Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 413–423
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Superstability of Pexiderized functional equations arising from distance measures

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Communicated by Choonkil Park

Abstract

In this paper, we obtain the superstability of the functional equation

$$f(pr, qs) + g(ps, qr) = \theta(pq, rs)h(p, q)k(r, s)$$

for all $p, q, r, s \in G$, where G is an Abelian group, f, g, h, k are functionals on G^2, and θ is a cocycle on G^2. This functional equation is a generalized form of the functional equation $f(pr, qs)+f(ps, qr) = f(p, q)\, f(r, s)$, which arises in the characterization of symmetrically compositive sum-form distance measures and the information measures, and also they can be represented as products of some multiplicative functions and the exponential functional equations. As corollaries, we obtain the superstability of the many functional equations (combination of three variables functions, for example: $f(pr, qs) + g(ps, qr) = \theta(pq, rs)h(p, q)g(r, s)$).

Keywords: Distance measure, superstability, multiplicative function, stability of functional equation.
2010 MSC: 39B82, 46S40

1. Introduction

Let $(G, \cdot)$ be a commutative semigroup. Let I denote the open unit interval $(0, 1)$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ be a set of positive real numbers and $\mathbb{R}_w = \{x \in \mathbb{R} \,|\, x > w > 0\}$ for some $k \in \mathbb{R}$.

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Received 2015-07-13

Further, let

$$\Gamma_n^o = \left\{ P = (p_1, p_2, ..., p_n) \,\middle|\, 0 < p_k < 1, \sum_{k=1}^{n} p_k = 1 \right\}$$

denote the set of all n-ary discrete complete probability distributions (without zero probabilities), that is Γ_n^o is the class of discrete distributions on a finite set Ω of cardinality n with $n \geq 2$. Almost all similarity, affinity or distance measures $\mu_n : \Gamma_n^o \times \Gamma_n^o \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented in the *sum-form*

$$\mu_n(P, Q) = \sum_{k=1}^{n} \phi(p_k, q_k), \tag{1.1}$$

where $\phi : I \times I \to \mathbb{R}$ is a real-valued function on unit square, or a monotonic transformation of the right side of (1.1), that is,

$$\mu_n(P, Q) = \psi \left(\sum_{k=1}^{n} \phi(p_k, q_k) \right), \tag{1.2}$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is an increasing function on $\mathbb{R}$. The function ϕ is called a *generating function.* It is also referred to as the *kernel* of $\mu_n(P, Q)$.

In information theory, for P and Q in Γ_n^o, the symmetric divergence of degree α is defined as

$$J_{n,\alpha}(P, Q) = \frac{1}{2^{\alpha-1} - 1} \left[\sum_{k=1}^{n} \left(p_k^{\alpha} q_k^{1-\alpha} + p_k^{1-\alpha} q_k^{\alpha} \right) - 2 \right].$$

For all $P, Q \in \Gamma_n^o$, we define the product

$$P \cdot R = (p_1 r_1, p_1 r_2, ..., p_1 r_m, p_2 r_1, ..., p_2 r_m, ..., p_n r_m).$$

Chung, Kannappan, Ng and Sahoo [1] characterized symmetrically compositive sum-form distance measures with a measurable generating function. The following functional equation

$$f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s) \tag{FE}$$

holding for all $p, q, r, s \in I$ was instrumental in the characterization of symmetrically compositive sum-form distance measures.

They obtained that the general solution of equation (FE) is represented by $f(p, q) = M_1(p)\, M_2(q) + M_1(q)\, M_2(p)$, where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.

