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Anti-periodic BVP for Volterra integro-differential equation of fractional order $1 < \alpha \leq 2$, involving Mittag-Leffler function in the kernel

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Abstract

In this paper, we consider an anti-periodic Boundary Value Problem for Volterra integro-differential equation of fractional order $1 < \alpha \leq 2$, with generalized Mittag-Leffler function in the kernel. Some existence and uniqueness results are obtained by using some well known fixed point theorems. We give some examples to exhibit our results.

Keywords: Fractional derivative, fractional integral, Caputo fractional derivative, boundary value problem, Caputo fractional boundary value problem, integral operators, Mittag-Leffler functions.
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1. Introduction

The roots of the problem to define non-integer order derivative and integral operators goes back to Leibniz and Bernoulli. In this direction the first step has been taken by Euler. He observed that the derivative of x^a has a meaning for non-integer order α. Later many well known mathematicians have contributed to the development of the theory of fractional order derivatives. So far different fractional order derivatives has been introduced by different mathematicians, among these definitions Reimann-Liouville and Caputo fractional derivatives are most used definitions.

Parallel to the generalization of derivative and integral to an arbitrary non-integer order, the ordinary calculus is naturally extended to the fractional calculus. After this extension the idea of finding meaningful

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solutions to existing problems by using fractional order derivative instead of ordinary derivatives brings evolution to different fields of science. Nowadays, fractional calculus is an active and attractive research area not only for mathematicians but also for engineers and physicists. Especially, solving fractional order linear or non-linear differential equations and boundary value problems gain popularity among researchers. Recently, various remarkable results have been published involving fractional order derivatives and q-derivatives. ([1], [2], [4], [5], [6], [8], [10], [11], [12], [20] and [30]).

Solutions of many problems involving fractional order differential equations can be written in terms of Mittag-Leffler functions. For instance, the fractional differential equation

$$(D_{0+}^{\beta} f)(x) = \mu f(x)$$

has the following solution

$$f(x) = x^{1-\beta} E_{\beta,\beta}(\mu x^{\beta}),$$

where

$$E_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + \mu)} \quad (z, \mu \in \mathbb{C},\ \operatorname{Re}(\rho) > 0\ ).$$

The function $E_{\lambda,\mu}(z)$ is known as the two parameter Mittag-Leffler function introduced by Wiman in [31] and including the one parameter function $E_{\lambda}(z) := E_{\lambda,1}(z)$ defined by Mittag-Leffler in [21]. Later the three parameter Mittag-Leffler function,

$$E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!} \qquad (\rho, \mu, \gamma, \lambda \in \mathbb{C},\ \operatorname{Re}(\rho) > 0\ ) \tag{1.1}$$

is defined by Prabhakar in [23], where

$$(\lambda)_k = \begin{cases} 1, & k = 0, \lambda \neq 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & k \in \mathbb{N}. \end{cases}$$

Obviously, $E_{\rho}(z)$ and $E_{\rho,\mu}(z)$ are special cases of $E_{\rho,\mu}^{\gamma}(z)$.

It was Prabhakar [23] who considered, the integral equation

$$\int_a^t (t-s)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(t-s)^{\rho}) f(s) ds = \psi(t)$$

and proved the existence and uniqueness of the problem on $[a, b]$. Later, by applying the method of successive approximation, Kilbas *et al.* [16] gave the solution of the following fractional integro-differential equation

$$(D_{a^+}^{\alpha})(x) = \lambda E_{\rho,\mu,\omega;a^+}^{\gamma} y(x) + f(x), \quad a < x \leq b,$$

where the integral operator

$$E_{\rho,\mu,\omega;a^+}^{\gamma} f(t) = \int_a^t (t-s)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(t-s)^{\rho}) f(s) ds, \quad (t > a), (\rho, \mu, \gamma, \omega \in \mathbb{C},\ \operatorname{Re}(\rho), \operatorname{Re}(\mu) > 0\ ) \tag{1.2}$$

