}. \end{aligned}$$

This implies that $Tv = TTv$. Therefore, $TBv = BTv = TTv$, that is, Bv is a common fixed point of B and T. Since $Sv = Tv$, we have

$$SSz = ASz = TSz = BSz$$

and so Sz is a common fixed point of S, T, A and B. We can obtain the uniqueness of common fixed point z, similarly, in Theorem 3.1. This completes the proof. □

If we take $p = 1$ and $\varphi(t) = t$ in Theorem 4.1, we have the following result.

Corollary 4.2. *Let (X, d) be a multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d(Sx,Ty) \le \Big[\max\Big\{d(Ax,By), \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ax,By)}, \frac{d(Ax,Ty)d(By,Ax)}{1+d(Ax,By)}\Big\}\Big]^\lambda$$

for all $x, y \in X$. Assume the following conditions are satisfied:

(a) *the pairs (S, A) and (T, B) are weakly compatible;*

(b) *the pairs (B, T) and (T, B) have the common limit with respect to the value of the mapping A (or B).*

Then S, T, A and B have a unique common fixed point in X.

Now, we give an example to illustrate Corollary 4.2.

Example 4.3. Let $X = [0, 64)$ be a usual metric space. Define a mapping $d : X \times X \longrightarrow R$ by $d(x, d) = e^{|x-y|}$ for all $x, y \in X$. Then (X, d) is a complete multiplicative metric space. Define the mappings $S, T, A, B : X \longrightarrow X$ by

$$Sx = \frac{1}{64}x, \quad Tx = \frac{1}{32}x, \quad Ax = x, \quad Bx = 2x.$$

Then we have have the following:

(1) $T(X) = [0,2) \subset [0,64) = A(X)$;

(2) Clearly, the pairs (S,A) and (T,B) have the common limit of the value of the mapping B;

(3) Clearly, the pairs (S,A) and (T,B) are weakly compatible mappings. For all $x, y \in X$,

$$d(Sx,Ty) = e^{|\frac{1}{64}x-\frac{1}{32}y|} \leq \Big[\max\Big\{e^{|x-2y|}, \frac{e^{|x-\frac{1}{64}x|}e^{|2y-\frac{1}{32}y|}}{1+e^{|x-2y|}}, \frac{e^{|x-\frac{1}{32}y|}e^{|2y-x|}}{1+e^{|x-2y|}}\Big\}\Big]^{\frac{2}{3}}.$$

Therefore, all the conditions of Corollary 4.2 are satisfied and, further, $S(0) = T(0) = A(0) = B(0) = 0$ and so 0 is a unique common fixed point of the maps S, T, A and B.

Note that (X,d), in Example 4.3, is not complete. Therefore, Theorem 4.1 cannot be applied.

As a consequence of Theorem 4.1, by putting $A = B = I_x$, we obtain the following result.

Corollary 4.4. *Let (X,d) be a multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d^p(Sx,Ty) \leq \Big[\varphi\Big(\max\Big\{d^p(x,y), \frac{d^p(x,Sx)d^p(y,Ty)}{1+d^p(x,y)}, \frac{d^p(x,Ty)d^p(y,Ax)}{1+d^p(x,y)}\Big\}\Big)\Big]^{\lambda}$$

for all $x, y \in X$ and $p \geq 1$, where $\varphi : [0,\infty) \to [0,\infty)$ is a monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Assume the following conditions are satisfied:

(a) *the pairs (S,A) and (T,B) are weakly compatible;*

(b) *the pairs (B,T) and (T,B) have the common limit with respect to the value of the mapping A (or B).*

Then S, T, A and B have a unique common fixed point in X.

If we take $\varphi(t) = t$ in Theorem 4.1, we have the following result.

Corollary 4.5. *Let (X,d) be a multiplicative metric space. Let $S, T, A, B : X \to X$ be single-valued mappings such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d^p(Sx,Ty) \leq \Big[\max\Big\{d^p(Ax,By), \frac{d^p(Ax,Sx)d^p(By,Ty)}{1+d^p(Ax,By)}, \frac{d^p(Ax,Ty)d^p(By,Ax)}{1+d^p(Ax,By)}\Big\}\Big]^{\lambda}$$

for all $x, y \in X$ and $p \geq 1$. Assume the following conditions are satisfied:

(a) *the pairs (S,A) and (T,B) are weakly compatible;*

(b) *the pairs (B,T) and (T,B) have the common limit with respect to the value of the mapping A (or B).*

Then S, T, A and B have a unique common fixed point in X.

If we take $p = 1$, $A = B$ and $S = T$ in Corollary 4.5, we have the following result.

Corollary 4.6. *Let (X,d) be a multiplicative metric space. Let $T, A : X \to X$ be single-valued mappings such that $T(X) \subset A(X)$ and there exists $\lambda \in (0, \frac{1}{2})$ such that*

$$d(Tx,Ty) \leq \Big[\max\Big\{d(Ax,Ay), \frac{d(Ax,Tx)d(Ay,Ty)}{1+d(Ax,Ay)}, \frac{d(Ax,Ty)d(Ay,Ax)}{1+d(Ax,Ay)}\Big\}\Big]^{\lambda}$$

for all $x, y \in X$. Assume the following conditions are satisfied:

(a) *the pair (T,A) is weakly compatible;*

(b) *the pair (T,A) has the common limit with respect to the value of the mapping A (or B).*

Then T and A have a unique common fixed point in X.

