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Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications

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Abstract

The purpose of this paper is to establish an improved version of companion of Ostrowski's type integral inequalities. The inequalities are obtained by using a newly developed special type of five steps quadratic kernel. The introduction of this new Kernel gives some new error bounds for various quadrature rules. Applications for composite quadrature rules and Cumulative Distributive Functions are considered.

Keywords: Ostrowski inequality, numerical integration, composite quadrature rule, cumulative distributive function.
2010 MSC: 26D15, 26D20, 26D99.

1. Introduction

The field of inequalities have applications in most of the domains of Mathematics. Their importance have increased during the past few decades and is now studied as a separate branch of Mathematics. A number of research papers and books have been written on inequalities and their applications (see for instance [5], [6], [8]-[20] and [14]-[19]). In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities such as Ostrowski are very useful for this purpose. Ostrowski type inequalities have immediate applications in numerical integration, optimization theory, statistics, and integral operator theory.

In 1938, Ostrowski [13] discovered the following useful integral inequality.

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Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e.*

$$\left\|f'\right\|_{\infty} = \sup_{t\in[a,b]} \left|f'(t)\right| < \infty$$

then for all $x \in [a,b]$

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_{\infty}. \tag{1.1}$$

We mention another inequality called Grüss inequality [12] which is stated as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals, which is given below.

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.2}$$

where $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in (1.2).

In [5], Dragomir and Wang combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequalities.

In [6], Guessab and Schmeisser proved the following Ostrowski's inequality:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ satisfy the Lipschitz condition i.e., $|f(t) - f(s)| \leq M|t-s|$. Then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M. \tag{1.3}$$

In (1.3), the point $x = \frac{3a+b}{4}$ yields the following trapezoid type inequality.

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}M. \tag{1.4}$$

The constant $\frac{1}{8}$ is sharp in (1.4).

In [3], Barnett *et al.* proved some Ostrowski and generalized trapezoid inequalities. Dragomir [4] and Liu [8] established some companions of Ostrowski type integral inequalities. Alomari [1] proved the following inequality:

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b). If $f' \in L^1[a,b]$ and $\gamma \leq f'(t) \leq \Gamma$, for all $t \in [a,b]$, then

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{1.5}$$

Recently, Liu [9] and Liu *et al.* [10] proved some Ostrowski type inequalities. In all references mentioned above, authors proved their results by using kernels with two or three steps.

2. Main Results

Before we prove our results for the 5-step quadratic kernel, we give the following lemma for 5-step linear kernel.

Lemma 2.1. *Consider the kernel*

$$K(x,t)=\begin{cases} t-a, & t\in\left(a,\frac{a+x}{2}\right],\\ t-\frac{3a+b}{4}, & t\in\left(\frac{a+x}{2},x\right],\\ t-\frac{a+b}{2}, & t\in\left(x,a+b-x\right],\\ t-\frac{a+3b}{4}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right],\\ t-b, & t\in\left(\frac{a+2b-x}{2},b\right] \end{cases} \tag{2.1}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, *then the following identity holds.*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K(x,t)f'(t)\,dt\\ &\quad=\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{2.2}$$

Proof. From (2.1), we have

$$\begin{aligned} \int_a^b K(x,t)f'(t)\,dt &= \int_a^{\frac{a+x}{2}}(t-a)\,f'(t)\,dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)f'(t)\,dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt\\ &\quad+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)f'(t)\,dt+\int_{\frac{a+2b-x}{2}}^{b}(t-b)\,f'(t)\,dt\\ &=\frac{1}{2}(x-a)\,f\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)f(x)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)f\left(\frac{a+x}{2}\right)\\ &\quad-\left(x-\frac{a+b}{2}\right)f(a+b-x)-\left(x-\frac{a+b}{2}\right)f(x)\\ &\quad-\frac{1}{2}\left(x-\frac{a+b}{2}\right)f\left(\frac{a+2b-x}{2}\right)+\left(x-\frac{3a+b}{4}\right)f(a+b-x)\\ &\quad+\frac{1}{2}(x-a)\,f\left(\frac{a+2b-x}{2}\right)-\int_a^b f(t)\,dt. \end{aligned}$$

Hence after simplification, we get (2.2). □

Now with the help of above identity, we state and prove the following theorem.