is known as the Mittag-Leffler integral operator. The properties such as

$$\begin{aligned} I_{0+}^{\alpha} E_{\rho,\mu,\omega;a^+}^{\gamma} \varphi &= E_{\rho,\mu+\alpha,\omega;a^+}^{\gamma} \varphi, \\ E_{\rho,\mu,\omega;a^+}^{\gamma} E_{\rho,\upsilon,\omega;a^+}^{\sigma} \varphi &= E_{\rho,\mu+\upsilon,\omega;a^+}^{\gamma+\sigma} \varphi, \\ \left\| E_{\rho,\mu,\omega;0^+}^{\gamma}(\varphi) \right\|_{C[0,1]} &\leq E_{\rho,\mu+1}^{\gamma}(|\omega|) \left\| \varphi \right\|_{C[0,1]}, \end{aligned}$$

the left inverse and boundedness of the integral operator (1.2) on $C[0,1]$ are studied by Kilbas *et al.*

in [17]. It should be mentioned that $E^{0}_{\rho,\mu,\omega;a^+}$ is the Riemann-Liouville fractional integral operator of order μ. Therefore, the operator (1.2) and its inverse can be considered as generalization of fractional integral and derivative operators involving (1.1) in their kernels.

Recently, different authors have studied fractional integro-differential equations involving generalized Mittag-Leffler function (1.1) in the kernel (see [14], [25], [26], [27] and [28]).

Recall that the Caputo fractional derivative operator is defined by;

$$C^{\alpha}_{0+}x(t) = \frac{1}{\Gamma(n-\alpha)}\int\limits_{a}^{t}\frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,\ (x>a,\ \operatorname{Re}\alpha>0)\ ,\ n=-\left[-\operatorname{Re}\alpha\right], \tag{1.3}$$

where $[\sigma]$ represents the greatest integer less than or equal to σ.

On the other hand, there has been a great interest to anti-periodic boundary problems see ([3], [7], [9] and [13]). The main motivation of the present paper is to consider an anti-periodic boundary value problem involving Caputo fractional derivative and the fractional integral operator $E^{\gamma}_{\rho,\mu,\omega;a^+}$. More precisely, in this paper we shall study the existence and uniqueness of the solution $x(t)$, of the following boundary value problem (BVP),

$$\begin{cases} C^{\alpha}_{0+}x(t) = E^{\gamma}_{\rho,\mu,\omega;0^+}f(t,x(t)), \quad t\in[0,1]\,,\\ x(0)=-x(1)\\ C^{\beta}_{0+}x(0) = -C^{\beta}_{0+}x(1),\ 0<\beta<1 \end{cases} \tag{1.4}$$

with $\alpha\in(1,2]$ and $\rho,\mu,\gamma,\omega\in\mathbb{R}$, $(\rho,\mu>0)\,,\ f(t,x),\ x(t)\in C\left[0,1\right].$

More details about fractional calculus and fractional differential equations can be found in ([18], [22], [24] and [29]).

Recall that Riemann-Liouville fractional integral of $x(t)$ is denoted by $I^{\alpha}_{0+}x(t)$ and defined as;

$$I^{\alpha}_{0+}x(t) = \frac{1}{\Gamma(\alpha)}\int\limits_{a}^{t}\frac{x(s)}{(t-s)^{1-\alpha}}ds, \quad (x>a,\ \operatorname{Re}\alpha>0).$$

For $0<\alpha\in\mathbb{R}$, we have,

$$\|I^{\alpha}_{0+}\varphi(t)\|_{C[0,1]} \le \frac{1}{|\Gamma(\alpha+1)|}\|\varphi\|_{C[0,1]}$$

and

$$(I^{\alpha}_{0+}\,C^{\alpha}_{0+}x)(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^{k} \tag{1.5}$$

where n is given in (1.3).

As a direct consequence of the definitions of $(\gamma)_k$ and $k!$ we can state the following lemma,

Lemma 1.1. *Assume that $\lambda\ge 0$ is fixed and $\rho,\mu,\gamma>0$ then*
(i) If $0\le\gamma\le 1$, then

$$E^{\gamma}_{\rho,\mu}(\lambda) \le E_{\rho,\mu}(\lambda).$$

(ii) If $\gamma\ge 1$, then

$$E^{\gamma}_{\rho,\mu}(\lambda) \ge E_{\rho,\mu}(\lambda).$$

Lemma 1.2. *Let $\rho,\mu,\gamma,\omega\in\mathbb{R}$, $(\rho,\mu>0,\ \mu>\alpha\ge 0)$ then for a continuous function $\varphi\in C\left[0,1\right]$*

$$C^{\alpha}_{0+}\left(E^{\gamma}_{\rho,\mu,\omega,0}\varphi\right) = E^{\gamma}_{\rho,\mu-\alpha,\omega,0}\varphi.$$