The stability of the functional equation (FE), as well as the four generalizations of (FE), namely,

$$f(pr, qs) + f(ps, qr) = f(p, q) g(r, s), \tag{FE_{fg}}$$

$$f(pr, qs) + f(ps, qr) = g(p, q) f(r, s), \tag{FE_{gf}}$$

$$f(pr, qs) + f(ps, qr) = g(p, q) g(r, s), \tag{FE_{gg}}$$

$$f(pr, qs) + f(ps, qr) = g(p, q) h(r, s) \tag{FE_{gh}}$$

for all $p, q, r, s \in G$, were studied by Kim and Sahoo in [13], [12]. For other functional equations similar to (FE), the interested reader should refer to [4], [5], [15]. J. Tabor [16] investigated the cocycle property. The definition of a cocycle as follows:

Definition 1.1. A function $\theta : G^2 \to \mathbb{R}$ is a cocycle if it satisfies the equation

$$\theta(a, bc)\theta(b, c) = \theta(ab, c)\theta(a, b), \quad \forall\, a, b, c \in G.$$

For example, if $F(x,y) = \frac{f(x)f(y)}{f(xy)}$ for a function $f : \mathbb{R} \to \mathbb{R}_+$, then F is a cocycle. Also if $\theta(x,y) = \ln(x)\ln(y)$ for a function $\theta : \mathbb{R}_+^2 \to (\mathbb{R},+)$, then θ is a cocycle, that is, $\theta(a,bc)+\theta(b,c) = \theta(ab,c)+\theta(a,b)$, and in this case, it is well known that $\theta(x,y)$ is represented by $B(x,y)+M(xy)-M(x)-M(y)$ where B is an arbitrary skew-symmetric biadditive function and M is some function [2]. If $\theta(x,y) = a^{\ln(x)\ln(y)}$, then $\theta : \mathbb{R}_+^2 \to (\mathbb{R},\cdot)$ is a cocycle and in this case, $\theta(x,y)$ is represented by $e^{B(x,y)}e^{M(xy)-M(x)-M(y)}$.

Let us consider the generalized characterization of a symmetrically compositive sum-form related to distance measures with a cocycle:

$$f(pr,qs)+f(ps,qr) = \theta(pq,rs)\, f(p,q)\, f(r,s), \tag{CDM}$$

for all $p,q,r,s \in G$ and where f, θ are functionals on G^2, which can be represented as exponential functional equation in reduction.

In fact, if $f(x,y) = \frac{1}{x}+\frac{1}{y}$, then $f(pr,qs)+f(ps,qr) = f(p,q)\, f(r,s)$, and also if $f(x,y) = a^{\ln xy}$, and $\theta(x,y) = 2$ then f,θ satisfy the equation $f(pr,qs)+f(ps,qr) = \theta(pq,rs)\, f(p,q)\, f(r,s)$.

The superstability of (CDM) and four generalized functional equations of (CDM) namely,

$$f(pr,qs)+f(ps,qr) = \theta(pq,rs)f(p,q)g(r,s), \tag{CM_{fffg}}$$
$$f(pr,qs)+f(ps,qr) = \theta(pq,rs)g(p,q)f(r,s), \tag{CM_{ffgf}}$$
$$f(pr,qs)+f(ps,qr) = \theta(pq,rs)g(p,q)g(r,s), \tag{CM_{ffgg}}$$
$$f(pr,qs)+f(ps,qr) = \theta(pq,rs)g(p,q)h(r,s) \tag{CM_{ffgh}}$$

for all $p,q,r,s \in G$, were studied by Lee and Kim in [14].

The present work continues the study for the superstability of the more generalized Pexider type functional equation

$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)h(p,q)k(r,s) \tag{CDM_{fghk}}$$

than (CM_{ffgh}) considered in Lee and Kim [14].

As corollaries, due to a combination of three variable functions, we obtain the superstability of the following functional equations, namely,

$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)g(p,q)k(r,s), \tag{CDM_{fggk}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)f(p,q)k(r,s), \tag{CDM_{fgfk}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)h(p,q)g(r,s), \tag{CDM_{fghg}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)h(p,q)f(r,s), \tag{CDM_{fggf}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)h(p,q)h(r,s), \tag{CDM_{fghh}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)h(p,q)f(r,s), \tag{CDM_{fghf}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)f(p,q)f(r,s), \tag{CDM_{fgff}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)f(p,q)g(r,s), \tag{CDM_{fgfg}}$$
$$f(pr,qs)+g(ps,qr) = \theta(pq,rs)g(p,q)g(r,s). \tag{CDM_{fggg}}$$

We will skip appearance for half of above equations and remainder equations.

In reduction, the above equations can be represented as a (hyperbolic) cosine(sine, trigonometric) functional equation, exponential, and Jensen functional equation, respectively.

