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On the Ulam stability of an n-dimensional quadratic functional equation

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Abstract

In the present paper, we construct a new n-dimensional quadratic functional equation with constant coefficients

$$\sum_{i,j=1}^{n} f(x_i + x_j) = 2 \sum_{1\le i<j\le n} f(x_i - x_j) + 4f\Big(\sum_{i=1}^{n} x_i\Big).$$

And then, we study the Ulam stability of the preceding equation in a real normed space and a non-Archimedean space, respectively.

Keywords: Ulam stability, n-dimensional quadratic functional equation, normed space, non-Archimedean space.
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1. Introduction

The investigation of the Ulam stability of functional equations originated from the following question proposed by Ulam [17] concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ such that $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

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In the following, Hyers [8] gave a first affirmative partial answer to the question mentioned above for Banach spaces. Afterwards, Hyers' theorem was generalized by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. It is worth mentioning that Rassias' work has a great impact on the development of the Ulam stability (It mainly includes Hyers-Ulam stability and Hyers-Ulam-Rassias stability) of functional equations. Hereafter, Găvruta [7] replaced the unbounded Cauchy difference in the result of Rassias and made a further generalization for Rassias' theorem by using a more general majorant function. Since then, the Ulam stability of various types of functional equations in different types of abstract spaces have been widely investigated by many authors. The vast majority of the results are included in the several important monographs, the reader can refer to [6, 9, 10, 16].

In order to study the Ulam stability of functional equations from a more general perspective, some generalized or n-dimensional functional equations were constructed and its Ulam stability was discussed, which probably included some basic quadratic or cubic functional equations as special cases [1, 2, 4].

In 2001, Bae and Jun [1] constructed the following n-dimensional quadratic functional equation and studied its Ulam stability

$$f\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{1\le i<j\le n}^{n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i).$$

Later, Bae and Park [2] and Nakmahachalasint [12] considered the Ulam stability of another n-dimensional quadratic functional equation

$$f\Big(\sum_{i=1}^{n} x_i\Big) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n}^{n} f(x_i + x_j).$$

Note that the coefficients of the preceding two equations depend on the number n of variables. Unlike this types of equations, in this paper, we construct an n-dimensional quadratic functional equation with constant coefficients, as follows

$$\sum_{i,j=1}^{n} f(x_i + x_j) = 2\sum_{1\le i<j\le n}^{n} f(x_i - x_j) + 4f\Big(\sum_{i=1}^{n} x_i\Big). \tag{1.1}$$

The objective of the present paper is to study the Ulam stability of Eq. (1.1) in a real normed space and a non-Archimedean space, respectively.

2. Ulam stability of Eq. (1.1) in a real Banach space

Throughout this section, let X and $(Y, \|\cdot\|)$ denote a real vector space and a real Banach space, respectively. Let $\mathbb{N}$ denote the set of all positive integers. For a given function $f : X \to Y$, we define the following difference operator

$$D_q f(x_1, x_2, \cdots, x_n) := \sum_{i,j=1}^{n} f(x_i + x_j) - 2\sum_{1\le i<j\le n}^{n} f(x_i - x_j) - 4f\Big(\sum_{i=1}^{n} x_i\Big). \tag{2.1}$$

Theorem 2.1. *Let $\varphi : X^n \to [0,\infty)$ be a function such that*

$$\Phi(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{\infty} 4^{-k}\varphi(2^k x_1, 2^k x_2, \cdots, 2^k x_n) < \infty \tag{2.2}$$

for all $x_1, x_2, \cdots, x_n \in X$. Assume that the function $f : X \to Y$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\| \le \varphi(x_1, x_2, \cdots, x_n) \tag{2.3}$$

for all $x_1, x_2, \cdots, x_n \in X$*. Then*

$$Q(x) = \lim_{m\to\infty} 4^{-m} f(2^m x)$$

exists for each $x \in X$ *and defines a unique quadratic function satisfying* (1.1) *such that*

$$\|f(x) - f(0) - Q(x)\| \leq \frac{1}{4}\Phi(x, 0, \cdots, 0) + \frac{1}{3}\varphi(0, 0, \cdots, 0) \tag{2.4}$$

for all $x \in X$*.*

Proof. Setting $x_1 = x_2 = \cdots = x_n = 0$ in (2.3), we have

$$\|(n-4)f(0)\| \leq \varphi(0, 0, \cdots, 0). \tag{2.5}$$

Putting $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$ in (2.3) gives

$$\|f(2x) - 4f(x) + (n-1)f(0)\| \leq \varphi(x, 0, \cdots, 0). \tag{2.6}$$

By (2.5) and (2.6), we can infer that

$$\begin{aligned}\|f(2x) - 4f(x) + 3f(0)\| &\leq \|f(2x) - 4f(x) + (n-1)f(0))\| + \| - (n-4)f(0)\| \\ &\leq \varphi(x, 0, \cdots, 0) + \varphi(0, 0, \cdots, 0).\end{aligned} \tag{2.7}$$