Remark 4.7. In all results in this section, we don't need the completeness of a multiplicative metric space.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The author, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] M. Aamri, D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181–188. 1

[2] M. Abbas, G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric space*, J. Math. Anal. Appl., **341** (2008), 416–420. 1

[3] I. Altun, A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl, **2011** (2011), 10 pages. 1

[4] H. Aydi, E. Karapinar, *A Meir-Keeler common type fixed point theorem on partial metric spaces*, Fixed Point Theory Appl., **2012** (2012), 10 pages. 1

[5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations integrales*, Fund. Math., **3** (1922), 133–181. 1

[6] A. E. Bashirov, E. M. Kurplnara, A. Özyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337** (2008), 36–48. 1, 2.1

[7] S. Chauhan, S. Dalal, W. Sintunavarat, J. Vujaković, *Common property (E.A) and existence of fixed points in Menger spaces*, J. Inequal. Appl., **2014** (2014), 14 pages. 1

[8] X. He, M. Song, D. Chen, *Common fixed points for weak commutative mappings on a multiplicative metric space*, Fixed Point Theory Appl., **2014** (2014), 9 pages. 1, 2, 2.16

[9] L. G. Huang, X. Zhang, *Cone metric space and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. 1

[10] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly, **83** (1976), 261–263. 1, 2.8

[11] G. Jungck, *Compatible and common fixed points*, Internat. J. Math. Math. Sci., **9** (1986), 771–779. 1, 2.8, 2.9

[12] G. Jungck, P. P. Murthy, Y. J. Cho, *Compatible mappings of type* (A) *and common fixed points*, Math. Japonica, **38** (1993), 381–390. 1

[13] G. Jungck, B. E. Rhoades, *Fixed point for set-valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), 227–238. 2.8, 2.9

[14] S. M. Kang, P. Kumar, S. Kumar, P. Nagpal, S. K. Garg, *Common fixed points for compatible mappings and its variants in multiplicative metric spaces*, Int. J. Pure Appl. Math., **102** (2015), 383–406. 1

[15] E. Karapinar, *Generalizations of Caristi Kirk's theorem on partial metric spaces*, Fixed Point Theory Appl., **2011** (2011), 7 pages. 1

[16] A. Latif, S. A. Al-Mezel, *Fixed point results in quasimetric spaces*, Fixed Point Theory Appl., **2011** (2011), 8 pages. 1

[17] Z. Mustafa, B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory Appl., **2009** (2009), 10 pages. 1

[18] M. Özavşar, A. C. Çevikel, *Fixed point of multiplicative contraction mappings on multiplicative metric spaces*, arXiv preprint, (2012), 14 pages. 1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.14, 2.15

[19] H. K. Pathak, M. S. Khan, *Compatible mappings of type* (B) *and common fixed pont theorems of Gregu's type*, Czech. Math. J., **45** (1995), 685–698. 1

[20] V. Popa, M. Imdad, J. Ali, *Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces*, Bull. Malays. Math. Sci. Soc., **33** (2010), 105–120. 1

[21] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math., **32** (1982), 149–153. 2.8

[22] W. Shatanawi, *Fixed point theory for contractive mappings satisfying* Φ*-maps in G-metric spaces*, Fixed Point Theory Appl., **2010** (2010), 9 pages. 1

[23] Y. J. Shen, J. Lu, H. Zheng, *Common fixed point theorems for converse commuting mappings in generalized metric spaces*, J. Hangzhou Norm. Univ. Nat. Sci., **5** (2014), 542–547. 1

[24] X. T. Shi, F. Gu, *A new common fixed point for the mappings in cone metric space*, J. Hangzhou Norm. Univ. Nat. Sci., **2** (2012), 162–168. 1

[25] X. T. Shi, F. Gu, *Common fixed point theorem for two pairs of converse commuting mappings in a cone metric space*, J. Hangzhou Norm. Univ. Nat. Sci., **5** (2012), 433–435. 1

[26] S. L. Singh, B. M. L. Tiwari, V. K. Gupta, *Common fixed points of commuting mappings in 2-metric spaces and an application*, Math. Nachr., **95** (1980), 293–297. 1

[27] W. Sintunavarat, P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math., **2011** (2011), 14 pages. 1

[28] H. Q. Ye, J. Lu, F. Gu, *A new common fixed point theorem for noncompatible mappings of type* (A_f) *in G-metric spaces*, J. Hangzhou Norm. Univ. Nat. Sci., **12** (2013), 50–56. 1

[29] J. Yu, F. Gu, *Common Fixed Point Theorems for a Pair of Set Valued Maps and Two Pairs of Single-Valued Maps*, J. Hangzhou Norm. Univ. Nat. Sci., **2** (2012), 151–156. 1

[30] H. Zheng, Y. J. Shen, F. Gu, *A new common fixed point theorem for three pairs of self-maps satisfying common* $(E.A)$ *property*, J. Hangzhou Norm. Univ. Nat. Sci., **1** (2015), 77–81. 1