Indeed, it should be noted that many well known functional equations like d'Alembert functional equation, Wilson functional equation, Jensen functional equation can be obtained from the functional equation (CDM_{fghk}). For instance, letting $r = s = 1$, cocycle $\theta(pq,rs) = 1$ in (CDM_{fghk}), one obtains the equation

$$f(p,q)+g(p,q) = k(1,1)\, h(p,q), \quad \forall\, p,q \in J. \tag{1.3}$$

When $f(p,q) = \psi(p+q)$, $g(p,q) = \psi(p-q)$, and $k(1,1)\,h(p,q) = 2\,\psi(p)\,\psi(q)$, then the equation (1.3) yields the well known d'Alembert functional equation. Similarly, when $f(p,q) = \psi(p+q)$, $g(p,q) = \psi(p-q)$, and $k(1,1)\,h(p,q) = \psi(p)\,\phi(q)$, then (1.3) yields the Wilson functional equation. Letting $f(p,q) = \psi(p+q)$, $g(p,q) = \psi(p-q)$, and $k(1,1)\,h(p,q) = 2\,\psi(p)$ it is easy to see that (1.3) reduces to Jensen functional equation. For stability of related functional equations see [6], [7], [8], [9], [10], [11] and [14]. The book [3] is an excellent source for reference on stability of functional equations.

2. Superstability of equations

In this section, we investigate the superstability of the equation (CDM_{fghk}) bounded by the two variables cases $\phi(r,s), \phi(p,q)$. In this section, M and M' are some nonnegative constants.

Theorem 2.1. *Let $f,\ g,\ h,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr,qs) + g(ps,qr) - \theta(pq,rs)h(p,q)k(r,s)| \le \phi(r,s) \tag{2.1}$$

with $|h(p,q) - f(p,q)| \le M$, and $|h(p,q) - g(p,q)| \le M'$ for all $p,q,r,s \in G$. Then, either h is bounded or k is a solution of (CDM). In particular, if h satisfies (CDM), then k and h satisfy the equation

$$k(pr,qs) + k(ps,qr) = \theta(pq,rs)\ h(p,q)\ k(r,s),$$

without above bounded condition by M and M'.

Proof. Let h be an unbounded solution of inequality (2.1). Then, there exists a sequence $\{(x_n,y_n)|n \in N\}$ in G^2 such that $0 \ne |h(x_n,y_n)| \to \infty$ as $n \to \infty$.

Letting $p = x_n$, $q = y_n$ in (2.1) and dividing $|\theta(x_ny_n,rs)h(x_n,y_n)|$, we have

$$\Big|\frac{f(x_nr,y_ns) + g(x_ns,y_nr)}{\theta(x_ny_n,rs)h(x_n,y_n)} - k(r,s)\Big| \le \frac{\phi(r,s)}{w|h(x_n,y_n)|}.$$

Passing to the limit as $n \to \infty$, we obtain that

$$k(r,s) = \lim_{n\to\infty} \frac{f(x_nr,y_ns) + g(x_ns,y_nr)}{\theta(x_ny_n,rs)h(x_n,y_n)}. \tag{2.2}$$

Letting $p = x_np$, $q = y_nq$ in (2.1) and dividing $|h(x_n,y_n)|$, we have

$$\Big|\frac{f(x_npr,y_nqs) + g(x_nps,y_nqr)}{h(x_n,y_n)} - \frac{\theta(x_npy_nq,rs)h(x_np,y_nq)}{h(x_n,y_n)}k(r,s)\Big| \le \frac{\phi(r,s)}{|h(x_n,y_n)|} \to 0 \tag{2.3}$$

as $n \to \infty$. Letting $p = x_nq$, $q = y_np$ in (2.1) and dividing $|h(x_n,y_n)|$, we have

$$\Big|\frac{f(x_nqr,y_nps) + g(x_nqs,y_npr)}{h(x_n,y_n)} - \frac{\theta(x_nqy_np,rs)h(x_nq,y_np)}{h(x_n,y_n)}k(r,s)\Big| \le \frac{\phi(r,s)}{|h(x_n,y_n)|} \to 0 \tag{2.4}$$

