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On the Meir-Keeler-Khan set contractions

Chi-Ming Chen[a], Erdal Karapınar[b,*], Guang-Ting Chen[a]

[a]*Department of Applied Mathematics, National Hsinchu University of Education, Taiwan.*

[b]*Atılım University Department of Mathematics 06586 Incek, Ankara, Turkey.*

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Abstract

This report is aim to investigate the fixed points of two classes of Meir-Keeler-Khan set contractions with respect to the measure of noncompactness. The proved results extend a number of recently announced theorems on the topic.

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1. Introduction and preliminaries

Let X and Y be two Hausdorff topological spaces, and let $N(X)$ [respectively, $CL(X), B(X), K(X), CB(X)$] denote the family of nonempty subsets [respectively, closed, bounded, compact, closed and bounded] subsets of X. Let $T : X \to 2^Y$ be a set-valued mapping (in short SVM). If the graph of T, that is, $\mathcal{G}_{\mathcal{T}} = \{(x, y) \in X \times Y, y \in Tx\}$ is closed, then T is closed. A mapping $\mathcal{H} : CB(X) \times CB(X) \to [0, \infty)$

$$\mathcal{H}(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

forms a metric (is called the Hausdorff metric) induced by the standard metric d (see e.g. [13]), where $d(x, B) := \inf\{d(x, b) : b \in B\}$, and $A, B \in CB(X)$. A SVM $T : X \to CB(X)$ is called a contraction if

$$\mathcal{H}(Tx, Ty) \le kd(x, y)$$

for all $x, y \in X$ and $k \in [0, 1)$.

*Corresponding author
Email address: erdalkarapinar@yahoo.com (Erdal Karapınar)

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Let $\mathbb{R}_0^+$ be the set of all real non-negative numbers, and let $\mathbb{N}$ be the set of all natural numbers. Let (M,d) be a metric space, $X \subset M$ and $\gamma > 0$. Then we let $B_M(X,\gamma) = \{x \in M : d(x,X) \le \gamma\}$ and $N_M(X,\gamma) = \{x \in M : d(x,X) < \gamma\}$, and we define the convex hull of X as follows:

$$co(X) = \bigcap\{B \subset M : B \text{ is a closed ball in } M \text{ such that } X \subset B\}.$$

Recall that X is said to be subadmissible [7] if $co(A) \subset X$ for each $A \in < X >$. For the sake of completeness, let us recall the notion of the set measure of noncompactness in the framework of metric space.

Definition 1.1 ([14])**.** A mapping $\Phi : B(X) \to \mathbb{R}_0^+$ is called a measure of noncompactness defined on (X,d), if following properties are fulfilled:

1. $\Phi(D) = 0$ if and only if D is precompact;
2. $\Phi(D) = \Phi(\overline{D})$;
3. $\Phi(D_1 \cup D_2) = \max\{\Phi(D_1), \Phi(D_2)\}$;
4. $\Phi(D) = \Phi(co(D))$.

On what follows, we state the concept of the σ-measure that is a well-known measure of noncompactness in metric spaces.

Definition 1.2. Suppose that (X,d) is a standard metric space. A mapping $\sigma : B(X) \to \mathbb{R}_0^+$, defined as,

$$\sigma(D) = \inf\{\gamma > 0 : \ D \text{ can be covered by finitely many sets with diameter } \le \gamma\}$$

for each $D \in B(X)$, is called the Kuratowski measure of noncompactness (see, [5]).

In 1955, Darbo [10] used measure of noncompactness to generalize Schauder's theorem to prove the following theorem.

Theorem 1.3 ([10])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in (0,1)$ with the property*

$$\sigma(T(X)) \le k\sigma(X)$$

for any nonempty subset X of Ω. Then T has a fixed point in the set Ω.

The following theorem is an extension of Darbo's fixed point theorem that was introduced by Banas and Goebel [8].

Theorem 1.4 ([8])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in (0,1)$ with the property*

$$\sigma(T(X)) \le \psi(\sigma(X))$$

for any nonempty subset X of Ω, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Then T has a fixed point in the set Ω.

In recent years, measures of noncompactness have developed rapidly on metric spaces which are interesting for fixed point theory, see e.g. [1–6].