Dividing both sides of (2.7) by 4, we obtain

$$\left\|\frac{f(2x) - f(0)}{4} - (f(x) - f(0))\right\| \leq \frac{\varphi(x, 0, \cdots, 0)}{4} + \frac{\varphi(0, 0, \cdots, 0)}{4} \tag{2.8}$$

for all $x \in X$. Replacing x by $2^{m-1}x$ in (2.8) and dividing both sides of the resulting equality by 4^{m-1}, we get

$$\left\|\frac{f(2^m x) - f(0)}{4^m} - \frac{f(2^{m-1}x) - f(0)}{4^{m-1}}\right\| \leq \frac{\varphi(2^{m-1}x, 0, \cdots, 0)}{4^m} + \frac{\varphi(0, 0, \cdots, 0)}{4^m} \tag{2.9}$$

for all $x \in X$ and for all $m \in \mathbb{N}$. Then, it follows from (2.8) and (2.9) that

$$\left\|f(x) - f(0) - \frac{f(2^m x) - f(0)}{4^m}\right\| \leq \sum_{k=0}^{m-1} \frac{\varphi(2^k x, 0, \cdots, 0)}{4^{k+1}} + \sum_{k=0}^{m-1} \frac{\varphi(0, 0, \cdots, 0)}{4^{k+1}} \tag{2.10}$$

for all $x \in X$ and for all $m \in \mathbb{N}$.

Here we claim that the sequence $\{\frac{f(2^m x) - f(0)}{4^m}\}$ is a Cauchy sequence in the Banach space Y. Indeed, for all $m, l \in \mathbb{N}$, it follows from (2.10) that

$$\begin{aligned}\left\|\frac{f(2^m x) - f(0)}{4^m} - \frac{f(2^{l+m}x) - f(0)}{4^{l+m}}\right\| &= \frac{1}{4^m}\left\|f(2^m x) - f(0) - \frac{f(2^l \cdot 2^m x) - f(0)}{4^l}\right\| \\ &\leq \sum_{k=0}^{l-1} \frac{\varphi(2^{m+k}x, 0, \cdots, 0)}{4^{m+k+1}} + \sum_{k=0}^{l-1} \frac{\varphi(0, 0, \cdots, 0)}{4^{m+k+1}}\end{aligned} \tag{2.11}$$

for all $x \in X$. It is easy to see that the last expression of (2.11) tends to zero as $m \to \infty$. Thus the sequence $\{\frac{f(2^m x) - f(0)}{4^m}\}$ is Cauchy for all $x \in X$ and hence it is convergent due to the completeness of Y. Therefore, we can define

$$Q(x) := \lim_{m\to\infty} \frac{f(2^m x) - f(0)}{4^m} = \frac{f(2^m x)}{4^m}$$

for all $x \in X$.

Replacing $x_1, x_2, \cdots, x_n$ by $2^m x_1, 2^m x_2, \cdots, 2^m x_n$ in (2.3), respectively, and dividing both sides by 4^m, we can infer that

$$\frac{1}{4^m}\Big\| \sum_{i,j=1}^{n} f(2^m(x_i+x_j)) - 2\sum_{1\le i<j\le n}^{n} f(2^m(x_i-x_j)) - 4f\Big(2^m(\sum_{i=1}^{n} x_i)\Big)\Big\| \le \frac{1}{4^m}\varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n).$$

Taking the limit in the preceding inequality, since the condition (2.2) implies that the last expression tends to zero as $m \to \infty$, it can easily be shown that Q is a solution of Eq. (1.1).