Theorem 2.2. *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b). *If* $f'\in L^1[a,b]$ *and* $\gamma\le f'(t)\le\Gamma$, *for all* $t\in[a,b]$, *then the inequality*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{16}(b-a)(\Gamma-\gamma) \tag{2.3}$$

holds for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. As we know that for all $t\in[a,b]$ and $x\in\left[a,\frac{a+b}{2}\right]$, we have

$$x-\frac{3a+b}{4}\le K(x,t)\le x-a. \tag{2.4}$$

Applying Grüss inequality to the mappings $K(x,.)$ and $f'(.)$, for all $x\in\left[a,\frac{a+b}{2}\right]$, we obtain

$$\frac{1}{b-a}\int_a^b K(x,t)f'(t)dt-\frac{1}{(b-a)^2}\int_a^b K(x,t)dt\int_a^b f'(t)dt\le\frac{1}{16}(b-a)(\Gamma-\gamma).\tag{2.5}$$

We also have

$$\frac{1}{b-a}\int_a^b K(x,t)dt=0\tag{2.6}$$

and

$$\frac{1}{b-a}\int_a^b f'(t)dt=\frac{f(b)-f(a)}{b-a}.\tag{2.7}$$

Hence from (2.4)-(2.7), we get our required result (2.3). □

We now introduce the quadratic version of 5-step kernel that further generalize (2.3) and various previous results contained in [1], [2], [6], and [10]. With the help of this special 5-step quadratic kernel, a number of different types of useful and interesting results are obtained. Moreover, we describe new results by using Grüss inequality, Cauchy inequality and Diaz-Metcalf inequality. At the end, some obtained inequalities are applied for quadrature rules and cumulative distributive function.

Now we start our main result for the 5-step quadratic kernel. Firstly, we need to state the following lemma.

Lemma 2.3. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. Define the kernel $P(x,t)$ as:*

$$P(x,t)=\begin{cases}\frac{1}{2}(t-a)^2, & t\in\left(a,\frac{a+x}{2}\right],\\ \frac{1}{2}\left(t-\frac{3a+b}{4}\right)^2, & t\in\left(\frac{a+x}{2},x\right],\\ \frac{1}{2}\left(t-\frac{a+b}{2}\right)^2, & t\in(x,a+b-x],\\ \frac{1}{2}\left(t-\frac{a+3b}{4}\right)^2, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right],\\ \frac{1}{2}(t-b)^2, & t\in\left(\frac{a+2b-x}{2},b\right]\end{cases}\tag{2.8}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. Then the following identity

$$\begin{aligned}\frac{1}{b-a}\int_a^b P(x,t)f''dt=&\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\Bigg[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\\&+\left(x-\frac{5a+3b}{8}\right)\{f'(a+b-x)-f'(x)\}\\&+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]\end{aligned}\tag{2.9}$$

holds.

Now we present our results by imposing three different types of conditions on f'' and f'''.

2.1. $f''\in L^1[a,b]$:

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b), f' is absolutely continuous on $[a,b]$ and $\gamma\le f''(t)\le\Gamma$, $\forall\ t\in[a,b]$, then for all $x\in\left[a,\frac{a+b}{2}\right]$, we have*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\right.\tag{2.10}$$

$$
\begin{aligned}
&+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\\
&\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]+\frac{f'(b)-f'(a)}{(b-a)^2}\\
&\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\Bigg|\\
\leq&\,\upsilon(x)(b-a)(S-\gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigg|\frac{1}{4}\Bigg[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\\
&+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\\
&\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]+\frac{f'(b)-f'(a)}{(b-a)^2}\\
&\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\Bigg|\\
\leq&\,\upsilon(x)(b-a)(\Gamma-S),
\end{aligned}
\tag{2.11}
$$

where

$$S=\frac{f'(b)-f'(a)}{b-a}$$

and

$$
\begin{aligned}
\upsilon(x)=\frac{1}{96}\max\Big\{&\left|-a^2-13ab+15ax-4b^2+21bx-18x^2\right|,\\
&\left|14a^2+5ab-33ax-b^2-3bx+18x^2\right|,\left|-a^2+11ab-9ax+8b^2-27bx+18x^2\right|,\\
&\left|-10a^2-7ab-b^2+27ax+9bx-18x^2\right|,\left|13a^2+13a(b-3x)+4b^2-21bx+30x^2\right|\Big\}.
\end{aligned}
$$

Proof. The use of

$$\frac{1}{b-a}\int_a^b f''(t)dt=\frac{f'(b)-f'(a)}{b-a}, \tag{2.12}$$

$$\frac{1}{b-a}\int_a^b P(x,t)dt=\frac{1}{b-a}\left[\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right] \tag{2.13}$$

and (2.8) implies that

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b P(x,t)f''(t)dt-\frac{1}{(b-a)^2}\int_a^b P(x,t)dt\int_a^b f''(t)dt\\
&=\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\Bigg[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\\
&\quad+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)
\end{aligned}
\tag{2.14}
$$

$$\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]-\frac{f'(b)-f'(a)}{(b-a)^2}$$
$$\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}.$$

We suppose that

$$R_n(x)=\frac{1}{b-a}\int_a^b P(x,t)f''(t)dt-\frac{1}{(b-a)^2}\int_a^b P(x,t)dt.\int_a^b f''(t)dt. \tag{2.15}$$