A function $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a Meir-Keeler type, (in short, MKT [12]), if ξ fulfills

$$\forall \eta > 0 \ \exists \delta > 0 \ \forall t \in \mathbb{R}_0^+ \ (\eta \le t < \eta + \delta \Rightarrow \xi(t) < \eta).$$

Remark 1.5. By the definition, MKT function ξ provides the following inequality:

$$\xi(t) < t, \ \text{ for all } \ t \in \mathbb{R}_0^+.$$

A (c)-comparison function ψ is a nondecreasing self-mapping on $\mathbb{R}_0^+$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iteration of ψ. It is clear that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. We denote Ψ the family of all (c)-comparison functions.

Recently, Redjel and Dehici [15] introduced the concept of (α, ψ)-Meir-Keeler-Khan mappings (in short, (α, ψ)-MKK mappings), and they proposed two theorems for the existence of fixed points for such mappings.

Theorem 1.6 ([15])**.** *Suppose that the self-mapping f over a complete metric space (X, d) is continuous, α-admissible and (α, ψ)-MKK mapping, that is, there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}_0^+$ such that for every $\eta > 0$, there exists $\delta(\eta)$ such that if*

$$\eta \le \psi\left(\frac{d(x, fx)d(x, fy) + d(y, fy)d(y, fx)}{d(x, fy) + d(y, fx)}\right) < \eta + \delta(\eta)$$

for all $x, y \in X$, then

$$\alpha(x, y)d(fx, fy) \le \eta.$$

If there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in fx_0$, then f has a fixed point in X.

Definition 1.7 ([16])**.** Let (X, d) be a metric space, and let $T : X \to N(X)$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two mappings on X. Then T is called an α-admissible SVM if for any $x \in X$ and $y \in Tx$ with $\alpha(x, y) \ge 1$, we have

$$\alpha(y, z) \ge 1, \quad \text{for any} \quad z \in Ty.$$

Recently, Wang et al. [17] characterized the results of Redjel and Dehici [15] in the setting of set-valued mappings.

Theorem 1.8 ([17])**.** *Suppose that a set-valued mapping $T : X \to K(X)$ over a complete metric space (X, d) is α-admissible, continuous and (α, ψ)-Meir-Keeler-Khan, that is, there exist $\psi \in \Psi$ and $\alpha : X \times X \to (0, \infty)$ satisfying*

1. *T is an SVM;*
2. *for each $x, y \in X$,*

$$\mathcal{H}(Tx, Ty) \ne 0 \implies \alpha(x, y)\mathcal{H}(Tx, Ty) \le \psi(\mathcal{P}(x, y)),$$

where

$$\mathcal{P}(x, y) = \frac{dist(x, Tx)dist(x, Ty) + dist(y, Ty)dist(y, Tx)}{dist(x, Ty) + dist(y, Tx)}.$$

If there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$, then T has a fixed point in X.

2. Main results

We start with the following definition:

Definition 2.1. Let Y be a nonempty subset of a metric space (X, d). A set-valued mapping $T : Y \to 2^Y$ is called Meir-Keeler type contraction with respect to the measure σ (in short, $MKTC_\sigma$) if, for each bounded subset A of Y and for each $\eta > 0$ there exists $\delta > 0$ (where δ depends on A and η) such that

$$\eta \le \sigma(A) < \eta + \delta \Longrightarrow \sigma(T(A)) < \eta,$$

where $T(A)$ is bounded.

Remark 2.2. Note that if T is a $MKTC_\sigma$, then we have

$$\sigma(T(A)) \le \sigma(A)$$

for all bounded subsets A of Y.

At follows that we shall prove the existence of the fixed point of $MKTC_\sigma$ under the certain assumptions.

Theorem 2.3. *Let Y be a nonempty bounded subadmissible subset of a metric space (X, d). Suppose $T : Y \to 2^Y$ is $MKTC_\sigma$. Then Y contains a precompact subadmissible subset K with $T(K) \subset K$.*

Proof. Take $x_0 \in Y$. we define the sequence $\{Y_n\}$ of sets as follows:

$$Y_0 = Y \text{ and } Y_{n+1} = co(T(Y_n) \cup \{x_0\}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

So, we have

(1) Y_n is a subadmissible subset of Y;
(2) $Y_{n+1} \subset Y_n$;
(3) $T(Y_n) \subset Y_{n+1}$;

for all $n \in \mathbb{N} \cup \{0\}$.