Furthermore, we take the limit in (2.10) as $m \to \infty$, we obtain that

$$\begin{aligned}\Big\|f(x) - f(0) - Q(x)\Big\| &\le \sum_{k=0}^{\infty} \frac{\varphi(2^k x, 0, \cdots, 0)}{4^{k+1}} + \frac{1}{3}\varphi(0, 0, \cdots, 0)\\ &= \frac{1}{4}\Phi(x, 0, \cdots, 0) + \frac{1}{3}\varphi(0, 0, \cdots, 0)\end{aligned}$$

for all $x \in X$.

To prove the uniqueness of Q. Assume that Q' is another quadratic function satisfying the inequality (2.4). Since $Q(2^m x) = 4^m Q(x)$, $Q'(2^m x) = 4^m Q'(x)$, we have

$$\begin{aligned}\|Q(x) - Q'(x)\| &= 4^{-m}\|Q(2^m x) - Q'(2^m x)\|\\ &\le 4^{-m}\Big(\|Q(2^m x) - f(2^m x) + f(0)\| + \|f(2^m x) - f(0) - Q'(2^m x)\|\Big)\\ &\le \frac{1}{4^m}\Big(\frac{1}{2}\Phi(2^m x, 0, \cdots, 0) + \frac{2}{3}\varphi(0, 0, \cdots, 0)\Big).\end{aligned}$$

By the condition (2.2), it follows that the last expression of the preceding inequality tends to zero as $m \to \infty$. Thus, we conclude that $Q(x) \equiv Q'(x)$. The proof of the theorem is now completed. □

Corollary 2.2. *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ be real numbers with $p < 2$ and $\theta \ge 0$. Assume that a function $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x_1, x_2, \cdots, x_n)\| \le \theta \sum_{i=1}^{n} \|x_i\|^p \tag{2.12}$$

for all $x_1, x_2, \cdots, x_n \in X$ ($x_1, x_2, \cdots, x_n \in X \setminus \{0\}$ if $p < 0$). Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the equality (1.1) and

$$\|f(x) - f(0) - Q(x)\| \le \frac{\theta\|x\|^p}{4 - 2^p}$$

for all $x \in X$ ($x \in X \setminus \{0\}$ if $p < 0$), where $f(0) = 0$ if $0 < p < 2$.

Proof. Letting $\varphi(x_1, x_2, \cdots, x_n) = \theta\sum_{i=1}^{n} \|x_i\|^p$ and the result follows directly from Theorem 2.1. If $0 < p < 2$, then we can obtain that $f(0) = 0$ since $\varphi(0, 0, \cdots, 0) = 0$ by replacing $x_1 = x_2 = \cdots = x_n = 0$ in (2.12). □

Corollary 2.3. *Let X and Y be a real normed space and a real Banach space, respectively. Let ϵ be a positive number. Assume that a function $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x_1, x_2, \cdots, x_n)\| \le \epsilon$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the equality (1.1) and

$$\|f(x) - f(0) - Q(x)\| \le \frac{2\epsilon}{3}$$

for all $x \in X$.

Proof. Letting $\varphi(x_1, x_2, \cdots, x_n) = \epsilon$ and the result follows directly from Theorem 2.1. $\square$

Theorem 2.4. *Let $\psi : X^n \to [0, \infty)$ be a function such that*

$$\Psi(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{\infty} 4^k \psi(2^{-k}x_1, 2^{-k}x_2, \cdots, 2^{-k}x_n) < \infty \tag{2.13}$$

for all $x_1, x_2, \cdots, x_n \in X$. Assume that the function $f : X \to Y$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\| \leq \psi(x_1, x_2, \cdots, x_n) \tag{2.14}$$

for all $x_1, x_2, \cdots, x_n \in X$. Then

$$Q(x) = \lim_{m\to\infty} 4^m f(2^{-m}x)$$

exists for each $x \in X$ and defines a unique quadratic function satisfying (1.1) *such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{4}\Psi(x, 0, \cdots, 0) \tag{2.15}$$

for all $x \in X$.

Proof. Putting $x_1 = x_2 = \cdots = x_n = 0$ in (2.14), we have $f(0) = 0$ since $\Psi(0, 0, 0) = \sum_{k=1}^{\infty} 4^k\psi(0, 0, 0) < \infty$ implies that $\psi(0, 0, 0) = 0$.