as $n \to \infty$. Note that for any $a,\ b,\ c$ in G, $\theta(ba,c)\theta(b,a) = \theta(b,ac)\theta(a,c)$ by the definition of the cocycle. Letting $pq = a,\ x_ny_n = b$ and $rs = c$ we have

$$\frac{\theta(x_ny_npq,rs)\theta(x_ny_n,pq)}{\theta(x_ny_n,pqrs)} = \theta(pq,rs)$$

for any $p,\ q,\ r,\ s,\ x_n,\ y_n$ in G. Thus, from (2.2), (2.3) and (2.4), we obtain

$$\Big|k(pr,qs) + k(ps,qr) - \theta(pq,rs)k(p,q)k(r,s)\Big|$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big| \frac{f(x_npr, y_nqs) + g(x_nqs, y_npr) + f(x_nps, y_nqr) + g(x_nqr, y_nps)}{\theta(x_ny_n, prqs)h(x_n, y_n)} \\
&\qquad - \theta(pq, rs)k(p, q)k(r, s)\Big| \\
&\le \lim_{n\to\infty} \Big|\frac{1}{\theta(x_ny_n, prqs)}\Big| \cdot \Big| \frac{f(x_npr, y_nqs) + g(x_nps, y_nqr)}{h(x_n, y_n)} \\
&\qquad - \frac{\theta(x_npy_nq, rs)h(x_np, y_nq)k(r, s)}{h(x_n, y_n)}\Big| \\
&\quad + \lim_{n\to\infty} \Big|\frac{1}{\theta(x_ny_n, prqs)}\Big| \cdot \Big| \frac{f(x_nqr, y_nps) + g(x_nqs, y_npr)}{h(x_n, y_n)} \\
&\qquad - \frac{\theta(x_nqy_np, rs)h(x_nq, y_np)k(r, s)}{h(x_n, y_n)}\Big| \\
&\quad + |k(r, s)| \lim_{n\to\infty} \Big| \frac{\theta(x_ny_npq, rs)\theta(x_ny_n, pq)}{\theta(x_ny_n, pqrs)} \cdot \frac{h(x_np, y_nq) + h(x_nq, y_np)}{\theta(x_ny_n, pq)h(x_n, y_n)} \\
&\qquad - \theta(pq, rs)k(p, q)\Big| \\
&\le k(r, s)\theta(pq, rs) \lim_{n\to\infty} \Big| \frac{f(x_np, y_nq) + g(x_nq, y_np)}{\theta(x_ny_n, pq)h(x_n, y_n)} \\
&\qquad + \frac{(h - f)(x_np, y_nq) + (h - g)(x_nq, y_np)}{\theta(x_ny_n, pq)h(x_n, y_n)} - k(p, q)\Big| \\
&\le k(r, s)\theta(pq, rs) \lim_{n\to\infty} \Big| \frac{M + M'}{wh(x_n, y_n)}\Big| \\
&\quad + k(r, s)\theta(pq, rs) \lim_{n\to\infty} \Big| \frac{f(x_np, y_nq) + g(x_nq, y_np)}{\theta(x_ny_n, pq)h(x_n, y_n)} - k(p, q)\Big| \\
&= 0.
\end{aligned}
$$

□

Theorem 2.2. *Let $f,\ g,\ h,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)k(r, s)| \le \phi(p, q), \tag{2.5}$$

with $|k(p, q) - f(p, q)| \le M$,and $|k(p, q) - g(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either k is bounded or h is a solution of (CDM).

In addition, if k satisfies the equation (CDM), then h and k satisfies the equation

$$h(pr, qs) + h(ps, qr) = \theta(pq, rs)h(p, q)\, k(r, s),$$

without above bounded condition by M and M'.