From the argument above and by regarding the properties of the set measure σ together with Remark 2.2, we get that

$$\begin{aligned}\sigma(Y_1) &= \sigma(co(T(Y_0) \cup \{x_0\})) \\ &= \sigma(T(Y_0)) \\ &\le \sigma(Y_0).\end{aligned}$$

By iteration, we derive that

$$\begin{aligned}\sigma(Y_{n+1}) &= \sigma(co(T(Y_n) \cup \{x_0\})) \\ &= \sigma(T(Y_n)) \\ &\le \sigma(Y_n)\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus we deduce that the sequence $\{\sigma(Y_n)\}$ is both nonincreasing and bounded below. So, it converges to $\eta \ge 0$, that is,

$$\lim_{n\to\infty} \sigma(Y_n) = \eta.$$

Notice that $\eta = \inf\{\sigma(Y_n) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\eta = 0$. Suppose, on the the contrary, that $\eta > 0$. Since T is $MKTC_\eta$, there exist $\delta > 0$ and a natural number k such that

$$\eta \le \sigma(Y_k) < \eta + \delta \Longrightarrow \sigma(Y_{k+1}) = \sigma(T(Y_k)) < \eta.$$

It is a contradiction since $\eta = \inf\{\sigma(Y_n) : n \in \mathbb{N} \cup \{0\}\}$. Thus, we find

$$\lim_{n\to\infty} \sigma(Y_n) = 0.$$

Let us take $Y_\infty = \cap_{n \in \mathbb{N}\cup\{0\}} Y_n$. Then Y_∞ is a nonempty precompact subadmissible subset of Y, and, by (2), (3), we also have that $T(X_\infty) \subset Y_\infty$. □

In Theorem 2.3, we call the set Y_∞ a Meir-Keeler-inducing precompact subadmissible subset of Y.

Definition 2.4. Let (X, d) be a metric space, and let $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a Meir-Keeler mapping with $\sup_{t>0} \frac{\psi(t)}{t} < 1$. A set-valued mapping $T : X \to N(X)$ is called a $(\psi, \mathcal{L}(x, y))$-Meir-Keeler-Khan type contraction with respect to the measure σ (in short, $(\psi, \mathcal{L}(x, y)) - MKKTC_\sigma$) if

1. T is a MKT set contraction with respect to the measure σ;

2. T fulfills
$$\mathcal{H}(Tx,Ty) \neq 0 \implies \mathcal{H}(Tx,Ty) \leq \psi(\mathcal{L}(x,y)),$$
where
$$\mathcal{L}(x,y) = \frac{dist(x,Tx)dist(x,Ty) + dist(y,Ty)dist(y,Tx)}{dist(x,Ty) + dist(y,Tx)}$$
for each $x, y \in X$.

We investigate an existence theorem for fixed points of $(\psi, \mathcal{L}(x,y)) - MKKTC_\sigma$.

Theorem 2.5. *Let Y be a nonempty bounded subadmissible subset of a complete metric space (X,d), let $T : Y \to CL(Y)$ be a$(\psi, \mathcal{L}(x,y)) - MKKTC_\sigma$ and $\overline{T(Y)} \subset Y$. Suppose that T is continuous. Then T has a fixed point in Y.*

Proof. By applying Theorem 2.3 and it follows from above argument, we get a Meir-Keeler-inducing precompact subadmissible subset Y_∞ of X. Since $\overline{T(Y)} \subset Y$ and $T(Y_{n+1}) \subset T(Y_n) \subset T(Y)$, we have that $\overline{T(Y_{n+1})} \subset \overline{T(Y_n)} \subset Y$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \sigma(\overline{T(Y_n)}) = 0$, we get that Y_∞ is a nonempty compact subset of X. Since Tx is closed, we also have that Tx is compact for each $x \in Y_\infty$.

Let $x_0 \in Y_\infty$. If $x_0 \in Tx_0$, then x_0 is a fixed point of T, and this proof is complete. Suppose that $x_0 \notin Tx_0$. Since Tx_0 is a compact subset of Y_∞, we have that $dist(x_0, Tx_0) > 0$. Let $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T, and subsequently, this proof is complete. Suppose that $x_1 \notin Tx_1$. Since Tx_1 is a compact subset of Y_∞, we have that $dist(x_1, Tx_1) > 0$. Since T is $(\psi, \mathcal{L}(x,y)) - MKKTC_\sigma$, we have
$$\begin{aligned}\mathcal{H}(Tx_0,Tx_1) &\leq \psi\left(\frac{dist(x_0,Tx_0)dist(x_0,Tx_1) + dist(x_1,Tx_1)dist(x_1,Tx_0)}{dist(x_0,Tx_1) + dist(x_1,Tx_0)}\right)\\ &= \psi(dist(x_0,Tx_0))\\ &< dist(x_0,Tx_0),\end{aligned}$$
and there exists $\eta_1 \in (0,\gamma]$, where $\gamma = \sup_{t>0} \frac{\psi(t)}{t}$, and obviously η_1 depends on x_0 and x_1 such that
$$\mathcal{H}(Tx_0,Tx_1) \leq \eta_1 \cdot dist(x_0,Tx_0).$$