Setting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (2.14), we get

$$\|f(2x) - 4f(x)\| \leq \psi(x, 0, \cdots, 0) \tag{2.16}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.16), we obtain that

$$\Big\|f(x) - 4f\big(\frac{x}{2}\big)\Big\| \leq \psi\Big(\frac{x}{2}, 0, \cdots, 0\Big) \tag{2.17}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{m-1}}$ in (2.17) and multiplying the resulting inequality by 4^{m-1}, we can infer that

$$\Big\|4^{m-1}f\big(\frac{x}{2^{m-1}}\big) - 4^m f\big(\frac{x}{2^m}\big)\Big\| \leq 4^{m-1}\psi\Big(\frac{x}{2^m}, 0, \cdots, 0\Big) \tag{2.18}$$

for all $x \in X$. From (2.17) and (2.18), it follows that

$$\begin{aligned}\Big\|f(x) - 4^m f\big(\frac{x}{2^m}\big)\Big\| &\leq \frac{1}{4}\sum_{k=1}^{m} 4^k\psi\Big(\frac{x}{2^k}, 0, \cdots, 0\Big)\\ &\leq \frac{1}{4}\Psi(x, 0, \cdots, 0)\end{aligned} \tag{2.19}$$

for all $x \in X$ and $m \in \mathbb{N}$. The rest of the proof is analogous to that of Theorem 2.1. $\square$

Corollary 2.5. *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ be real numbers with $p > 2$ and $\theta \geq 0$. Assume that a function $f : X \to Y$ satisfies the following inequality*

$$\|D_q f(x_1, x_2, \cdots, x_n)\| \leq \theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the equality (1.1) *and*

$$\|f(x) - Q(x)\| \leq \frac{\theta\|x\|^p}{2^p - 4}$$

for all $x \in X$.

Proof. Letting $\varphi(x_1, x_2, \cdots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ and the result follows directly from Theorem 2.4. $\square$

Remark 2.6. All results obtained in this section show that the Ulam stability of Eq. (1.1) is independent of the number n of variables. The inequalities (2.4) and (2.15) implies that the errors of approximation depend only on the first component of the control functions φ and ψ, respectively. These results provide a unified error estimations of Eq. (1.1) with the different number of variables.

Remark 2.7. In Eq. (1.1), taking $n = 2$ and letting $x_1 = x$, $x_2 = y$, we obtain that

$$2f(x + y) + 2f(x - y) = f(2x) + f(2y).$$

Furthermore, if we let $x + y = t$, $x - y = s$, we get

$$f(t + s) + f(t - s) = 2f(t) + 2f(s),$$

which means that the Eq. (1.1) reduces to the classical quadratic functional equation. Therefore, our results can be regarded as an extension of the Ulam stability of the classical quadratic functional equation obtained in [3, 5, 15].

3. Ulam stability of Eq. (1.1) in a non-Archimedean space

Here we recall some related concepts which are needed in the following [11, 13].

A *non-Archimedean field* means that a field $\mathbb{K}$ endowed with a function (valuation) $|\cdot|_A : \mathbb{K} \to [0, \infty)$ which satisfies the following conditions:

(i) $|r|_A = 0$ if and only if $r = 0$;
(ii) $|rs|_A = |r|_A |s|_A$;
(iii) $|r + s|_A \leq \max\{|r|_A, |s|_A\}$ for all $r, s \in \mathbb{K}$. From the foregoing conditions, it follows that $|1|_A = |-1|_A = 1$ and $|n|_A \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|_A$. A function $\|\cdot\|_A : X \to \mathbb{R}$ is called a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(NA1) $\|x\|_A = 0$ if and only if $x = 0$;
(NA2) $\|rx\|_A = |r|_A \|x\|_A$ for all $r \in \mathbb{K}$ and $x \in X$;
(NA3) $\|x + y\|_A \leq \max\{\|x\|_A, \|y\|_A\}$ for all $x, y \in X$ (strong triangle inequality (ultrametric)).

Moreover, the pair $(X, \|\cdot\|_A)$ is called a *non-Archimedean space*. From the condition (NA3), we can infer that

$$\|x_n - x_m\|_A \leq \max\{\|x_{j+1} - x_j\|_A : m \leq j \leq n - 1\}$$

for all $n, m \in \mathbb{N}$ with $n > m$. Thus, a sequence $\{x_n\}$ in a non-Archimedean space is *Cauchy* if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero. A sequence $\{x_n\}$ is said to be *convergent* if for any $\epsilon > 0$, there exist a natural number $n_0 \in \mathbb{N}$ and $x \in X$ such that $\|x_n - x\|_A \leq \epsilon$ for all $n \geq n_0$. Furthermore, a non-Archimedean space $(X, \|\cdot\|_A)$ is said to be *complete* if every Cauchy sequence in X is convergent.

