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Unbounded solutions of second order discrete BVPs on infinite intervals

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Abstract

In this paper, we study Sturm-Liouville boundary value problems for second order difference equations on a half line. By using the discrete upper and lower solutions, the Schäuder fixed point theorem, and the degree theory, the existence of one and three solutions are investigated. An interesting feature of our existence theory is that the solutions may be unbounded.

Keywords: Coincidence point, discrete boundary value problem, infinite interval, upper solution, lower solution, degree theory common fixed point.
2010 MSC: 47H10, 54H25, 54C60.

1. Introduction

In this paper, we study Sturm-Liouville boundary value problems for second order difference equations on an infinite interval

$$\begin{cases} -\triangle^2 x_{k-1} = f\big(k, x_k, \triangle x_{k-1}\big), & k \in \mathbb{N}, \\ x_0 - a\triangle x_0 = B, \qquad \triangle x_\infty = C, \end{cases} \tag{1.1}$$

where $\triangle x_k = x_{k+1} - x_k$ is the forward difference operator. $\mathbb{N} = \{1, 2, \cdots, \infty\}$ and $f : \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. $a > 0$, B, $C \in \mathbb{R}$, $\triangle x_\infty = \lim\limits_{k\to\infty} \triangle x_k$. Recall that the map $f : \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous if it maps continuously the topological space $\mathbb{N} \times \mathbb{R}^2$ into $\mathbb{R}$. The topology on $\mathbb{N}$ is the discrete topology. By a solution x of (1.1), we mean a sequence $x = (x_0, x_1, \cdots, x_n, \cdots)$ which satisfy (1.1). We will provide

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sufficient conditions on f so that the discrete boundary value problem (1.1) have one solution, and three solutions. An important aspect of our existence theory is that the solutions may be unbounded.

Recently, the existence of linear and nonlinear discrete boundary value problems has been studied by many authors. We refer here to some works using the upper and lower solutions method, e.g., see [3, 10, 11, 13, 14, 15, 18, 20, 21, 24, 25, 28, 29] for finite interval problems, and [1, 5, 7, 27] for infinite interval problems. Discrete infinite interval problems have also been studied by several other methods in [2, 4, 6, 8, 9, 12, 16, 17, 19, 21, 22, 26].

In [5], R. P. Agarwal and D. O'Regan studied the existence of nonnegative solutions to the boundary value problem

$$\begin{cases} \triangle^2 x(i-1) + f\big(i, x(i)\big) = 0, \\ x(0) = 0, \qquad \lim\limits_{i\to\infty} x(i) = 0. \end{cases}$$

They employed upper and lower solutions on finite intervals, and a diagnolization process, to prove the existence of at least one nonnegative (bounded) solution. Later, similar methods were used for the existence of solutions to such discrete BVP and on the time scales, see [1, 7].

In [25], Y. Tian, C. C. Tisdell and Weigao Ge established the existence of three (bounded) solutions of the discrete boundary value problem

$$\begin{cases} \triangle^2 x(n-1) - p\triangle x(n-1) - qx(n-1) + f\big(n, x(n), \triangle x(n)\big) = 0, \\ x(0) - \gamma x(l) = x_0, \qquad x(n) \ \text{ bounded on } [0,\infty). \end{cases}$$

For this, they assumed the existence of two pairs of upper and lower solutions on finite intervals, and used the sequential arguments and the degree theory.

As far as we know the existence of unbounded solutions for the discrete boundary value problems has not been studied. The only known work where unbounded positive solutions of second order nonlinear neutral delay difference equation have been established is a recent contribution of Zeqing Liu *et al.* [17].

Since the infinite interval is noncompact, the discussion here is more complicated compared to finite interval problems. In Section 2, we shall begin with the whole discrete infinite interval and introduce a new Banach space. Here discrete Arezà-Ascoli lemma is also established, which is necessary to prove that the summation mapping is compact. In Section 3, we will show that in the presence of a pair of upper and lower solutions the problem (1.1) has a solution. For this we shall apply the Schäuder fixed point theorem. Here to show how easily our result can be applied in practice two examples are also illustrated. In Section 4, we shall employ the topological degree theory to show that the problem (1.1), in the presence of two pairs of upper and lower solutions, has three solutions.

2. Definitions and Green's function

Let $\mathbb{N}_0$ be the set of all nonnegative integers and S be the space of sequences, i.e., by $x \in S$, we means $x = \{x_k\}_{k\in\mathbb{N}_0}$. For $x, y \in S$, we write $x \leqslant y$ if $x_k \leqslant y_k$ for all $k \in \mathbb{N}_0$. We consider

$$S_\infty = \left\{ x \in S : \lim_{k\to\infty} \triangle x_k \text{ exists} \right\}$$

endowed with the norm

$$\|x\| = \max\{\|x\|_1,\ \|\triangle x\|_\infty\},$$

where $\triangle x = \{\triangle x_k\}_{k\in\mathbb{N}_0}$, $\|x\|_1 = \sup\limits_{k\in\mathbb{N}_0} \frac{|x_k|}{1+k}$, $\|x\|_\infty = \sup\limits_{k\in\mathbb{N}_0} |x_k|$. Because $\lim\limits_{k\to\infty} \triangle x_k$ exists, $\{\triangle x_k\}_{k\in\mathbb{N}_0}$ is bounded. If we denote by $M = \sup\limits_{k\in\mathbb{N}_0} |\triangle x_k|$, then it follows that

$$\begin{aligned}\frac{|x_k|}{1+k} &= \frac{1}{1+k}\left|x_0 + \sum_{i=0}^{k-1} \triangle x_i\right| \\ &\leqslant \frac{|x_0|}{1+k} + \frac{1}{1+k}\sum_{i=0}^{k-1} |\triangle x_i| \\ &\leqslant \frac{|x_0|}{1+k} + \frac{k}{1+k} M.\end{aligned}$$

Hence, $\sup\limits_{k\in\mathbb{N}_0} \frac{|x_k|}{1+k} < \infty$. It is clear that $(S_\infty, \|\cdot\|)$ is a normed linear space. We claim that it is in fact a Banach space.