Proof. The proof is easily follow from [19, Property 1, Eq 9. pp. 5] □

Lemma 1.3 ([17])**.** *Let $\rho,\mu,\gamma,\omega\in\mathbb{R}$, $(\rho,\mu>0)$, then for a continuous function $\varphi\in C\left[0,1\right]$ and positive integer k, where $\mu>k$,*

$$\left(\frac{d}{dx}\right)^{k}E^{\gamma}_{\rho,\mu,\omega;0^{+}}\varphi=E^{\gamma}_{\rho,\mu-k,\omega;0^{+}}\varphi.$$

Lemma 1.4. *Assume that $\rho,\mu,\gamma,\omega\in\mathbb{R}$, $(\rho,\mu>0)$, $1<\alpha\in\mathbb{R}$ and $\varphi\in C\left[0,1\right]$. Then the solution $x(t)$ of the following boundary value problem*

$$\begin{cases} C^{\alpha}_{0^{+}}x(t)=E^{\gamma}_{\rho,\mu,\omega;0^{+}}\varphi(t), \quad t\in\left[0,1\right], \\ x(0)=-x(1) \\ C^{\beta}_{0^{+}}x(0)=-C^{\beta}_{0^{+}}x(1),\ 0<\beta<1, \end{cases} \tag{1.6}$$

can be represented by the following integral equation:

$$\begin{aligned} x(t)=&\int_{0}^{t}(t-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(t-s)^{\rho})\varphi(s)ds-\frac{1}{2}\int_{0}^{1}(1-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(1-s)^{\rho})\varphi(s)ds \\ &+\left(\frac{\Gamma\left(2-\beta\right)\left(1-2t\right)}{2}\right)\int_{0}^{1}(1-s)^{\mu+\alpha+\beta-1}\,E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})\varphi(s)ds. \end{aligned}$$

Proof. Applying $I^{\alpha}_{0^{+}}$ to both sides of (1.6) and using (1.5) we have,

$$x(t)=\int_{0}^{t}(t-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(t-s)^{\rho})\varphi(s)ds+x(0)+x'(0)t \tag{1.7}$$

which implies by Lemma 1.2 and (1.7) that,

$$C^{\beta}_{0^{+}}x(t)=\int_{0}^{t}(t-s)^{\alpha+\mu-\beta-1}E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(t-s)^{\rho})\varphi(s)ds+x'(0)\frac{t^{1-\beta}}{\Gamma\left(2-\beta\right)}. \tag{1.8}$$

Now using condition $C^{\beta}_{0^{+}}x(0)=-C^{\beta}_{0^{+}}x(1)$ and (1.8) we have

$$x'(0)=-\Gamma\left(2-\beta\right)\int_{0}^{1}(1-s)^{\alpha+\mu-\beta-1}E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})\varphi(s)ds$$

which gives,

$$\begin{aligned} x(t)=&\int_{0}^{t}(t-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(t-s)^{\rho})\varphi(s)ds+x(0) \\ &-t\,\Gamma\left(2-\beta\right)\int_{0}^{1}(1-s)^{\mu+\alpha-\beta-1}\,E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})\varphi(s)ds. \end{aligned} \tag{1.9}$$

Finally, using condition $x(0)=-x(1)$ and (1.9)

$$\begin{aligned} x(0)=&-\frac{1}{2}\int_{0}^{1}(1-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(1-s)^{\rho})\varphi(s)ds \\ &+\frac{\Gamma\left(2-\beta\right)}{2}\int_{0}^{1}(1-s)^{\mu+\alpha-\beta-1}\,E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})\varphi(s)ds, \end{aligned}$$

after some simplifications we get

$$x(t)=\int_0^t (t-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(t-s)^{\rho})\varphi(s)ds-\frac{1}{2}\int_0^1 (1-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(1-s)^{\rho})\varphi(s)ds$$
$$+\left(\frac{\Gamma(2-\beta)(1-2t)}{2}\right)\int_0^1 (1-s)^{\mu+\alpha-\beta-1}E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})\varphi(s)ds$$

which completes the proof. □

2. Solvability of the fractional boundary value problem

This section is devoted to the solvability of the BVP (1.4). For this purpose we shall obtain some existence and uniqueness results for the solution $x(t)$ of (1.4) by using some well known Banach fixed point theorems. Recall that, $C[0,1]$ is a Banach space with

$$\|x\|_{C[0,1]}=\max_{t\in[0,1]}|x(t)|.$$

Theorem 2.1 ([15])**.** *Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $W:=\{v\in X: v=\lambda Tv,\ 0<\lambda<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 2.2 ([15])**.** *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator such that;*

$$\|Tv\|\le\|v\|,\qquad \forall v\in\partial\Omega$$

then T has a fixed point in $\overline{\Omega}$.