Proof. For k to be an unbounded solution of inequality (2.5), we can choose a sequence $\{(x_n, y_n)|n \in N\}$ in G^2 such that $0 \neq |k(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$, $s = y_n$ in (2.5) and dividing $|\theta(pq, x_ny_n)k(x_n, y_n)|$, we have

$$\Big| \frac{f(px_n, qy_n) + g(py_n, qx_n)}{\theta(pq, x_ny_n)k(x_n, y_n)} - h(p, q)\Big| \le \frac{\phi(p, q)}{w|k(x_n, y_n)|}.$$

Passing to the limit as $n \to \infty$, we obtain that

$$h(p, q) = \lim_{n\to\infty} \frac{f(px_n, qy_n) + g(py_n, qx_n)}{\theta(pq, x_ny_n)k(x_n, y_n)}. \tag{2.6}$$

Replacing $r = rx_n$, $s = sy_n$ in (2.5) and dividing $|k(x_n, y_n)|$, we have

$$\Big|\frac{f(prx_n, qsy_n) + g(psy_n, qrx_n)}{k(x_n, y_n)} - \theta(pq, rx_n sy_n)h(p, q)\frac{k(rx_n, sy_n)}{k(x_n, y_n)}\Big| \le \frac{\phi(p, q)}{|k(x_n, y_n)|} \to 0 \tag{2.7}$$

as $n \to \infty$. Replacing $r = ry_n$, $s = sx_n$ in (2.5) and dividing $|k(x_n, y_n)|$, we have

$$\Big|\frac{f(pry_n, qsx_n) + g(psx_n, qry_n)}{k(x_n, y_n)} - h(p, q)\theta(pq, ry_n sx_n)\frac{k(ry_n, sx_n)}{k(x_n, y_n)}\Big| \le \frac{\phi(p, q)}{|k(x_n, y_n)|} \to 0 \tag{2.8}$$

as $n \to \infty$. Thus from (2.6), (2.7), and (2.8), we obtain

$$\begin{aligned}
&\Big|h(pr, qs) + h(ps, qr) - \theta(pq, rs)h(p, q)h(r, s)\Big| \\
&= \lim_{n\to\infty}\Big|\frac{f(prx_n, qsy_n) + g(pry_n, qsx_n) + f(psx_n, qry_n) + g(psy_n, qrx_n)}{\theta(prqs, x_ny_n)k(x_n, y_n)} \\
&\qquad - \theta(pq, rs)h(p, q)h(r, s)\Big| \\
&\le \lim_{n\to\infty}\Big|\frac{1}{\theta(pqrs, x_ny_n)}\Big| \cdot \Big|\frac{f(prx_n, qsy_n) + g(psy_n, qrx_n)}{k(x_n, y_n)} \\
&\qquad - h(p, q)\theta(pq, rx_n sy_n)\frac{k(rx_n, sy_n)}{k(x_n, y_n)}\Big| \\
&\quad + \lim_{n\to\infty}\Big|\frac{1}{\theta(pqrs, x_ny_n)}\Big| \cdot \Big|\frac{f(pry_n, qsx_n) + g(psx_n, qry_n)}{k(x_n, y_n)} \\
&\qquad - h(p, q)\theta(pq, ry_n sx_n)\frac{k(ry_n, sx_n)}{k(x_n, y_n)}\Big| \\
&\quad + |h(p, q)| \lim_{n\to\infty}\Big|\frac{\theta(pq, rx_n sy_n)\theta(rs, x_ny_n)}{\theta(pqrs, x_ny_n)} \cdot \frac{k(rx_n, sy_n) + k(ry_n, sx_n)}{\theta(rs, x_ny_n)k(x_ny_n)} \\
&\qquad - \theta(pq, rs)h(r, s)\Big| \\
&= |h(p, q)|\theta(pq, rs) \quad \lim_{n\to\infty}\Big|\frac{(k - f)(rx_n, sy_n) + (k - g)(ry_n, sx_n)}{\theta(rs, x_ny_n)k(x_n, y_n)} \\
&\quad + \frac{f(rx_n, sy_n) + g(ry_n, sx_n)}{\theta(rs, x_ny_n)k(x_n, y_n)} - h(r, s)\Big| \\
&\le |h(p, q)|\theta(pq, rs)\frac{M + M'}{w|k(x_n, y_n)|} \\
&\quad + |h(p, q)|\theta(pq, rs) \lim_{n\to\infty}\Big|\frac{f(rx_n, sy_n) + g(ry_n, sx_n)}{\theta(rs, x_ny_n)k(x_n, y_n)} - h(r, s)\Big| \\
&= 0.
\end{aligned}$$

□

We can obtain many corollaries by reducing of functions in above two theorems. Namely, the reduced functional equations are maked by three functions, two functions and one function. In here, we only will represent the equations reduced by three functions. The representation of the other reduced equations will be skip.