Lemma 2.1. $(S_\infty, \|\cdot\|)$ *is a Banach space.*

Proof. We shall prove its completeness. Suppose $\{x^{(n)}\}_{n=1}^\infty \subset S_\infty$ is a Cauchy sequence. Then $\{y^{(n)} : y_k^{(n)} = \frac{x_k^{(n)}}{1+k},\ k \in \mathbb{N}_0\}$ and $\{z^{(n)} :\ z_k^{(n)} = \triangle x_k^{(n)},\ k \in \mathbb{N}_0\}$ are bounded for each $n \in \mathbb{N}$. Now since for any $k \in \mathbb{N}_0$, $\{y_k^{(n)}\}_{n\in\mathbb{N}}$ and $\{z_k^{(n)}\}_{n\in\mathbb{N}}$ are Cauchy sequences in R, there exist two sequences y^* and z^* in S such that

$$\|y^{(n)} - y^*\|_\infty \to 0, \quad \text{and} \quad \|z^{(n)} - z^*\|_\infty \to 0, \quad \text{as } n \to \infty.$$

Clearly, y^* and z^* are bounded. Let $x_k^* = (1+k)y_k^*$ and $x^* = (x_1^*, x_2^*, \cdots, x_k^*, \cdots)$, then

$$\|x^{(n)} - x^*\|_1 \to 0, \quad \text{as } n \to \infty.$$

But this means that for each $k \in \mathbb{N}_0$,

$$\lim_{n\to\infty} x_k^{(n)} = x_k^*.$$

Further,

$$\begin{aligned}\triangle x_k^* = x_{k+1}^* - x_k^* &= \lim_{n\to\infty} x_{k+1}^{(n)} - \lim_{n\to\infty} x_k^{(n)} = \lim_{n\to\infty} (x_{k+1}^{(n)} - x_k^{(n)}) \\ &= \lim_{n\to\infty} \triangle x_k^{(n)} = z_k^*, \qquad k = 1, 2, \ldots\end{aligned}$$

Hence, $\|x^{(n)} - x^*\| \to 0 (n \to \infty)$. The proof is completed. □

Lemma 2.2. *If* $e = \{e_k\}_{k\in\mathbb{N}}$ *satisfy* $\sum\limits_{k=1}^\infty e_k < \infty$*, then the linear discrete BVP*

$$\begin{cases} -\triangle^2 x_{k-1} = e_k, & k \in \mathbb{N}, \\ x_0 - a\triangle x_0 = B, & \triangle x_\infty = C, \end{cases} \tag{2.1}$$

has a unique solution in S_∞*. Further, this solution can be expressed as*

$$x_k = aC + B + kC + \sum_{i=1}^\infty G(k,i)e_i, \qquad k \in \mathbb{N}_0,$$

where

$$G(k,i) = \begin{cases} a+i, & i \leqslant k, \\ a+k, & i > k. \end{cases} \tag{2.2}$$

We define $T : S_\infty \to S$ by

$$(Tx)_k = aC + B + kC + \sum_{i=1}^\infty G(k,i) f(i, x_i, \triangle x_{i-1}), \qquad k \in \mathbb{N}_0.$$

Clearly, x is a solution of problem (1.1) if and only if x is a fixed point of the mapping T.

Definition 2.3. A function $\alpha \in S$ is called a lower solution of (1.1) provided

$$\begin{cases} -\triangle^2\alpha_{k-1} \leqslant f\big(k,\alpha_k,u\big), \\ \alpha_0 - a\triangle\alpha_0 \leqslant B, \qquad \triangle\alpha_\infty < C \end{cases} \tag{2.3}$$

for all $k \in \mathbb{N}$ and $u \leqslant \triangle\alpha_{k-1}$. If all inequalities are strict, it will be called a strict lower solution.

Definition 2.4. A function $\beta \in S$ is called an upper solution of (1.1) provided

$$\begin{cases} -\triangle^2\beta_{k-1} \geqslant f\big(k,\beta_k,v\big), \\ \beta_0 - a\triangle\beta_0 \geqslant B, \qquad \triangle\beta_\infty > C \end{cases} \tag{2.4}$$

for all $k \in \mathbb{N}$ and $v \geqslant \triangle\beta_{k-1}$. If all inequalities are strict, it will be called a strict upper solution.

Definition 2.5. Let α, β be the lower and upper solutions for the problem (1.1) satisfying $\alpha \leqslant \beta$. We say that f satisfies a discrete Bernstein Nagumo condition with respect to α and β if there exist positive functions $\psi \in C(\mathbb{N})$ and $h \in C[0,+\infty)$ such that

$$|f(k,x_k,y)| \leqslant \psi(k)h(|y|)$$

for all $k \in \mathbb{N}$ and $\alpha_k \leqslant x_k \leqslant \beta_k$ with h nondecreasing, and

$$\sum_{i=1}^{\infty}\psi(i) < \infty, \qquad \int^{+\infty}\frac{s}{h(s)}ds = \infty.$$

We will use the Schäuder fixed point theorem to obtain a fixed point of the mapping T. To show the mapping is compact, the following generalized discrete Arezà-Ascoli lemma will be used.