Theorem 2.3 (Banach fixed point theorem)**.** *Let (X,d) be a complete metric space, and let $F:X\to X$, be a contraction mapping, then F has a unique fixed point.*

Consider the operator $T:C[0,1]\to C[0,1]$ by

$$Tx(t):=\int_0^t (t-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(t-s)^{\rho})f(t,x(t))ds-\frac{1}{2}\int_0^1 (1-s)^{\mu+\alpha-1}E^{\gamma}_{\rho,\mu+\alpha}(\omega(1-s)^{\rho})f(s,x(s))ds$$
$$+\left(\frac{\Gamma(2-\beta)(1-2t)}{2}\right)\int_0^1 (1-s)^{\mu+\alpha-\beta-1}E^{\gamma}_{\rho,\mu+\alpha-\beta}(\omega(1-s)^{\rho})f(s,x(s))ds$$

then, $x\in C[0,1]$ is a solution of (1.4) if and only if x is a fixed point of T.

Theorem 2.4. *Assume that there exists a positive constant M_1 such that $|f(t,x)|\le M_1$ for $t\in[0,1]$, $x\in C([0,1],\mathbb{R})$. Then the problem (1.4) has at least one solution.*

Proof. First of all the operator T is continuous on $C[0,1]$. Now assume that $\Omega\subset C[0,1]$ is a bounded subset then using the assumption $|f(t,x)|\le M_1$ for $t\in[0,1]$, $x\in C([0,1],\mathbb{R})$ we have,

$$\begin{aligned}|Tx(t)|&\le\frac{M_1}{2}\left[3\left\|E^{\gamma}_{\rho,\mu+\alpha,\omega;0^+}(1)\right\|+\Gamma(2-\beta)\left\|E^{\gamma}_{\rho,\mu+\alpha-\beta,\omega;0^+}(1)\right\|\right]\\&\le\frac{M_1}{2}\left[3E^{\gamma}_{\rho,\mu+\alpha+1}(|\omega|)\ +\Gamma(2-\beta)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}(|\omega|)\right]=M_2\end{aligned}\tag{2.1}$$

which implies that $\|Tx(t)\| \leq M_2$. On the other hand, by Lemma 1.3, we easily get that

$$\begin{aligned}\left|T'x(t)\right| &\leq M_1\left[\left\|E^{\gamma}_{\rho,\mu+\alpha-1,\omega,0}\left(1\right)\right\| + \Gamma\left(2-\beta\right)\left\|E^{\gamma}_{\rho,\mu+\alpha-\beta,\omega,0}\left(1\right)\right\|\right] \\ &\leq M_1\left[E^{\gamma}_{\rho,\mu+\alpha}\left(|\omega|\right) + \Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}\left(|\omega|\right)\right] \leq M_3.\end{aligned}$$

For arbitrary $t_1, t_2 \in [0,1]$, we get,

$$|Tx(t_1) - Tx(t_2)| \leq \int_{t_1}^{t_2}\left|T'x(s)\right| ds \leq M_3\left(t_2 - t_1\right).$$

Therefore T is equicontinuous, moreover as a consequence of Arzela-Ascoli theorem it is also completely continuous. Consider the set $W := \{v \in C[0,1] : v = \lambda Tv,\ 0 < \lambda < 1\}$ and let x be an arbitrary element of W then $x = \lambda Tx$, for some $\lambda \in (0,1)$. Since,

$$|x(t)| = \lambda\,|Tx(t)| \leq |Tx(t)| \leq M_2, \quad \forall t \in [0,1],$$

where M_2 is given in (2.1) we have

$$\|x\|_{C[0,1]} \leq M_2.$$

This implies that W is bounded. Therefore as a consequence of Theorem 2.1, T has at least one fixed point and (1.4) has at least one solution. □