Throughout this section, let G be an additive group and let $(X, \|\cdot\|_A)$ be a complete non-Archimedean space.

Theorem 3.1. *Let $\varphi : G^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} |4|_A^{-m} \varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n) = 0 \tag{3.1}$$

for all $x_1, x_2, \cdots, x_n \in G$ and the limits

$$\Phi(x) := \lim_{m\to\infty} \max\{|4|_A^{-k} \varphi(2^k x, 0, \cdots, 0) : 0 \leq k < m\} \tag{3.2}$$

exists for each $x \in G$. Assume that the function $f : G \to X$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\|_A \leq \varphi(x_1, x_2, \cdots, x_n) \tag{3.3}$$

for all $x_1, x_2, \cdots, x_n \in G$. Then there exists a quadratic function $Q : G \to X$ such that

$$\|f(x) - Q(x)\|_A \leq \frac{1}{|4|_A}\Phi(x) \tag{3.4}$$

for all $x \in G$. Furthermore, if

$$\lim_{l\to\infty}\lim_{m\to\infty} \max\{|4|_A^{-k}\varphi(2^k x, 0, \cdots, 0) : l \leq k < l+m\} = 0, \tag{3.5}$$

then Q is the unique quadratic function satisfying the Eq. (1.1).

Proof. From (3.1), it follows that $f(0) = 0$. Indeed, letting $x_1 = x_2 = \cdots = x_n = 0$ in (3.3) we can obtain that $f(0) = 0$ since the condition (3.1) implies that $\lim_{m\to\infty} |4|_A^{-m}\varphi(0, 0, \cdots, 0) = 0$ and due to the fact $|4|_A^{-m} \geq 1$ for all $m \in \mathbb{N}$.

Using the analogous method as in the proof of Theorem 2.1, we have

$$\Big\|\frac{f(2^m x)}{4^m} - \frac{f(2^{m-1}x)}{4^{m-1}}\Big\|_A \leq \frac{1}{|4|_A^m}\varphi(2^{m-1}x, 0, \cdots, 0) \tag{3.6}$$

for all $x \in G$ and $m \in \mathbb{N}$. By (3.1) and (3.6), we know that the sequence $\{\frac{f(2^m x)}{4^m}\}$ is Cauchy in X. Due to the completeness of X, we can define

$$Q(x) := \lim_{m\to\infty}\frac{f(2^m x)}{4^m}$$

for all $x \in G$.

Next, we claim that

$$\Big\|f(x) - \frac{f(2^m x)}{4^m}\Big\|_A \leq \frac{1}{|4|_A}\max\{|4|_A^{-k}\varphi(2^k x, 0, \cdots, 0) : 0 \leq k < m\} \tag{3.7}$$

for all $x \in G$ and $m \in \mathbb{N}$.

Letting $m = 1$ in (3.6), it is easy to verify that the inequality (3.7) holds true for $m = 1$. By mathematical induction, we can assume that the inequality (3.7) is true for some $m \in \mathbb{N}$. Therefore, we can infer that

$$\begin{aligned}
\Big\|f(x) - \frac{f(2^{m+1}x)}{4^{m+1}}\Big\|_A &\leq \max\Big\{\Big\|f(x) - \frac{f(2^m x)}{4^m}\Big\|_A, \Big\|\frac{f(2^m x)}{4^m} - \frac{f(2^{m+1}x)}{4^{m+1}}\Big\|_A\Big\}\\
&\leq \max\Big\{\frac{1}{|4|_A}\max\{|4|_A^{-k}\varphi(2^k x, 0, \cdots, 0) : 0 \leq k < m\}, \frac{1}{|4|_A^{m+1}}\varphi(2^m x, 0, \cdots, 0)\Big\}\\
&= \frac{1}{|4|_A}\max\{|4|_A^{-k}\varphi(2^k x, 0, \cdots, 0) : 0 \leq k < m+1\},
\end{aligned}$$

which proves the validity of the inequality (3.7) for $m + 1$. Thus, letting $m \to \infty$ in (2.6) and using (3.2), it can be shown that the inequality (3.4) is true.