Lemma 2.6 ([4])**.** *$M \subset B_\infty = \{x \in S : \lim_{k\to\infty} x_k$ exists.$\}$ is relatively compact if it is uniformly bounded and uniformly convergent at infinity.*

Lemma 2.7. *$M \subset S_\infty$ is relatively compact if it is uniformly bounded and uniformly convergent at infinity, that is, for each $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that*

$$\left|\frac{x_k}{1+k} - \lim_{k\to\infty}\frac{x_k}{1+k}\right| < \epsilon, \qquad and \qquad |\triangle x_k - \triangle x_\infty| < \epsilon, \qquad k > K$$

for all $x \in M$.

Proof. $M \subset S_\infty$ is relatively compact if every sequence of M has a convergent subsequence. First, we will show that $\lim_{k\to\infty}\frac{x_k}{1+k}$ exists for any $x = \{x_k\}_{k\in\mathbb{N}_0} \in S_\infty$. Since $\lim_{k\to\infty}\triangle x_k$ exists, we can denote its limit by c. Now $\forall\epsilon > 0$, there exists a $K = K(\epsilon,x) > 0$ such that

$$|\triangle x_k - c| < \epsilon, \qquad \forall k > K,$$

which implies that

$$x_{K+1} + (k-K-1)c - (k-K-1)\epsilon < x_k < x_{K+1} + (k-K-1)c + (k-K-1)\epsilon,$$

and hence either $\{x_k\}_{k\in\mathbb{N}_0}$ is bounded or x_k tends to infinity as $k \to \infty$. For the later case, by using the Stolz rule (the discrete L'Hospital rule), we have

$$\lim_{k\to\infty}\frac{x_k}{1+k} = \lim_{k\to\infty}\triangle x_k \text{ exists.}$$

Now, consider the sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subset S_\infty$. Since

$$\left\{ y^{(n)}:\ y_k^{(n)} = \frac{x_k^{(n)}}{1+k},\ k\in\mathbb{N}_0 \right\}_{n\in\mathbb{N}} \subset B_\infty$$

and

$$\left\{ z^{(n)}:\ z_k^{(n)} = \triangle x_k^{(n)},\ k\in\mathbb{N}_0 \right\}_{n\in\mathbb{N}_0} \subset B_\infty,$$

the conditions of Lemma 2.7 guarantee that they both have convergent subsequences. Without loss of generality, we write these convergent subsequences as $y^{(n)}$ and $z^{(n)}$ satisfying

$$\lim_{n\to\infty} y^{(n)} = y^*, \quad \text{and} \quad \lim_{n\to\infty} z^{(n)} = z^*.$$

Now following the discussion as in Lemma 2.1, we can show that

$$\|x^{(n)} - x^*\| \to 0, \quad n\to\infty,$$

where $x^* = \{x_k^*\}_{k\in\mathbb{N}_0}$, $x_k^* = (1+k)y_k^*$ and $\triangle x_k^* = z_k^*$. □

3. The existence of a solution

Theorem 3.1. *Assume that*

(H_1) The discrete boundary value problem (1.1) *has one pair of upper and lower solution α and β in S_∞ satisfying $\alpha \leqslant \beta$.*

(H_2) $f\in C(\mathbb{N}_0\times\mathbb{R}^2,\mathbb{R})$ satisfies the Bernstein-Nagumo condition with respect to α and β.

(H_3) There exists $\gamma>1$ such that $\sup_{k\in\mathbb{N}}(1+k)^\gamma\psi(k)<\infty$.

Then the discrete boundary value problem (1.1) *has at least one solution x satisfying*

$$\alpha \leqslant x \leqslant \beta, \quad \|\triangle x\|_\infty \leqslant R,$$

where $R>0$ is a constant (independent of the solution x).

Proof. We choose $\eta, R > C$ such that

$$\eta \geqslant \max\left\{ \sup_{k\in\mathbb{N}} \frac{\beta_k-\alpha_0}{k},\ \sup_{k\in\mathbb{N}} \frac{\beta_0-\alpha_k}{k} \right\},$$
$$\int_\eta^R \frac{s}{h(s)}ds > M\left(\sup_{k\in\mathbb{N}} \frac{\beta_k}{(1+k)^\gamma} - \inf_{k\in\mathbb{N}} \frac{\alpha_k}{(1+k)^\gamma} + N\sum_{i=0}^{\infty} \frac{(2+i)^\gamma-(1+i)^\gamma}{(2+i)^{\gamma-1}(1+i)^\gamma} \right),$$

where C is the nonhomgeneous boundary value, and $M=\sup_{k\in\mathbb{N}}(1+k)^\gamma\psi(k)$, $N=\max\{\|\alpha\|,\ \|\beta\|\}$.