If $C\in\mathbb{R}$ is an arbitrary constant, then we have

$$R_n(x)=\frac{1}{b-a}\int_a^b \left(f''(t)-C\right)\left[P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right]dt. \tag{2.16}$$

Furthermore, we have

$$|R_n(x)|\le\frac{1}{b-a}\max_{t\in[a,b]}\left|P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right|\int_a^b\left|f''(t)-C\right|dt. \tag{2.17}$$

Now

$$\max\left|P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right| \tag{2.18}$$
$$=\max\Bigg\{\left|\frac{1}{2}\left(\frac{x-a}{2}\right)^2-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2-\frac{\lambda(x)}{b-a}\right|,$$
$$\left|\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{8}\left(x-\frac{a+b}{2}\right)^2-\frac{\lambda(x)}{b-a}\right|,\frac{\lambda(x)}{b-a}\Bigg\},$$

where

$$\lambda(x)=\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3$$

and

$$\upsilon(x)=\frac{1}{96}\max\left\{\left|-a^2-13ab+15ax-4b^2+21bx-18x^2\right|,\right. \tag{2.19}$$
$$\left|14a^2+5ab-33ax-b^2-3bx+18x^2\right|,\left|-a^2+11ab-9ax+8b^2-27bx+18x^2\right|,$$
$$\left.\left|-10a^2-7ab-b^2+27ax+9bx-18x^2\right|,\left|13a^2+13a(b-3x)+4b^2-21bx+30x^2\right|\right\}.$$

We also have

$$\int_a^b\left|f''(t)-\gamma\right|dt=(S-\gamma)(b-a) \tag{2.20}$$

and

$$\int_a^b\left|f''(t)-\Gamma\right|dt=(\Gamma-S)(b-a). \tag{2.21}$$

Therefore, we obtain (2.10) and (2.11) by using (2.12) to (2.21) and choosing $C=\gamma$ and $C=\Gamma$ in (2.17), respectively. □

Corollary 2.5. *Substitution of $x = a$ in (2.10) and (2.11) gives*

$$\left|\frac{f(a)+f(b)}{2}-(b-a)\frac{f'(b)-f'(a)}{12}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}(b-a)^3(S-\gamma), \tag{2.22}$$

$$\left|\frac{f(a)+f(b)}{2}-(b-a)\frac{f'(b)-f'(a)}{12}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}(b-a)^3(\Gamma-S). \tag{2.23}$$

Corollary 2.6. *Substitution of $x = \frac{a+b}{2}$, in (2.10) and (2.11) gives*

$$\begin{aligned}&\left|\frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right\}+\frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right.\\&\left.+\frac{1}{96}(b-a)\{f'(b)-f'(a)\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{48}(b-a)^3(S-\gamma),\end{aligned} \tag{2.24}$$

$$\begin{aligned}&\left|\frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right\}+\frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right.\\&\left.+\frac{1}{96}(b-a)\{f'(b)-f'(a)\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{48}(b-a)^3(\Gamma-S).\end{aligned} \tag{2.25}$$

Corollary 2.7. *Substitution of $x = \frac{3a+b}{4}$, in (2.10) and (2.11) gives*

$$\begin{aligned}&\left|\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right\}\right.\\&-\frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}-\frac{5}{768}(b-a)\\&\left.\times\{f'(b)-f'(a)\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{19}{768}(b-a)^3(S-\gamma),\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\left|\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right\}\right.\\&-\frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}-\frac{5}{768}(b-a)\\&\left.\times\{f'(b)-f'(a)\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{19}{768}(b-a)^3(\Gamma-S).\end{aligned} \tag{2.27}$$

2.2. $f''' \in L^2[a,b]$

Theorem 2.8. *Let $f : [a,b] \to \mathbb{R}$ be three times differentiable function on (a,b). If $f''' \in L^2[a,b]$, then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\right. \tag{2.28}$$

$$+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)$$
$$\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]+\frac{f'(b)-f'(a)}{(b-a)^2}$$
$$\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\Bigg|$$
$$\le\frac{1}{\pi}\left\|f'''\right\|_2\Bigg[\frac{1}{320}(x-a)^5+\frac{1}{10}\left(x-\frac{3a+b}{4}\right)^5-\frac{33}{320}\left(x-\frac{a+b}{2}\right)^5 \tag{2.29}$$
$$-\frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}^2\Bigg]^{\frac{1}{2}}.$$

Proof. Let $R_n(x)$ be defined by (2.15). From (2.14), we get

$$R_n(x)=\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{4}\Bigg[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right) \tag{2.30}$$
$$+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)$$
$$\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\Bigg]-\frac{f'(b)-f'(a)}{(b-a)^2}$$
$$\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}.$$