Theorem 2.5. *Let* $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$. *Then the BVP* (1.4) *has at least one solution.*

Proof. By the assumption that $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$, $\exists\ s, \delta > 0$, such that $|f(t,x(t)| \leq \delta\,|x|$ for $|x| < s$, and

$$\frac{\delta}{2}\left[3E^{\gamma}_{\rho,\mu+\alpha+1}(|\omega|)\ + \Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}(|\omega|)\right] \ \leq 1.$$

Now define the set $V := \{x \in C[0,1] : \|x\| < s\}$ and let $x \in \partial V$ be arbitrary. Then

$$|Tx(t)| \leq \frac{1}{2}\left[3E^{\gamma}_{\rho,\mu+\alpha+1}(|\omega|)\ + \Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}(|\omega|)\right]\delta\,\|x\|.$$

Above inequality implies that $\|Tx\| \leq \|x\|$, $\forall x \in \partial V$. Therefore as a conclusion of Theorem 2.2, T has at least one fixed point and (1.4) has at least one solution. □

Theorem 2.6. *Suppose that the following conditions holds:*

$$\text{there exists } 0 < M < \frac{2}{\left[3E^{\gamma}_{\rho,\mu+\alpha+1}(|\omega|)\ + \Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}(|\omega|)\right]} \tag{2.2}$$

and

$$|f(t,x) - f(t,y)| \leq M\,|x-y| \ \text{ for } t \in [0,1] \ \text{ and } x, y \in \mathbb{R}.$$

Then the boundary value problem (1.4) *has a unique solution.*

Proof. By the definition of the operator T we have,

$$\begin{aligned}|T(x(t)) - T(y(t))| &\leq \frac{M\,\|x-y\|}{2}\left[3\left\|E^{\gamma}_{\rho,\mu+\alpha,\omega;0^+}\left(1\right)\right\| + \Gamma\left(2-\beta\right)\left\|E^{\gamma}_{\rho,\mu+\alpha-\beta,\omega;0^+}\left(1\right)\right\|\right] \\ &\leq \frac{M}{2}\left[3E^{\gamma}_{\rho,\mu+\alpha+1}\left(|\omega|\right) + \Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}\left(|\omega|\right)\right]\|x-y\| \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) we can get that

$$\|T(x) - T(y)\|_{C[0,1]} \leq L\,\|x-y\|_{C[0,1]}.$$

Therefore T is a contraction. By the Theorem 2.3, T has a unique fixed point and boundary value problem (1.4) has a unique solution. □

Finally, we shall illustrate our results on suitable examples.

Example 2.7. Consider the following anti-periodic boundary value problem,

$$\begin{cases} C_{0^+}^{\frac{3}{2}}x(t) = E_{1,\frac{1}{2},2;0^+}^{\frac{2}{3}}(3t^2(1+\cos^2 x), & t\in[0,1]\,, \\ x(0) = -x(1), \quad C_{0^+}^{\frac{1}{2}}x(0) = -C_{0^+}^{\frac{1}{2}}x(1)\ . \end{cases} \tag{2.4}$$

Using Lemma 1.1, the fact that $\Gamma(k+2) \le \Gamma\left(k+\frac{5}{2}\right)$ for $k = 0, 1, \cdots$ and choose M_1, M_2 and M_3 as follows,

$$|f(t,x)| = 3t^2(1+\cos^2 x) \le 6 = M_1.$$

$$\begin{aligned} &\frac{M_1}{2}\left[3E_{\rho,\mu+\alpha+1}^{\gamma}(|\omega|) \ + \Gamma(2-\beta)\, E_{\rho,\mu+\alpha-\beta+1}^{\gamma}(|\omega|)\right] \\ &\quad = 3\left[3E_{1,3}^{\frac{2}{3}}(2) \ + \Gamma\left(\frac{3}{2}\right) E_{1,\frac{5}{2}}^{\frac{2}{3}}(2)\right] \le 3\left[3E_{1,3}(2) \ + \frac{\sqrt{\pi}}{2} E_{1,\frac{5}{2}}(2)\right] \\ &\quad \le 3\left[\frac{3}{4}(e^2-3) \ + \frac{\sqrt{\pi}}{4}(e^2-1)\right] = \left[\left(\frac{9+3\sqrt{\pi}}{4}\right)e^2 - \left(\frac{27+3\sqrt{\pi}}{4}\right)\right] = M_2 \end{aligned}$$