Define the auxiliary functions $F_0,\ F_1:\ \mathbb{N}\times\mathbb{R}^2\to\mathbb{R}$ as follows

$$F_0(k,x,y) = \begin{cases} f\big(k,\beta_k,y\big) - \frac{x-\beta_k}{k^2(1+|x-\beta_k|)}, & x>\beta_k, \\ f\big(k,x,y\big), & \alpha_k\leqslant x\leqslant\beta_k, \\ f\big(k,\alpha_k,y\big) + \frac{x-\alpha_k}{k^2(1+|x-\alpha_k|)}, & x<\alpha_k, \end{cases}$$

and

$$F_1(k,x,y) = \begin{cases} F_0\big(k,x,R\big), & y>R, \\ F_0\big(k,x,y\big), & -R\leqslant y\leqslant R, \\ F_0\big(k,x,-R\big), & y<-R. \end{cases}$$

Clearly, F_1 is a continuous function on $\mathbb{N}\times\mathbb{R}^2$ and satisfies

$$|F_1(k,x,y)| \leqslant \psi(k)h(R)+\frac{1}{k^2}, \quad \text{for all} \quad (k,x,y)\in\mathbb{N}\times\mathbb{R}^2.$$

Consider the modified boundary value problem

$$\begin{cases} -\triangle^2 x_{k-1} = F_1\big(k,x_k,\triangle x_{k-1}\big), & k\in\mathbb{N},\\ x_0 - a\triangle x_0 = B, \qquad \triangle x_\infty = C. \end{cases} \tag{3.1}$$

To complete the proof, it suffices to show that (3.1) has at least one solution $x=\{x_k\}_{k\in N_0}$ such that

$$\alpha_k \leqslant x_k \leqslant \beta_k, \quad \text{and} \quad |\triangle x_k|\leqslant R, \qquad k\in\mathbb{N}_0.$$

We divided the proof into the following three steps.

Step 1. Problem (3.1) has a solution.

To show that the problem (3.1) has a solution, we define the operator $T_1 : S_\infty \to S$ as

$$(T_1x)_k = aC + B + kC + \sum_{i=1}^{\infty} G(k,i)F_1\big(i,x_i,\triangle x_{i-1}\big), \quad k\in\mathbb{N}_0. \tag{3.2}$$

From Lemma 2.2, we can see that the fixed points of T_1 are the solutions of (3.1). We will prove that $T_1 : S_\infty \to S_\infty$ is completely continuous and has at least one fixed point from the Schäuder fixed point theorem.

For any $x\in S_\infty$, because

$$\left|\sum_{i=1}^{\infty} G(k,i)F_1(i,x_i,\triangle x_{i-1})\right| \leqslant (a+k)\sum_{i=1}^{\infty}\left(\psi(i)h(R)+\frac{1}{i^2}\right)<\infty,$$

for any $k\in\mathbb{N}_0$, from the definition of T_1, we have

$$\begin{aligned}\lim_{k\to\infty}\triangle(T_1x)_k &= \lim_{k\to\infty}\big((T_1x)_{k+1}-(T_1x)_k\big)\\ &= C+\sum_{i=1}^{\infty}\lim_{k\to\infty}\big(G(k+1,i)-G(k,i)\big)F_1\big(i,x_i,\triangle x_{i-1}\big)\\ &= C.\end{aligned}$$

Thus, $T_1S_\infty \subset S_\infty$.

Next, for any convergent sequence $x^{(m)}\to x$ as $m\to\infty$ in S_∞, we have

$$\begin{aligned}\|T_1x^{(m)}-T_1x\|_1 &= \sup_{k\in\mathbb{N}_0}\left|\frac{(T_1x^{(m)})_k}{1+k}-\frac{(T_1x)_k}{1+k}\right|\\ &\leqslant \sup_{k\in\mathbb{N}_0}\sum_{i=1}^{\infty}\frac{G(k,i)}{1+k}\left|F_1\big(i,x_i^{(m)},\triangle x_{i-1}^{(m)}\big)-F_1\big(i,x_i,\triangle x_{i-1}\big)\right|\\ &\leqslant \max\{a,1\}\sum_{i=1}^{\infty}\left|F_1\big(i,x_i^{(m)},\triangle x_{i-1}^{(m)}\big)-F_1\big(i,x_i,\triangle x_{i-1}\big)\right|\\ &\to 0, \ \text{ as } m\to\infty\end{aligned}$$

and

$$\begin{aligned}&\|\triangle(T_1x^{(m)})-\triangle(T_1x)\|_\infty\\ &= \sup_{k\in\mathbb{N}_0}\left|\sum_{i=1}^{\infty}(G(k+1,i)-G(k,i))\Big(F_1\big(i,x_i^{(m)},\triangle x_{i-1}^{(m)}\big)-F_1\big(i,x_i,\triangle x_{i-1}\big)\Big)\right|\\ &\leqslant \sum_{i=1}^{\infty}\left|F_1\big(i,x_i^{(m)},\triangle x_{i-1}^{(m)}\big)-F_1\big(i,x_i,\triangle x_{i-1}\big)\right|\\ &\to 0, \ \text{ as } m\to\infty.\end{aligned}$$

Therefore, T_1 is continuous. Finally, we will show that T_1 is compact, that is, T_1 maps bounded subsets of S_∞ into relatively compact sets. For this, let B be any bounded subset of S_∞, then there exists a constant $r > 0$ such that $\|x\| \leqslant r,\ \forall x \in B$.

For any $x \in B$, we have

$$\begin{aligned}
\|T_1x\|_1 &= \sup_{k\in\mathbb{N}_0}\left|\frac{(T_1x)_k}{1+k}\right| \\
&= \sup_{k\in\mathbb{N}_0}\left|\frac{aC+B+Ck}{1+k}+\sum_{i=1}^{\infty}\frac{G(k,i)}{1+k}F_1\big(i,x_i,\triangle x_{i-1}\big)\right| \\
&\leqslant \max\{|aC+B|,|C|\}+\max\{a,1\}\sum_{i=1}^{\infty}\left(h(r)\psi(i)+\frac{1}{i^2}\right) \\
&< +\infty
\end{aligned}$$

and

$$\begin{aligned}
\|\triangle(T_1x)\|_\infty &= \sup_{k\in\mathbb{N}_0}|\triangle(T_1x)_k| \\
&= \sup_{k\in\mathbb{N}_0}\left|C+\sum_{i=1}^{\infty}(G(k+1,i)-G(k,i))F_1\big(i,x_i,\triangle x_{i-1}\big)\right| \\
&\leqslant |C|+\sum_{i=1}^{\infty}|F_1\big(i,x_i,\triangle x_{i-1}\big)| \\
&\leqslant |C|+\sum_{i=1}^{\infty}\left(h(r)\psi(i)+\frac{1}{i^2}\right).
\end{aligned}$$