If we choose $C=f''\left(\frac{a+b}{2}\right)$ in (2.16) and use the Cauchy Inequality, then we get

$$|R_n(x)|\le\frac{1}{b-a}\int_a^b\left|f''(t)-f''\left(\frac{a+b}{2}\right)\right|\left|P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right|dt \tag{2.31}$$
$$\le\frac{1}{b-a}\left[\int_a^b\left(f''(t)-f''\left(\frac{a+b}{2}\right)\right)^2dt\right]^{\frac{1}{2}}\times\left[\int_a^b\left(P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right)^2dt\right]^{\frac{1}{2}}.$$

We use the Diaz-Metcalf inequality from [11] or [20] to get

$$\int_a^b\left(f''(t)-f''\left(\frac{a+b}{2}\right)\right)^2dt\le\frac{(b-a)^2}{\pi^2}\left\|f'''\right\|_2^2$$

and

$$\int_a^b\left(P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right)^2dt \tag{2.32}$$
$$=\int_a^b P(x,t)^2dt-\frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}^2$$

$$=\frac{1}{320}(x-a)^5+\frac{1}{10}\left(x-\frac{3a+b}{4}\right)^5-\frac{33}{320}\left(x-\frac{a+b}{2}\right)^5$$
$$-\frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}^2.$$

Therefore, using the above relations (2.30)-(2.32), we obtain (2.28). □

Corollary 2.9. *Substitution of $x=a$ in (2.28) gives*

$$\left|\frac{f(a)+f(b)}{2}-(b-a)\frac{f'(b)-f'(a)}{12}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{\|f'''\|_2}{\pi}(b-a)^{\frac{5}{2}}\frac{1}{64}\sqrt{\frac{89}{45}}.$$

Corollary 2.10. *Substitution of $x=\frac{a+b}{2}$ in (2.28) gives*

$$\left|\frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right\}+\frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right.$$
$$\left.-\frac{1}{96}(b-a)\left\{f'(b)-f'(a)\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{\|f'''\|_2}{\pi}(b-a)^{\frac{5}{2}}\frac{1}{\sqrt{16645}}. \tag{2.33}$$

Corollary 2.11. *Substitution of $x=\frac{3a+b}{4}$ in (2.28) gives*

$$\left|\frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right\}-\left(\frac{b-a}{32}\right)\left\{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right\}\right.$$
$$\left.-\frac{5}{768}(b-a)\left\{f'(b)-f'(a)\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{\|f'''\|_2}{\pi}(b-a)^{\frac{5}{2}}\frac{\sqrt{97}}{768}. \tag{2.34}$$

2.3. $f''\in L^2[a,b]$

Theorem 2.12. *Let $f:[a,b]\to R$ be an absolutely continuous function on (a,b) with $f''\in L^2[a,b]$. Then, we have*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\right. \tag{2.35}$$
$$+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)$$
$$\left.\times\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right]+\frac{f'(b)-f'(a)}{(b-a)^2}$$
$$\left.\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\le\frac{\sqrt{\sigma(f'')}}{b-a}\left[\frac{1}{320}(x-a)^5+\frac{1}{10}\left(x-\frac{3a+b}{}\right)^5-\frac{33}{320}\left(x-\frac{a+b}{2}\right)^5\right.$$
$$\left.-\frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}.$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, *where*

$$\sigma\left(f''\right) = \left\|f''\right\|_2^2 - \frac{\left(f'(b) - f'(a)\right)^2}{b-a} = \left\|f''\right\|_2^2 - S^2(b-a), \tag{2.36}$$

and S *is defined in* Theorem 2.4.

Proof. Let $R_n(x)$ is defined as in (2.15). If we choose $C = \frac{1}{b-a}\int_a^b f''(s)\,ds$ in (2.16) and use the Cauchy inequality and (2.32), then we get

$$\begin{aligned}
|R_n(x)| &\le \frac{1}{b-a}\int_a^b \left| f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds \right| \left| P(x,t) - \frac{1}{b-a}\int_a^b P(x,s)ds \right| dt \\
&\le \frac{1}{b-a}\left[\int_a^b \left(f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds \right)^2 dt\right]^{\frac{1}{2}} \times \left[\int_a^b \left(P(x,t) - \frac{1}{b-a}\int_a^b P(x,s)ds \right)^2 dt\right]^{\frac{1}{2}} \\
&= \frac{\sqrt{\sigma\left(f''\right)}}{b-a}\left[\frac{1}{320}(x-a)^5 + \frac{1}{10}\left(x - \frac{3a+b}{4}\right)^5 - \frac{33}{320}\left(x - \frac{a+b}{2}\right)^5\right. \\
&\quad \left. - \frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3 + \frac{1}{3}\left(x - \frac{3a+b}{4}\right)^3 - \frac{3}{8}\left(x - \frac{a+b}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Hence proved (2.35). □