Replacing $x_1, x_2, \cdots, x_n$ by $2^m x_1, 2^m x_2, \cdots, 2^m x_n$ in (3.3), respectively, and dividing both sides by $|4|_A^m$, we obtain that

$$\begin{aligned}
&\Big\|\frac{1}{4^m}\sum_{i,j=1}^{n} f(2^m(x_i + x_j)) - \frac{2}{4^m}\sum_{1\leq i<j\leq n} f(2^m(x_i - x_j)) - \frac{4}{4^m}f\Big(2^m(\sum_{i=1}^{n} x_i)\Big)\Big\|_A\\
&\leq \frac{1}{|4|_A^m}\varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n).
\end{aligned}$$

By letting $m \to \infty$ in the foregoing expression and using (3.1), it is easy to see that the function Q satisfies Eq. (1.1).

To show the uniqueness of Q. Let us assume that Q' is another quadratic function which satisfies the inequality (3.4). Since $Q(2^l x) = 4^l Q(x)$, $Q(2^l x) = 4^l Q(x)$, we have

$$\begin{aligned}\|Q(x) - Q'(x)\|_A &= |4|_A^{-l} \|Q(2^l x) - Q'(2^l x)\|_A \\ &\leq |4|_A^{-l} \max\{\|Q(2^l x) - f(2^l x)\|_A, \|f(2^l x) - Q'(2^l x)\|_A\} \\ &\leq |4|_A^{-(l+1)} \Phi(2^l x) \\ &= |4|_A^{-(l+1)} \lim_{m\to\infty} \max\{|4|_A^{-k} \varphi(2^{k+l} x, 0, \cdots, 0) : 0 \leq k < m\} \\ &= \frac{1}{|4|_A} \lim_{m\to\infty} \max\{|4|_A^{-k} \varphi(2^k x, 0, \cdots, 0) : l \leq k < l + m\}.\end{aligned}$$

Taking the limit as $l \to \infty$ in the preceding expression and using (3.5), we conclude that $Q(x) \equiv Q'(x)$. This completes the proof. □

Corollary 3.2. *Let θ be a positive number and let $\rho : [0, \infty) \to [0, \infty)$ be a function such that*

$$\rho(|2|_A t) \leq \rho(|2|_A)\rho(t)$$

for all $t \in [0, \infty)$ and with $\rho(|2|_A) < |4|_A$. Assume that the function $f : G \to X$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\|_A \leq \theta \sum_{i=1}^{n} \rho(\|x_i\|_A)$$

for all $x_1, x_2, \cdots, x_n \in G$. Then there exists a unique quadratic function $Q : G \to X$ satisfying Eq. (1.1) such that

$$\|f(x) - Q(x)\|_A \leq \frac{\theta\rho(\|x\|_A)}{|4|_A}$$

for all $x \in G$.

Proof. Letting $\varphi(x_1, x_2, \cdots, x_n) = \theta \sum_{i=1}^{n} \rho(\|x_i\|_A)$. From Theorem 3.1, it suffices to verify that all conditions are satisfied. By (3.1), we can obtain

$$\begin{aligned}\lim_{m\to\infty} |4|_A^{-m} \varphi(2^m x_1, 2^m x_2, \cdots, 2^m x_n) &= \lim_{m\to\infty} |4|_A^{-m} \theta \sum_{i=1}^{n} \rho(|2|_A^m \|x_i\|_A) \\ &\leq \lim_{m\to\infty} |4|_A^{-m} [\rho(|2|_A)]^m \theta \sum_{i=1}^{n} \rho(\|x_i\|_A) \\ &= 0.\end{aligned}$$

From (3.2), it follows that

$$\begin{aligned}\Phi(x) &= \lim_{m\to\infty} \max\{|4|_A^{-k} \varphi(2^k x, 0, \cdots, 0) : 0 \leq k < m\} \\ &= \varphi(x, 0, \cdots, 0).\end{aligned}$$

By (3.5), we can infer that

$$\begin{aligned}\lim_{l\to\infty} \lim_{m\to\infty} \max\{|4|_A^{-k} \varphi(2^k x, 0, \cdots, 0) : l \leq k < l + m\} &= \lim_{l\to\infty} |4|_A^{-l} \varphi(2^l x, 0, \cdots, 0) \\ &= 0.\end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied and hence the foregoing results hold true. □

Remark 3.3. It can easily be verified that the function $\rho(t) = t^{2p}$ $(p > 1)$ is an appropriate choice for all $t \geq 0$, and with the further assumption that $|2|_A < 1$ in this case.