Thus, T_1B is uniformly bounded. From Lemma 2.7, if T_1B is uniformly convergent at infinity, then T_1B is relatively compact. In fact,

$$\begin{aligned}
&\left|\frac{(T_1x)_k}{1+k}-\lim_{k\to\infty}\frac{(T_1x)_k}{1+k}\right| \\
&\quad= \left|\frac{aC+B+Ck}{1+k}-C+\sum_{i=1}^{\infty}\left(\frac{G(k,i)}{1+k}-1\right)F_1\big(i,x_i,\triangle x_{i-1}\big)\right| \\
&\quad\leqslant \left|\frac{aC+B+Ck}{1+k}-C\right|+\sum_{i=1}^{\infty}\left|\frac{G(k,i)}{1+k}-1\right|\left(h(r)\psi(i)+\frac{1}{i^2}\right) \\
&\quad\to 0,\ \text{ as } k\to+\infty
\end{aligned}$$

and

$$\begin{aligned}
\left|\triangle(T_1x)_k-\lim_{k\to\infty}\triangle(T_1x)_k\right| &= \left|\sum_{i=1}^{\infty}(G(k+1,i)-G(k,i))F_1\big(i,x_i,\triangle x_{i-1}\big)\right| \\
&\leqslant \sum_{i=1}^{\infty}|G(k+1,i)-G(k,i)|\left(h(r)\psi(i)+\frac{1}{i^2}\right) \\
&\to 0,\ \text{ as } k\to+\infty.
\end{aligned}$$

Hence, we find that T_1B is relatively compact. Therefore, $T_1 : S_\infty \to S_\infty$ is completely continuous.

Now choose $N_1 > \max\{L_1,\ \|\alpha\|, \|\beta\|\}$, where

$$L_1 = \max\{|aC+B|,|C|\}+\max\{a,1\}\sum_{i=1}^{\infty}\left(\psi(i)h(R)+\frac{1}{i^2}\right) \tag{3.3}$$

and set $\Omega_1 = \{x \in S_\infty,\ \|x\| < N_1\}$. Then for any $x \in \overline{\Omega}_1$, it is easy to see that $\|T_1 x\| < N_1$, and thus $T_1\overline{\Omega}_1 \subset \Omega_1$. The Schäuder fixed point theorem now guarantees that the operator T_1 has at least one fixed point in S_∞, which is a solution of BVP (3.1).

Step 2: Every solution x of the problem (3.1) satisfies $\alpha \leqslant x \leqslant \beta$.

We assume that the right hand inequality does not hold. Then $x - \beta$ has a positive maximum in $\mathbb{N}_0$. The positive maximum does not occur at infinity because $\lim\limits_{k\to\infty} \triangle(x_k - \beta_k) < 0$. If the positive maximum occurs at 0, then $\triangle(x_0 - \beta_0) \leqslant 0$. However, we have

$$x_0 - a\triangle x_0 - (\beta_0 - a\triangle\beta_0) = (x_0 - \beta_0) - a\triangle(x_0 - \beta_0) > 0,$$

which is a contradiction to the left boundary condition.

If the positive maximum occurs at $k_0 \in \mathbb{N}$, then

$$x_{k_0} - \beta_{k_0} > 0, \quad \triangle(x_{k_0-1} - \beta_{k_0-1}) \geqslant 0, \quad \triangle(x_{k_0} - \beta_{k_0}) \leqslant 0$$

and

$$\triangle^2(x_{k_0-1} - \beta_{k_0-1}) \leqslant 0.$$

However, it follows from (2.4) and (3.1) that

$$\begin{aligned} -\triangle^2 x_{k_0-1} &= F_1\big(k_0, x_{k_0}, \triangle x_{k_0-1}\big) \\ &= f\big(k_0, \beta_{k_0}, \triangle x_{k_0-1}\big) - \frac{x_{k_0} - \beta_{k_0}}{k_0^2(1 + |x_{k_0} - \beta_{k_0}|)} \\ &\leqslant -\triangle^2\beta_{k_0-1} - \frac{x_{k_0} - \beta_{k_0}}{k_0^2(1 + |x_{k_0} - \beta_{k_0}|)} < -\triangle^2\beta_{k_0-1}, \end{aligned}$$

which is a contradiction. Thus, $x_k \leqslant \beta_k$ hold for all $k \in \mathbb{N}_0$. The proof for $x \geqslant \alpha$ is similar.

Step 3: If the solution x of the problem (3.1) satisfies $\alpha \leqslant x \leqslant \beta$, then $|\triangle x_k| \leqslant R, \quad \forall k \in \mathbb{N}_0$.

We claim that $|\triangle x_k| > \eta$ does not hold for all $k \in \mathbb{N}_0$. Otherwise, without loss of generality, we can suppose that $\triangle x_k > \eta$ for all $k \in \mathbb{N}_0$, but then it follows that

$$\frac{\beta_k - \alpha_0}{k} \geqslant \frac{x_k - x_0}{k} = \frac{1}{k}\sum_{i=0}^{k-1} \triangle x_i > \eta \geqslant \frac{\beta_k - \alpha_0}{k}, \quad \forall k \in \mathbb{N},$$

which is a contraction. Thus there must exist a $k_1 \in \mathbb{N}_0$ such that $|\triangle x_{k_1}| \leqslant \eta$.