Corollary 2.13. *Substitution of* $x = a$, *in* (2.35) *gives*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{6}(b-a)\left\{f'(b) - f'(a)\right\} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \sqrt{\sigma\left(f''\right)}\,(b-a)^{\frac{3}{2}}\,\frac{1}{64}\sqrt{\frac{89}{45}}. \tag{2.37}$$

Corollary 2.14. *Substitution of* $x = \frac{a+b}{2}$, *in* (2.35) *gives*

$$\begin{aligned}
&\left| \frac{1}{2}f\left(\frac{a+b}{2}\right) + \frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right\} + \frac{1}{32}(b-a)\left\{f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right)\right\} \right. \\
&\left. - \frac{1}{96}(b-a)\left\{f'(b) - f'(a)\right\} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \sqrt{\sigma\left(f''\right)}\,(b-a)^{\frac{3}{2}}\sqrt{\frac{1}{16645}}.
\end{aligned} \tag{2.38}$$

Corollary 2.15. *Substitution of* $x = \frac{3a+b}{4}$, *in* (2.35) *gives*

$$\begin{aligned}
&\left| \frac{1}{4}\left\{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right\} - \left(\frac{b-a}{32}\right)\left\{f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right)\right\} \right. \\
&\left. - \frac{5}{768}(b-a)\left\{f'(b) - f'(a)\right\} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \sqrt{\sigma\left(f''\right)}\,(b-a)^{\frac{3}{2}}\,\frac{\sqrt{97}}{768}.
\end{aligned} \tag{2.39}$$

3. An application to composite quadrature rules

Let $I_n : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$ a sequence of intermediate points $\xi_i \in [x_i, x_{i+1}]$ and $h = x_{i+1} - x_i$ $(i = 0, 1, \ldots, n-1)$. We have the following quadrature formula:

Consider the perturbed composite quadrature rules

$$
\begin{aligned}
A_n\left(f, f', \xi, I_n\right) =& \frac{1}{4} h_i\left[f\left(\xi_i\right)+f\left(x_i+x_{i+1}-\xi_i\right)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2 x_{i+1}-\xi_i}{2}\right)\right. \\
& +\left(\xi_i-\frac{5 x_i+3 x_{i+1}}{8}\right)\left\{f'\left(x_i+x_{i+1}-\xi_i\right)-f'\left(\xi_i\right)\right\} \\
& \left.+\frac{1}{2}\left(\xi_i-\frac{3 x_i+x_{i+1}}{4}\right) \times\left\{f'\left(\frac{x_i+2 x_{i+1}-\xi_i}{2}\right)-f'\left(\frac{x_i+\xi_i}{2}\right)\right\}\right] \\
& +\frac{f'(x_{i+1})-f'(x_i)}{h_i} \\
& \times\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3 x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}
\end{aligned} \tag{3.1}
$$

for all $\xi_i \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right]$.

Theorem 3.1. *Let $f : [a, b] \to R$ be such that f' is absolutely continuous function. If $f'' \in L^1[a, b]$ and $\gamma \le f''(x) \le \Gamma$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then we have the following quadrature formulae:*

$$
\int_a^b f(t)dt = A_n\left(f, f', \xi, I_n\right) + R_n^1\left(f, f', \xi, I_n\right), \tag{3.2}
$$

$$
\int_a^b f(t)dt = A_n\left(f, f', \xi, I_n\right) + R_n^2\left(f, f', \xi, I_n\right), \tag{3.3}
$$

where $A_n\left(f, f', \xi, I_n\right)$ is defined in (3.1) and remainder satisfies the estimation

$$
\left|R_n^1\left(f, f', \xi, I_n\right)\right| \le (S-\gamma) \sum_{i=0}^{n-1} h_i^2 \upsilon\left(\xi_i\right) \tag{3.4}
$$

and

$$
\left|R_n^2\left(f, f', \xi, I_n\right)\right| \le (\Gamma - S) \sum_{i=0}^{n-1} h_i^2 \upsilon\left(\xi_i\right). \tag{3.5}
$$

Proof. Apply (2.10) and (2.11) on the interval $[x_i, x_{i+1}]$ to get

$$
\begin{aligned}
\left|R_n^1\left(f, f', \xi, I_n\right)\right| =& \left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4} h_i\left[f\left(\xi_i\right)+f\left(x_i+x_{i+1}-\xi_i\right)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2 x_{i+1}-\xi_i}{2}\right)\right.\right. \\
& +\left(\xi_i-\frac{5 x_i+3 x_{i+1}}{8}\right)\left\{f'\left(x_i+x_{i+1}-\xi_i\right)-f'\left(\xi_i\right)\right\} \\
& \left.+\frac{1}{2}\left(\xi_i-\frac{3 x_i+x_{i+1}}{4}\right) \times\left\{f'\left(\frac{x_i+2 x_{i+1}-\xi_i}{2}\right)-f'\left(\frac{x_i+\xi_i}{2}\right)\right\}\right] - \frac{f'(x_{i+1})-f'(x_i)}{h_i} \\
& \left. \times\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3 x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}\right| \le (S-\gamma) h_i^2 \upsilon\left(\xi_i\right)
\end{aligned}
$$