Corollary 2.3. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)h(r, s)| \le \phi(r, s) or \phi(p, q), \tag{2.9}$$

with $|h(p, q) - f(p, q)| \le M$, and $|h(p, q) - g(p, q)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either h is bounded or h is a solution of (CDM).

Corollary 2.4. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)f(r, s)| \le \phi(r, s), \tag{2.10}$$

with $|h(p, q) - f(p, q)| \le M$, and $|h(p, q) - g(p, q)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either h is bounded or f is a solution of (CDM). In particular, if h satisfies (CDM), then f and h satisfy the solutions of (CM_{ffgf}) without above bounded condition by M and M' , that is

$$f(pr, qs) + f(ps, qr) = \theta(pq, rs)\ h(p, q)\ f(r, s).$$

Corollary 2.5. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)f(r, s)| \le \phi(p, q), \tag{2.11}$$

with $|f(p, q) - g(q, p)| \le M$, and $|f(p, q) - g(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either f is bounded or h is a solution of (CDM).

In addition, if f satisfies the equation (CDM), then h and f satisfies the equation

$$h(pr, qs) + h(ps, qr) = \theta(pq, rs)h(p, q)\ f(r, s),$$

without above bounded condition by M and M'.

Corollary 2.6. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)g(r, s)| \le \phi(r, s), \tag{2.12}$$

with $|h(p, q) - f(p, q)| \le M$, and $|h(p, q) - g(p, q)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either h is bounded or g is a solution of (CDM). In particular, if h satisfies (CDM), then g and h satisfy the equation

$$g(pr, qs) + g(ps, qr) = \theta(pq, rs)\ h(p, q)\ g(r, s),$$

without above bounded condition by M and M'.

Corollary 2.7. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)g(r, s)| \le \phi(p, q), \tag{2.13}$$

with $|f(p, q) - g(p, q)| \le M$, and $|f(p, q) - g(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either g is bounded or h is a solution of (CDM).

In addition, if g satisfies the equation (CDM), then h and g satisfies the equation

$$h(pr, qs) + h(ps, qr) = \theta(pq, rs)h(p, q)\ g(r, s),$$

without above bounded condition by M and M'.

Corollary 2.8. *Let $f,\ g,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)f(p, q)k(r, s)| \le \phi(r, s), \tag{2.14}$$

$|f(p, q) - g(p, q)| \le M$ for all $p, q, r, s \in G$. Then, either f is bounded or k is a solution of (CDM). In particular, if f satisfies (CDM), then k and f satisfy the equation

$$k(pr, qs) + k(ps, qr) = \theta(pq, rs)\ f(p, q)\ k(r, s),$$

without above bounded condition by M.

Corollary 2.9. *Let $f,\ g,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)f(p, q)k(r, s)| \le \phi(p, q), \tag{2.15}$$

with $|k(p, q) - f(p, q)| \le M$,and $|k(p, q) - g(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either k is bounded or f is a solution of (CDM).

In addition, if k satisfies the equation (CDM), then f and k satisfies the equation

$$f(pr, qs) + f(ps, qr) = \theta(pq, rs)f(p, q)\,k(r, s),$$

without above bounded condition by M and M'.

Corollary 2.10. *Let $f,\ g,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)g(p, q)k(r, s)| \le \phi(r, s), \tag{2.16}$$

with $|g(p, q) - f(p, q)| \le M$ for all $p,\ q,\ r,\ s \in G$. Then, either g is bounded or k is a solution of (CDM). In particular, if g satisfies (CDM), then k and g satisfy the equation

$$k(pr, qs) + k(ps, qr) = \theta(pq, rs)\ g(p, q)\ k(r, s),$$

without above bounded condition by M.

Corollary 2.11. *Let $f,\ g,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)g(p, q)k(r, s)| \le \phi(p, q), \tag{2.17}$$

with $|k(p, q) - f(p, q)| \le M$, and $|k(p, q) - g(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either k is bounded or g is a solution of (CDM).