If $|\triangle x_k| \leqslant R$ does not hold for all $k \in \mathbb{N}$, then there exists $k_2 \in \mathbb{N}_0$ such that $|\triangle x_{k_2}| > R$. Proceeding with this argument, we may suppose $k_2 > k_1$ and

$$0 \leqslant \triangle x_{k_1} \leqslant \eta < R \leqslant \triangle x_{k_2}, \qquad \eta \leqslant \triangle x_k \leqslant R, \qquad k_1 < k < k_2.$$

Let $I = \{i :\ k_1 < i \leqslant k_2,\ \triangle x_{k_i} > \triangle x_{k_i - 1}\}$ and $\bar{I} = (k_1, k_2] \cap N_0 \setminus I$. Then, we have

$$\begin{aligned} \int_\eta^R \frac{s}{h(s)} ds &\leq \sum_{i=k_1+1}^{k_2} \int_{\triangle x_{i-1}}^{\triangle x_i} \frac{s}{h(s)} ds = \sum_{i\in I} \int_{\triangle x_{i-1}}^{\triangle x_i} \frac{s}{h(s)} ds - \sum_{j\in \bar{I}} \int_{\triangle x_j}^{\triangle x_{j-1}} \frac{s}{h(s)} ds \\ &\leq \sum_{i\in I} \frac{1}{h(\triangle x_{i-1})} \int_{\triangle x_{i-1}}^{\triangle x_i} s ds - \sum_{j\in \bar{I}} \frac{1}{h(\triangle x_{j-1})} \int_{\triangle x_j}^{\triangle x_{j-1}} s ds \\ &= \sum_{i=k_1+1}^{k_2} \frac{(\triangle x_i + \triangle x_{i-1})}{2} \frac{\triangle^2 x_{i-1}}{h(\triangle x_{i-1})} \\ &\leq \sum_{i=k_1+1}^{k_2} \frac{(\triangle x_i + \triangle x_{i-1})}{2} \psi(i) \leqslant \frac{M}{2}\left(\sum_{i=k_1+1}^{k_2} \frac{\triangle x_i}{(1+i)^\gamma} + \sum_{i=k_1}^{k_2-1} \frac{\triangle x_i}{(1+i)^\gamma}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M}{2}\left(\frac{x_{k_2+1}}{(2+k_2)^\gamma}+\frac{x_{k_2}}{(1+k_2)^\gamma}-\frac{x_{k_1+1}}{(2+k_1)^\gamma}-\frac{x_{k_1}}{(1+k_1)^\gamma}\right)\\
&\quad +M\sup_{k\in\mathbb{N}_0}\frac{x_{k+1}}{2+k}\sum_{i=0}^{\infty}\frac{(2+i)^\gamma-(1+i)^\gamma}{(2+i)^{\gamma-1}(1+i)^\gamma)}\\
&\leq M\left(\sup_{k\in\mathbb{N}}\frac{\beta_k}{(1+k)^\gamma}-\inf_{k\in\mathbb{N}}\frac{\alpha_k}{(1+k)^\gamma}+N\sum_{i=0}^{\infty}\frac{(2+i)^\gamma-(1+i)^\gamma}{(2+i)^{\gamma-1}(1+i)^\gamma}\right),
\end{aligned}
$$

which is a contradiction. Hence, $\triangle x_k \leqslant R,\ k\in\mathbb{N}_0$. Here we note that the series

$$\sum_{i=0}^{\infty}\frac{(2+i)^\gamma-(1+i)^\gamma}{(2+i)^{\gamma-1}(1+i)^\gamma}$$

is convergent. In a similarly way, we can also show that $\triangle x_k \geqslant -R$ for all $k\in\mathbb{N}_0$. Hence there exists a $R>0$, independent of every solution x of (1.1), such that $\|\triangle x\|_\infty \leqslant R$. □

Example 1. Consider the Sturm-Liouville boundary value problem involving the second order difference equation

$$
\begin{cases}
\triangle^2 x_{k-1}+\dfrac{(3/2-\triangle x_{k-1})(2k+x_k)}{(1+k)^4}=0, & k\in\mathbb{N},\\
x_0-\triangle x_0=1, \qquad \triangle x_\infty = 1.
\end{cases}
\tag{3.4}
$$

Clearly, BVP (3.4) is a particular case of (1.1) with

$$f(k,x,y)=-\frac{(3/2-y)(2k+x)}{(1+k)^4},$$

and $a=1,\ B=1, C=1$. Consider the upper and lower solutions of (3.4) defined by

$$\alpha_k=-k,\ \beta_k=2k+3, \qquad k\in\mathbb{N}_0.$$

Here the function f is continuous and we will show that it satisfies the Bernstein Nagumo condition with respect to α and β. In fact, when $k\in\mathbb{N}_0,\ -k\leqslant x_k\leqslant 2k+3,\ y\in\mathbb{R}$, it follows that

$$
\begin{aligned}
|f(t,x_k,y)| &= \left|\frac{(3/2-y)(2k+x_k)}{(1+k)^4}\right|\\
&\leqslant \frac{1}{(1+k)^2}\left(\sup_{n\in\mathbb{N}_0}\frac{4k+3}{(1+k)^2}\right)(|y|+3/2)\\
&\leqslant \frac{3}{(1+k)^2}(|y|+3/2).
\end{aligned}
$$