and

$$
\left|R_n^2\left(f, f', \xi, I_n\right)\right| = \left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4} h_i\left[f\left(\xi_i\right)+f\left(x_i+x_{i+1}-\xi_i\right)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2 x_{i+1}-\xi_i}{2}\right)\right.\right.
$$

$$+\left(\xi_i-\frac{5x_i+3x_{i+1}}{8}\right)\left\{f'\left(x_i+x_{i+1}-\xi_i\right)-f'\left(\xi_i\right)\right\}$$
$$+\frac{1}{2}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)\times\left\{f'\left(\frac{x_i+2x_{i+1}-\xi_i}{2}\right)-f'\left(\frac{x_i+\xi_i}{2}\right)\right\}\Bigg]-\frac{f'(x_{i+1})-f'(x_i)}{h_i}$$
$$\times\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}\Bigg|\leq\left(\Gamma-S\right)h_i^2\upsilon\left(\xi_i\right)$$

for $(i=1,2,\ldots,n-1)$.

Summing over i from 0 to $n-1$, and using the triangle inequality, we get (3.4) and (3.5). □

Theorem 3.2. *Let $f:[a,b]\to R$ be a thrice continuously differentiable mappingin (a,b) with $f'''\in L^2[a,b]$. Then for all $x\in\left[a,\frac{a+b}{2}\right]$, we have*

$$\int_a^b f(t)dt=A_n\left(f,f',\xi,I_n\right)+R_n^3\left(f,f',\xi,I_n\right), \tag{3.6}$$

where $A_n\left(f,f',\xi,I_n\right)$ is defined by formula (3.1) and the remainder $R_n^3\left(f,f',\xi,I_n\right)$ satisfies the estimation

$$\left|R_n^3\left(f,f',\xi,I_n\right)\right|\leq\frac{1}{\pi}\left\|f'''\right\|_2\sum_{i=0}^{n-1}h_i\Bigg[\frac{1}{320}\left(\xi_i-x_i\right)^5+\frac{1}{10}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^5-\frac{33}{320}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^5$$
$$-\frac{1}{h_i}\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}^2\Bigg]^{\frac{1}{2}}. \tag{3.7}$$

Proof. Applying (2.28) to the interval $[x_i,x_{i+1}]$, then we get

$$\Bigg|\int_{x_i}^{x_{i+1}}f(t)dt-\frac{1}{4}h_i\Bigg[f\left(\xi_i\right)+f\left(x_i+x_{i+1}-\xi_i\right)+f\left(\frac{x_i+\xi_i}{2}\right)+f\left(\frac{x_i+2x_{i+1}-\xi_i}{2}\right)$$
$$+\left(\xi_i-\frac{5x_i+3x_{i+1}}{8}\right)\left\{f'\left(x_i+x_{i+1}-\xi_i\right)-f'\left(\xi_i\right)\right\}$$
$$+\frac{1}{2}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)\times\left\{f'\left(\frac{x_i+2x_{i+1}-\xi_i}{2}\right)-f'\left(\frac{x_i+\xi_i}{2}\right)\right\}\Bigg]-\frac{f'(x_{i+1})-f'(x_i)}{h_i}$$
$$\times\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}\Bigg|$$
$$\leq\frac{1}{\pi}\left\|f'''\right\|_2h_i\Bigg[\frac{1}{320}\left(\xi_i-x_i\right)^5+\frac{1}{10}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^5-\frac{33}{320}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^5$$
$$-\frac{1}{h_i}\left\{\frac{1}{24}\left(\xi_i-x_i\right)^3+\frac{1}{3}\left(\xi_i-\frac{3x_i+x_{i+1}}{4}\right)^3-\frac{3}{8}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^3\right\}^2\Bigg]^{\frac{1}{2}}.$$

for $(i=0,1,\ldots,n-1)$.

Now summing over i from 0 to $n-1$, using the triangle inequality, we get (3.7). □

Theorem 3.3. *Let $f:[a,b]\to R$ be such that f' is absolutely continuous on $[a,b]$ with $f''\in L^2[a,b]$. Then for all $x\in\left[a,\frac{a+b}{2}\right]$, we have*

$$\int_a^b f(x)dx=A_n\left(f,f',\xi,I_n\right)+R_n^4\left(f,f',\xi,I_n\right) \tag{3.8}$$

where $A_n(f, f', \xi, I_n)$ is defined by formula (3.1) *and the remainder $R_n^4(f, f', \xi, I_n)$ satisfies the estimation*

$$\left|R_n^4(f, f', \xi, I_n)\right| \leq \sqrt{\sigma(f'')} \sum_{i=0}^{n-1} \left[\frac{1}{320}(\xi_i - x_i)^5 + \frac{1}{10}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^5 - \frac{33}{320}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^5 \right.$$
$$\left. -\frac{1}{h_i}\left\{\frac{1}{24}(\xi_i - x_i)^3 + \frac{1}{3}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^3 - \frac{3}{8}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}. \tag{3.9}$$