In addition, if k satisfies the equation (CDM), then g and k satisfies the equation

$$g(pr, qs) + g(ps, qr) = \theta(pq, rs)g(p, q)\,k(r, s),$$

without above bounded condition by M and M'.

Corollary 2.12. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + f(ps, qr) - \theta(pq, rs)g(p, q)h(r, s)| \le \phi(r, s), \tag{2.18}$$

with $|g(p, q) - f(p, q)| \le M$ for all $p,\ q,\ r,\ s \in G$. Then, either g is bounded or h is a solution of (CDM). In particular, if g satisfies (CDM), then h and g satisfy the equation

$$h(pr, qs) + h(ps, qr) = \theta(pq, rs)\ g(p, q)\ h(r, s),$$

without above bounded condition by M.

Corollary 2.13. *Let $f,\ g,\ h : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + f(ps, qr) - \theta(pq, rs)g(p, q)h(r, s)| \le \phi(p, q), \tag{2.19}$$

with $|h(p, q) - f(p, q)| \le M$,and $|h(p, q) - f(q, p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either h is bounded or g is a solution of (CDM).

In addition, if h satisfies the equation (CDM), then g and h satisfies the equation

$$g(pr, qs) + g(ps, qr) = \theta(pq, rs)g(p, q)\,h(r, s),$$

without above bounded condition by M and M'.

Corollary 2.14. *Let $f,\ g,\ h,\ k : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$|f(pr, qs) + g(ps, qr) - \theta(pq, rs)h(p, q)k(r, s)| \le \varepsilon, \tag{2.20}$$

with $|k(p,q) - f(p,q)| \le M$ and $|k(p,q) - g(q,p)| \le M'$ for all $p,\ q,\ r,\ s \in G$. Then, either h (or k) is bounded or k (or h) is a solution of (CDM), respectively. In addition,

(i) If h satisfies (CDM), then k and h satisfy (CDM_{kkhk}) without above bounded condition by M and M',

$$k(pr, qs) + k(ps, qr) = \theta(pq, rs)\ h(p, q)\ k(r, s).$$

(ii) If k satisfies (CDM), then h and k satisfies CDM_{hhhk} without above bounded condition by M and M',

$$h(pr, qs) + h(ps, qr) = \theta(pq, rs)h(p, q)\, k(r, s).$$

Corollary 2.15. *Let $(S; \diamond)$ be a commutative semigroup with operation $\diamond$. Let $f,\ g,\ h,\ k : S^2 \to \mathbb{R}$ and $\phi : S^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(p \diamond r, q \diamond s) + g(p \diamond s, q \diamond r) - \theta(p \diamond q, r \diamond s)h(p, q)k(r, s)| \le \begin{cases} (i) & \phi(r, s) \\ (ii) & \phi(p, q) \end{cases} \tag{2.21}$$

for all $p,\ q,\ r,\ s \in S$

(a) In case (i), let $|h(p,q) - f(p,q)| \le M$ and $|h(p,q) - g(p,q)| \le M'$.

Then, either h is bounded or k is a solution of (CDM). In particular, if h satisfies (CDM), then k and h satisfy the equation

$$k(p \diamond r, q \diamond s) + k(p \diamond s, q \diamond r) = \theta(p \diamond q, r \diamond s)\ h(p, q)\ k(r, s),$$

without above bounded condition by M and M'.

(b) In case (ii), let $|k(p,q) - f(p,q)| \le M$ and $|k(p,q) - g(q,p)| \le M'$.

Then, either k is bounded or h is a solution of (CDM). In addition, if k satisfies the equation (CDM), then h and k satisfies the equation

$$h(p \diamond r, q \diamond s) + h(p \diamond s, q \diamond r) = \theta(p \diamond q, r \diamond s)h(p, q)\, k(r, s),$$

without above bounded condition by M and M'.

Remark 2.16. (i) As Corollary 2.14, letting $\phi(r,s) = \phi(p,q) = \varepsilon$ in all corollary, then we obtain the same type results.