Set $\psi(k)=\frac{1}{(1+k)^2}$, $h(s)=3(s+3/2)$ and $1<\gamma\leqslant 2$, then

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{(1+k)^2}<\infty,\\
&\sup_{k\in\mathbb{N}}(1+k)^\gamma\frac{1}{(1+k)^2}=\sup_{k\in\mathbb{N}}\frac{1}{(1+k)^{2-\gamma}}\leqslant 1<\infty,\\
&\int^{\infty}\frac{s}{h(s)}ds=\frac{1}{3}\int^{\infty}\frac{s}{s+3/2}ds=\infty.
\end{aligned}
$$

Hence, all conditions of Theorem 3.1 are satisfied, and thus the problem (3.4) has at least one nontrival solution x satisfying $-k\leqslant x_k\leqslant 2k+3$ for all $k\in\mathbb{N}$.

Example 2. Consider the Sturm-Liouville boundary value problem involving the second order difference equation

$$\begin{cases} \triangle^2 x_{k-1} - \dfrac{\triangle x_{k-1}+1}{(1+k)^4}\left(\sin x_k + x_k^2 + \sqrt{|x_k|}\right) = 0, \qquad k \in \mathbb{N}, \\ x_0 - 3\triangle x_0 = 0, \qquad \triangle x_\infty = \dfrac{1}{2}. \end{cases} \tag{3.5}$$

Clearly, BVP (3.5) is a particular case of (1.1) with

$$f(k,x,y) = -\frac{y+1}{(1+k)^4}\left(\sin x + x^2 + \sqrt{|x|}\right),$$

and $a = 3,\ B = 0, C = \frac{1}{2}$. Consider the upper and lower solutions of (3.5) defined by

$$\alpha_k = -k-4,\ \beta_k = k+4, \qquad k \in \mathbb{N}_0.$$

Here the function f is continuous and we will show that it satisfies the Bernstein Nagumo condition with respect to α and β. In fact, when $k \in \mathbb{N}_0,\ -k-4 \leqslant x_k \leqslant k+4,\ y \in \mathbb{R}$, it follows that

$$\begin{aligned} |f(t,x_k,y)| &= \left|\frac{y+1}{(1+k)^4}\left(\sin x + x^2 + \sqrt{|x|}\right)\right| \\ &\leqslant \frac{1}{(1+k)^2}\left(1 + \sup_{n\in\mathbb{N}_0}\frac{(k+4)^2}{(1+k)^2} + \sup_{n\in\mathbb{N}_0}\frac{\sqrt{k+4}}{(1+k)^2}\sqrt{|x|}\right)(|y|+1) \\ &\leqslant \frac{19}{(1+k)^2}(|y|+1). \end{aligned}$$

Set $\psi(k) = \frac{1}{(1+k)^2}$, $h(s) = 19(s+1)$ and $1 < \gamma \leqslant 2$, then

$$\begin{aligned} &\sum_{k=1}^{\infty}\frac{1}{(1+k)^2} = \frac{\pi^2}{6} - 1 < \infty, \\ &\sup_{k\in\mathbb{N}}(1+k)^\gamma\frac{1}{(1+k)^2} = \sup_{k\in\mathbb{N}}\frac{1}{(1+k)^{2-\gamma}} \leqslant 1 < \infty, \\ &\int^{\infty}\frac{s}{h(s)}ds = \frac{1}{19}\int^{\infty}\frac{s}{s+1}ds = \infty. \end{aligned}$$

Hence, Theorem 3.1 guarantees that problem (3.5) has at least one nontrival solution x satisfying $-k-4 \leqslant x_k \leqslant k+4$ for all $k \in \mathbb{N}$.

4. The multiplicity results

Here we shall show that in the presence of two pairs of upper and lower solutions the problem (1.1) has at least three solutions.

Theorem 4.1. *Suppose that the following condition holds.*

(H_4) *The discrete boundary value problem* (1.1) *has two pairs of upper and lower solutions* $\beta^{(j)},\ \alpha^{(j)},\ j = 1,2$ *in* S_∞ *with* $\alpha^{(2)},\ \beta^{(1)}$ *strict and*

$$\alpha_k^{(1)} \leqslant \alpha_k^{(2)} \leqslant \beta_k^{(2)}, \quad \alpha_k^{(1)} \leqslant \beta_k^{(1)} \leqslant \beta_k^{(2)}, \quad \alpha_k^{(2)} \nleqslant \beta_k^{(1)}, \quad k \in \mathbb{N}_0.$$

Suppose further that conditions (H_2) *and* (H_3) *hold with* $\alpha,\ \beta$ *replaced by* $\alpha^{(1)},\ \beta^{(2)}$ *respectively. Then the problem* (1.1) *has at least three solutions* $x^{(1)},\ x^{(2)}$ *and* $x^{(3)}$ *satisfying*

$$\alpha_k^{(j)} \leqslant x_k^{(j)} \leqslant \beta_k^{(j)}(j=1,2), \quad x_k^{(3)} \nleqslant \beta_k^{(1)} \ \text{ and } \ x_k^{(3)} \ngeqslant \alpha_k^{(2)}, \quad k \in \mathbb{N}_0.$$

Proof. Define the truncated function F_2, the same as F_1 in Theorem 3.1 with $\alpha,\ \beta$ replaced by $\alpha^{(1)}$ and $\beta^{(2)}$ respectively. Consider the modified difference equation

$$\begin{cases} -\triangle^2 x_{k-1} = F_2\big(k, x_k, \triangle x_{k-1}\big), & k \in \mathbb{N}, \\ x_0 - a\triangle x_0 = B, \quad \triangle x_\infty = C. \end{cases} \tag{4.1}$$

To show that the problem (4.1) has at least three solutions, we define a mapping $T_2 : S_\infty \to S_\infty$

$$(T_2 x)_k = aC + B + kC + \sum_{i=1}^{\infty} G(k,i) F_2\big(i, x_i, \triangle x_{i-1}\big), \quad k \in \mathbb{N}_0. \tag{4.2}$$

As in Theorem 3.1, T_2 is completely continuous. By using the degree theory, we will show that T_2 has at least three fixed points which coincide with the solutions of (4.1).