Proof. Applying (2.35) to the interval $[x_i, x_{i+1}]$, then we get

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}h_i \left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f\left(\frac{x_i + \xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right)\right.\right.$$
$$+ \left(\xi_i - \frac{5x_i + 3x_{i+1}}{8}\right)\{f'(x_i + x_{i+1} - \xi_i) - f'(\xi_i)\}$$
$$\left. + \frac{1}{2}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right) \times \left\{f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{x_i + \xi_i}{2}\right)\right\}\right] - \frac{f'(x_{i+1}) - f'(x_i)}{h_i}$$
$$\left. \times \left\{\frac{1}{24}(\xi_i - x_i)^3 + \frac{1}{3}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^3 - \frac{3}{8}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right\}\right|$$
$$\leq \sqrt{\sigma(f'')} \left[\frac{1}{320}(\xi_i - x_i)^5 + \frac{1}{10}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^5 - \frac{33}{320}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^5\right.$$
$$\left. -\frac{1}{h_i}\left\{\frac{1}{24}(\xi_i - x_i)^3 + \frac{1}{3}\left(\xi_i - \frac{3x_i + x_{i+1}}{4}\right)^3 - \frac{3}{8}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right\}^2\right]^{\frac{1}{2}}.$$

for $(i = 0, 1, \ldots, n-1)$.

Now summing over i from 0 to $n-1$, using the triangle inequality, we get (3.9). □

4. An application to cumulative distribution function

Let X be a random variable taking values in the finite interval $[a, b]$ with the probability density function $f : [a, b] \to [0, 1]$ and cumulative distributive function

$$F(x) = \Pr(X \leq x) = \int_a^x f(t)\, dt, \tag{4.1}$$

$$F(b) = \Pr(X \leq b) = \int_a^b f(u)\, du = 1. \tag{4.2}$$

Theorem 4.1. *With the assumptions of Theorem 2.4, we have the following inequality which holds*

$$\left|\frac{b - E(X)}{b - a} - \frac{1}{4}\left[F(x) + F(a + b - x) + F\left(\frac{a + x}{2}\right) + F\left(\frac{a + 2b - x}{2}\right)\right.\right.$$
$$+ \left(x - \frac{5a + 3b}{8}\right)\{f\ (a + b - x) - f(x)\}$$
$$\left. + \frac{1}{2}\left(x - \frac{3a + b}{4}\right)\left\{f\ \left(\frac{a + 2b - x}{2}\right) - f\ \left(\frac{a + x}{2}\right)\right\}\right] - \frac{f\ (b) - f\ (a)}{(b - a)^2} \tag{4.3}$$

$$\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}\Bigg|\le \upsilon(x)(b-a)(S-\gamma),$$

$$\begin{aligned}&\Bigg|\frac{b-E(X)}{b-a}-\frac{1}{4}\Bigg[F(x)+F(a+b-x)+F\left(\frac{a+x}{2}\right)+F\left(\frac{a+2b-x}{2}\right)\\&+\left(x-\frac{5a+3b}{8}\right)\{f(a+b-x)-f(x)\}\\&+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f\left(\frac{a+2b-x}{2}\right)-f\left(\frac{a+x}{2}\right)\right\}\Bigg]-\frac{f(b)-f(a)}{(b-a)^2}\\&\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}\Bigg|\le \upsilon(x)(b-a)(\Gamma-S),\end{aligned}\tag{4.4}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. *Where* $E(X)$ *is the expectation of* X.

Proof. By (2.10) and (2.11) on choosing $f=F$ and using the fact

$$E(X)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt,\tag{4.5}$$

we obtain (4.3) and (4.4). □

Corollary 4.2. *Under the assumptions of Theorem 4.1, if we put* $x=\frac{3a+b}{4}$ *in* (4.3) *and* (4.4) *then we get*

$$\begin{aligned}&\Bigg|\frac{b-E(X)}{b-a}-\frac{1}{4}\Bigg[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)+F\left(\frac{7a+b}{8}\right)+F\left(\frac{a+7b}{8}\right)\\&-\frac{b-a}{8}\left\{f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right\}\Bigg]-\frac{5}{768}(b-a)\{f(b)-f(a)\}\Bigg|\le\frac{19}{768}(b-a)^3(S-\gamma),\end{aligned}\tag{4.6}$$

$$\begin{aligned}&\Bigg|\frac{b-E(X)}{b-a}-\frac{1}{4}\Bigg[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)+F\left(\frac{7a+b}{8}\right)+F\left(\frac{a+7b}{8}\right)\\&-\frac{b-a}{8}\left\{f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right\}\Bigg]-\frac{5}{768}(b-a)\{f(b)-f(a)\}\Bigg|\le\frac{19}{768}(b-a)^3(\Gamma-\gamma).\end{aligned}\tag{4.7}$$