and

$$\begin{aligned} M_1\left[E_{1,2}^{\frac{2}{3}}(2) + \Gamma\left(\frac{3}{2}\right) E_{1,\frac{5}{2}}^{\frac{2}{3}}(2)\right] &\le 6\left[E_{1,2}(2) + \Gamma\left(\frac{3}{2}\right) E_{1,\frac{5}{2}}(2)\right] \\ &\le \frac{3}{2}(e^2-1)\left(2+\sqrt{\pi}\right) = M_3. \end{aligned}$$

As a consequence of Theorem 2.4, the BVP (2.4) has at least one solution.

Example 2.8. Consider the following anti-periodic boundary value problem,

$$\begin{cases} C_{0^+}^{\frac{3}{2}}x(t) = E_{1,\frac{1}{2},1;0^+}^{\frac{5}{2}}(x^3\left(t^2+1\right)), & t\in[0,1]\,, \\ x(0) = -x(1), \quad C_{0^+}^{1/2}x(0) = -C_{0^+}^{1/2}x(1). \end{cases} \tag{2.5}$$

Obviously,

$$\lim_{x\to 0} \frac{x^3\left(t^2+1\right)}{x} = 0.$$

If we choose δ as follows:

$$\begin{aligned} \delta &\le \frac{2}{3E_{1,3}^{\frac{5}{2}}(1) \ + \Gamma(2-\beta)\, E_{1,\frac{5}{2}}^{\frac{5}{2}}(1)} \le \frac{2}{3E_{1,3}(1) \ + \Gamma(2-\beta)\, E_{1,\frac{5}{2}}(1)} \\ &\le \frac{2}{3E_{1,3}(1) \ + \Gamma(2-\beta)\, E_{1,3}(1)} = \frac{2}{E_{1,3}(1)\,(3 \ + \Gamma(2-\beta))} \\ &\le \frac{2}{(e-2)\left(3+\frac{\sqrt{\pi}}{2}\right)} \le \frac{4}{(e-2)\,(6+\sqrt{\pi})}, \end{aligned}$$

then by Theorem 2.5, BVP (2.5) has at least one solution.

Example 2.9. Consider the following anti-periodic boundary value problem,

$$\begin{cases} C_{0^+}^{\frac{5}{4}}x(t) = E_{2,\frac{3}{4},1;0^+}^{\frac{3}{4}}\left(\frac{t^2}{10}\sin x\right), & t\in[0,1]\,, \\ x(0) = -x(1), \ C_{0^+}^{\frac{1}{2}}x(0) = -C_{0^+}^{\frac{1}{2}}x(1). \end{cases} \tag{2.6}$$

Then choose M as follows,

$$\begin{aligned}
&\frac{2}{\left[3E^{\gamma}_{\rho,\mu+\alpha+1}(|\omega|)\ +\Gamma\left(2-\beta\right)E^{\gamma}_{\rho,\mu+\alpha-\beta+1}(|\omega|)\right]}\\
&\quad=\frac{2}{3E^{\frac{3}{4}}_{2,3}(1)\ +\Gamma\left(\frac{3}{2}\right)E^{\frac{3}{4}}_{2,\frac{5}{2}}(1)}\geq\frac{2}{3E_{2,3}(1)\ +\Gamma\left(\frac{3}{2}\right)E_{2,\frac{5}{2}}(1)}\\
&\quad\geq\frac{2}{3E_{2,3}(1)\ +\Gamma\left(\frac{3}{2}\right)E_{2,2}(1)}\geq\frac{2}{3e\ +\frac{\sqrt{\pi}}{2}e}\\
&\quad>\frac{2}{e\left(3+\frac{\sqrt{\pi}}{2}\right)}=\frac{4}{e\left(6+\sqrt{\pi}\right)}=M,
\end{aligned}$$

which also satisfies,

$$|f(t,x)-f(t,y)|=\frac{t^2}{10}\left|\sin x-\sin y\right|\leq\frac{t^2}{10}\left|x-y\right|\leq\frac{1}{10}\left|x-y\right|\leq M\left|x-y\right|.$$

Therefore, as a consequence of Theorem 2.6, BVP (2.6) has a unique solution.

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