Choose $N_2 > \max\{L_2,\ \|\alpha^{(1)}\|, \|\beta^{(2)}\|\}$, where L_2 has the same expression as L_1 in (3.3) except that R given by $\alpha,\ \beta$ is now defined by $\alpha^{(1)},\ \beta^{(2)}$. Set $\Omega_2 = \{x \in S_\infty,\ \|x\| < N_2\}$. Then for any $x \in \overline{\Omega}_2$, it follows that $\|T_2 x\| < N_2$. Thus, $T_2\overline{\Omega}_2 \subset \Omega_2$, and so we have $\deg(I - T_2, \Omega_2, 0) = 1$.

Set

$$\Omega_{\alpha^{(2)}} = \left\{ x \in \Omega_2,\ x_k > \alpha_k^{(2)},\ k \in \mathbb{N}_0 \right\},$$
$$\Omega^{\beta^{(1)}} = \left\{ x \in \Omega_2,\ x_k < \beta_k^{(1)},\ k \in \mathbb{N}_0 \right\}.$$

Because $\alpha^{(2)} \not\leqslant \beta^{(1)},\ \alpha^{(1)} \leqslant \alpha^{(2)} \leqslant \beta^{(2)}$ and $\alpha^{(1)} \leqslant \beta^{(1)} \leqslant \beta^{(2)}$, we have

$$\Omega_{\alpha^{(2)}} \neq \emptyset, \quad \Omega^{\beta^{(1)}} \neq \emptyset, \quad \Omega_2 \setminus \overline{\Omega_{\alpha^{(2)}} \cup \Omega^{\beta^{(1)}}} \neq \emptyset, \quad \Omega_{\alpha^{(2)}} \cap \Omega^{\beta^{(1)}} = \emptyset.$$

Noticing that $\alpha^{(2)},\ \beta^{(1)}$ are strict lower and upper solutions, there is no solution on $\partial\Omega_{\alpha^{(2)}} \cup \partial\Omega^{\beta^{(1)}}$. Therefore

$$\begin{aligned} \deg(I - T_2, \Omega_2, 0) =& \deg(I - T_2, \Omega_2 \setminus \overline{\Omega_{\alpha^{(2)}} \cup \Omega^{\beta^{(1)}}}, 0) \\ &+ \deg(I - T_2, \Omega_{\alpha^{(2)}}, 0) + \deg(I - T_2, \Omega^{\beta^{(1)}}, 0). \end{aligned}$$

To show that

$$\deg(I - T_2, \Omega_{\alpha^{(2)}}, 0) = 1,$$

we define another mapping $T_3 :\ \overline{\Omega}_2 \to \overline{\Omega}_2$ by

$$(T_3 x)_k = aC + B + kC + \sum_{i=1}^{\infty} G(k,i) F_3\big(i, x_i, \triangle x_{i-1}\big), \quad k \in \mathbb{N}_0.$$

where the function F_3 is similar to F_2 except $\alpha^{(1)}$ is replaced by $\alpha^{(2)}$. Similar to the proof of Theorem 3.1, we find that x is a fixed point of T_3 only if $\alpha_k^{(2)} \leqslant x_k \leqslant \beta_k^{(2)},\ k \in \mathbb{N}_0$. So $\deg(I - T_3, \Omega \setminus \overline{\Omega_{\alpha^{(2)}}}, 0) = 0$. Thus from the Schäuder fixed point theorem and $T_3\overline{\Omega}_2 \subset \Omega_2$, we have $\deg(I - T_2, \Omega_2, 0) = 1$. Furthermore,

$$\begin{aligned} \deg(I - T_2, \Omega_{\alpha^{(2)}}, 0) =& \deg(I - T_3, \Omega_{\alpha^{(2)}}, 0) \\ =& \deg(I - T_3, \Omega_2, 0) + \deg(I - T_3, \Omega_2 \setminus \overline{\Omega_{\alpha^{(2)}}}, 0) = 1. \end{aligned}$$

Similarly, we have $\deg(I - T_2, \Omega^{\beta^{(1)}}, 0) = 1$. And then

$$\deg(I - T_2, \Omega_2 \setminus \overline{\Omega_{\alpha^{(2)}} \cup \Omega^{\beta^{(1)}}}, 0) = -1.$$

Using the properties of the degree, we conclude that T_2 has at least three fixed points $x^{(1)} \in \Omega_{\alpha^{(2)}}$, $x^{(2)} \in \Omega^{\beta^{(1)}}$ and $x^{(3)} \in \Omega_2 \setminus \overline{\Omega_{\alpha^{(2)}} \cup \Omega^{\beta^{(1)}}}$, which are the claimed three different solutions of the BVP (4.1). Similarly, we can show that $\alpha^{(1)} \leqslant x^{(i)} \leqslant \beta^{(2)}$ and $\|\triangle x^{(i)}\|_\infty \leqslant R$. Thus they are the solutions of BVP (1.1). $\square$

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