Theorem 4.3. *With the assumptions of Theorem 2.8, we have the following inequality which holds*

$$\begin{aligned}&\Bigg|\frac{b-E(X)}{b-a}-\frac{1}{4}\Bigg[F(x)+F(a+b-x)+F\left(\frac{a+x}{2}\right)+F\left(\frac{a+2b-x}{2}\right)\\&+\left(x-\frac{5a+3b}{8}\right)\{f(a+b-x)-f(x)\}\\&+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)f\left(\frac{a+2b-x}{2}\right)-f\left(\frac{a+x}{2}\right)\Bigg]-\frac{f(b)-f(a)}{(b-a)^2}\\&\times\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}\Bigg|\\&\le\frac{1}{\pi}\left\|f'''\right\|_2\Bigg[\frac{1}{320}(x-a)^5+\frac{1}{10}\left(x-\frac{3a+b}{4}\right)^5-\frac{33}{320}\left(x-\frac{a+b}{2}\right)^5\\&-\frac{1}{b-a}\left\{\frac{1}{24}(x-a)^3+\frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3-\frac{3}{8}\left(x-\frac{a+b}{2}\right)^3\right\}^2\Bigg]\end{aligned}\tag{4.8}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, *where* $E(X)$ *is the expectation of* X.

Proof. Using (2.28) and the same conditions that we used in above theorem, we get the required inequality (4.8). □

Corollary 4.4. *Under the assumptions of Theorem 4.3, if we put* $x = \frac{3a+b}{4}$ *in* (4.8)*, then we get*

$$\begin{aligned} &\left| \frac{b-E(X)}{b-a} - \frac{1}{4}\left[F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) \right.\right. \\ &+F\left(\frac{7a+b}{8}\right) + F\left(\frac{a+7b}{8}\right) - \frac{b-a}{8}\left\{ f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right) \right\} \Bigg] \\ &\left. -\frac{5}{768}(b-a)\{f(b)-f(a)\} \right| \leq \frac{\|f'''\|_2}{\pi}(b-a)^{\frac{5}{2}} \frac{\sqrt{97}}{768}. \end{aligned} \tag{4.9}$$

Theorem 4.5. *With the assumptions of Theorem 2.12, we have the following inequality which holds*

$$\begin{aligned} &\left| \frac{b-E(X)}{b-a} - \frac{1}{4}\left[F(x) + F(a+b-x) + F\left(\frac{a+x}{2}\right) + F\left(\frac{a+2b-x}{2}\right) \right.\right. \\ &+\left(x - \frac{5a+3b}{8}\right)\{f(a+b-x) - f(x)\} \\ &\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{ f\left(\frac{a+2b-x}{2}\right) - f\left(\frac{a+x}{2}\right) \right\} \right] - \frac{f(b)-f(a)}{(b-a)^2} \\ &\left.\times\left\{ \frac{1}{24}(x-a)^3 + \frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3 - \frac{3}{8}\left(x-\frac{a+b}{2}\right)^3 \right\} \right| \\ \leq &\frac{\sqrt{\sigma(f'')}}{b-a}\left[\frac{1}{320}(x-a)^5 + \frac{1}{10}\left(x-\frac{3a+b}{4}\right)^5 - \frac{33}{320}\left(x-\frac{a+b}{2}\right)^5 \right. \\ &\left. -\frac{1}{b-a}\left\{ \frac{1}{24}(x-a)^3 + \frac{1}{3}\left(x-\frac{3a+b}{4}\right)^3 - \frac{3}{8}\left(x-\frac{a+b}{2}\right)^3 \right\}^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

Proof. Applying (2.35) and using the same conditions that we used in Theorem 4.1, we get the required inequality (4.10). □

Corollary 4.6. *Under the assumptions of Theorem 4.5, if we put* $x = \frac{3a+b}{4}$ *in* (4.10)*, then we get*

$$\begin{aligned} &\left| \frac{b-E(X)}{b-a} - \frac{1}{4}\left[F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right) + F\left(\frac{7a+b}{8}\right) + F\left(\frac{a+7b}{8}\right) \right.\right. \\ &\left.\left. -\frac{1}{8}(b-a)\left\{ f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right) \right\} \right] - \frac{5}{768}(b-a)\{f(b)-f(a)\} \right| \\ \leq &\sqrt{\sigma(f'')}(b-a)^{\frac{3}{2}} \frac{\sqrt{97}}{768}. \end{aligned} \tag{4.11}$$

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