(ii) For the following equations reduced to two functions : (CDM_{fgfg}), (CDM_{fgff}), (CDM_{fggf}), (CDM_{fggg}), (CDM_{fghf}), (CM_{fffg}), (CM_{ffgf}), (CM_{ffgg}), (CM_{ffgh}), (FE_{gf}), (FE_{gg}), (FE_{fg}), and (FE_{gh}) under cocycle condition $\theta(pq, rs) = 1$, we can obtain the same results. In this case, note that f is bounded iff g is bounded, by using of this, we can obtain more good results(see [12], [13]).

(iii) For example of a cocycle function $\theta : \mathbb{R}_+^2 \to \mathbb{R}_+$, apply $\theta(pq, rs) = k^{\ln(pq)\ln(rs)}$, $\theta(pq, rs) = c$: constant, etc,.

(iv) In Corollary 2.15, replacing the operation $\diamond$ on S to $+$ in all results, then we obtain same results for each corollary.

3. Extension of results to Banach algebra

All results in Section 2 can be extended to the superstability on the Banach algebra. In this section, let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra.

Theorem 3.1. *Let $f,\ g,\ h,\ k : G^2 \to E$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$\|f(pr,qs) + g(ps,qr) - \theta(pq,rs)h(p,q)k(r,s)\| \le \phi(r,s), \quad \forall\, p,q,r,s \in G \tag{3.1}$$

and let $\|h(p,q) - f(p,q)\| \le M$, and $\|h(p,q) - g(p,q)\| \le M'$ for all $p,\ q,\ r,\ s \in G$.

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ h$ is bounded or k satisfies the equation (CDM).

In particular, the superposition $x^* \circ h$ satisfies the equation (CDM), then k and h satisfy the equation

$$k(pr,qs) + k(ps,qr) = \theta(pq,rs)\ h(p,q)\ k(r,s),$$

without above bounded condition by M and M'.

Proof. Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E$. As well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}\phi(r,s) &\ge \|f(pr,qs) + g(ps,qr) - \theta(pq,rs)h(p,q)k(r,s)\| \\ &\ge \big|x^*\big(f(pr,qs)\big) + x^*\big(g(ps,qr)\big) - \theta(pq,rs)x^*\big(h(p,q)\big)x^*\big(k(r,s)\big)\big|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, $x^* \circ h$, and $x^* \circ k$ yield solutions of inequality (3.1). Since the superposition $x^* \circ h$ is unbounded, an appeal to Theorem 2.1 shows that the function $x^* \circ k$ solves the equation (CDM). In other words, bearing the linear multiplicativity of x^* in mind, for all $p,q,r,s \in G$, the difference

$$\mathcal{D}GM_{fghk}(p,q,r,s) := k(pr,qs) + k(ps,qr) - \theta(pq,rs)k(p,q)k(r,s),$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}GM_{fghk}(p,q,r,s) \in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $p,\ q,\ r,\ s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$k(pr,qs) + k(ps,qr) - \theta(pq,rs)k(p,q)k(r,s) = 0, \qquad \text{for all} \quad p,q,r,s \in G$$

as claimed.

The additional case also can be check easily. □

Theorem 3.2. *Let $f,\ g,\ h,\ k : G^2 \to E$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_w$ be a cocycle satisfying*

$$\|f(pr,qs) + g(ps,qr) - \theta(pq,rs)h(p,q)k(r,s)\| \le \phi(p,q), \quad \forall\, p,q,r,s \in S$$

and let $\|k(p,q) - f(p,q)\| \le M$and $\|k(p,q) - g(q,p)\| \le M'$ for all $p,\ q,\ r,\ s \in G$.

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ k$ is bounded or h satisfies the equation (CDM).

In particular, if the superposition $x^* \circ h$ satisfies the equation (CDM), then h and k satisfies the equation

$$h(pr,qs) + h(ps,qr) = \theta(pq,rs)h(p,q)\,k(r,s),$$

without above bounded condition by M and M'.

Remark 3.3. As Theorems 3.1 and 3.2, All results of the section 2 can be extended to the Banach algebra.

Acknowledgment

The first author and the second author of this work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2010-0010243) and (Grant No. 2011-0021253), respectively.

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