Theorem 3.4. *Let $\varphi : G^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} |4|_A^m \psi(2^{-m}x_1, 2^{-m}x_2, \cdots, 2^{-m}x_n) = 0 \tag{3.8}$$

for all $x_1, x_2, \cdots, x_n \in G$ and the limits

$$\Psi(x) := \lim_{m\to\infty} \max\{|4|_A^k \psi(2^{-k}x, 0, \cdots, 0) : 0 < k \leq m\} \tag{3.9}$$

exists for each $x \in G$. Assume that the function $f : G \to X$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\|_A \leq \psi(x_1, x_2, \cdots, x_n) \tag{3.10}$$

for all $x_1, x_2, \cdots, x_n \in G$ and $f(0) = 0$. Then there exists a quadratic function $Q : G \to X$ such that

$$\|f(x) - Q(x)\|_A \leq \frac{1}{|4|_A} \Psi(x) \tag{3.11}$$

for all $x \in G$. Furthermore, if

$$\lim_{l\to\infty} \lim_{m\to\infty} \max\{|4|_A^k \varphi(2^{-k}x, 0, \cdots, 0) : l < k \leq l + m\} = 0, \tag{3.12}$$

then Q is the unique quadratic function satisfying the Eq. (1.1).

Proof. Letting $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$ in (3.10) and using $f(0) = 0$, we get

$$\|f(2x) - 4f(x)\|_A \leq \psi(x, 0, \cdots, 0) \tag{3.13}$$

for all $x \in G$. Replacing x by $\frac{x}{2^m}$ in (3.13) and multiplying the resulting inequality by $|4|_A^{m-1}$, we obtain that

$$\left\| 4^{m-1} f\Big(\frac{x}{2^{m-1}}\Big) - 4^m f\Big(\frac{x}{2^m}\Big) \right\|_A \leq |4|_A^{m-1} \psi\Big(\frac{x}{2^m}, 0, \cdots, 0\Big) \tag{3.14}$$

for all $x \in G$ and for all $m \in \mathbb{N}$. The condition (3.8) implies that the right-hand side of (3.14) tends to zero as $m \to \infty$. Then, we know that the sequence $\left\{4^m f\big(\frac{x}{2^m}\big)\right\}$ is Cauchy in X. According to the completeness of X, we can define

$$Q(x) := \lim_{m\to\infty} 4^m f\Big(\frac{x}{2^m}\Big)$$

for each $x \in G$. By mathematical induction, we can infer that

$$\left\| f(x) - 4^m f\Big(\frac{x}{2^m}\Big) \right\|_A \leq \frac{1}{|4|_A} \max\{|4|_A^k \psi(2^{-k}x, 0, \cdots, 0) : 0 < k \leq m\}$$

for all $x \in G$ and for all $m \in \mathbb{N}$. The rest of the proof is similar to that of Theorem 3.1. □

Corollary 3.5. *Let θ be a positive number and let $\rho : [0, \infty) \to [0, \infty)$ be a function such that*

$$\rho\Big(\frac{t}{|2|_A}\Big) \leq \frac{\rho(t)}{\rho(|2|_A)}$$

for all $t \in [0, \infty)$ and with $\rho(|2|_A) > |4|_A$. Assume that the function $f : G \to X$ satisfies

$$\|D_q f(x_1, x_2, \cdots, x_n)\|_A \leq \theta \sum_{i=1}^{n} \rho(\|x_i\|_A)$$

for all $x_1, x_2, \cdots, x_n \in G$ and $f(0) = 0$. Then there exists a unique quadratic function $Q : G \to X$ satisfying Eq. (1.1) *such that*

$$\|f(x) - Q(x)\|_A \leq \frac{\theta \rho(\|x\|_A)}{\rho(|2|_A)}$$

for all $x \in G$.

Remark 3.6. Similar to Corollary 3.2, it is easy to show that the function $\rho(t) = t^{2p}$ $(0 < p < 1)$ is suitable for Corollary 3.4. Certainly, the further assumption $|2|_A < 1$ is also necessary.

Remark 3.7. From Remark 2.7, our results obtained in this section provide a more general theoretical frame of a quadratic functional equation with n-variables in a non-Archimedean space, which include as a special case the Ulam stability results of the classical quadratic functional equation obtained by Moslehian and Rassias [11].

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