By the definition of the Hausdorff metric and above inequality, we obtain that
$$dist(x_1,Tx_1) \leq \mathcal{H}(Tx_0,Tx_1) \leq \eta_1 \cdot dist(x_0,Tx_0).$$

Since Tx_1 is a compact subset of Y_∞, there exists $x_2 \in Tx_1$ such that
$$d(x_1,x_2) = dist(x_1,Tx_1).$$

Thus, we have
$$d(x_1,x_2) \leq \eta_1 \cdot dist(x_0,Tx_0).$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T, and this proof is complete. Suppose that $x_2 \notin Tx_2$. Since T is $(\psi, \mathcal{L}(x,y)) - MKKTC_\sigma$, we have
$$\begin{aligned}\mathcal{H}(Tx_1,Tx_2) &\leq \psi\left(\frac{dist(x_1,Tx_1)dist(x_1,Tx_2) + dist(x_2,Tx_2)dist(x_2,Tx_1)}{dist(x_1,Tx_2) + dist(x_2,Tx_1)}\right)\\ &= \psi(dist(x_1,Tx_1))\\ &< dist(x_1,Tx_1),\end{aligned}$$
and there exists $\eta_2 \in (0,\gamma]$, where $\gamma = \sup_{t>0} \frac{\psi(t)}{t}$, and obviously η_2 depends on x_1 and x_2 such that
$$\mathcal{H}(Tx_1,Tx_2) \leq \eta_2 \cdot dist(x_1,Tx_1).$$

By the definition of the Hausdorff metric, we obtain that

$$dist(x_2, Tx_2) \leq \mathcal{H}(Tx_1, Tx_2) \leq \eta_2 \cdot dist(x_1, Tx_1).$$

Since Tx_2 is a compact subset of X, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) = dist(x_2, Tx_2).$$

Thus, we also have

$$d(x_2, x_3) \leq \eta_2 \cdot dist(x_1, Tx_1) \leq \eta_2\eta_1 \cdot d(x_0, x_1).$$

By the induction, we can obtain a sequence $\{x_n\}$ of X satisfying

$$x_{n+1} \in Tx_n, \quad x_{n+1} \notin Tx_{n+1},$$

and for each $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{H}(Tx_n, Tx_{n+1}) &\leq \psi\left(\frac{dist(x_n, Tx_n)dist(x_n, Tx_{n+1}) + dist(x_{n+1}, Tx_{n+1})dist(x_{n+1}, Tx_n)}{dist(x_n, Tx_{n+1}) + dist(x_{n+1}, Tx_n)}\right)\\ &= \psi(dist(x_n, Tx_n))\\ &< dist(x_n, Tx_n),\end{aligned}$$

and there exists $\eta_{n+1} \in (0, \gamma]$, where $\gamma = \sup_{t>0} \frac{\psi(t)}{t}$. It is clear that η_{n+1} depends both on x_n and x_{n+1} such that

$$\mathcal{H}(Tx_n, Tx_{n+1}) \leq \eta_{n+1} \cdot dist(x_n, Tx_n).$$

By the definition of the Hausdorff metric with inequality above, we obtain

$$dist(x_{n+1}, Tx_{n+1}) \leq \mathcal{H}(Tx_n, Tx_{n+1}) \leq \eta_{n+1} \cdot dist(x_n, Tx_n),$$

for each $n \in \mathbb{N}$. Since Tx_{n+1} is a compact subset of X, there exists $x_{n+2} \in Tx_{n+1}$ such that

$$d(x_{n+1}, x_{n+2}) = dist(x_{n+1}, Tx_{n+1}).$$

Thus, we have

$$\begin{aligned}d(x_{n+1}, x_{n+2}) &\leq \eta_{n+1} \cdot dist(x_n, Tx_n)\\ &\leq \eta_{n+1}\eta_n \cdot dist(x_{n-1}, Tx_{n-1})\\ &\vdots\\ &\leq \eta_{n+1}\eta_n \cdots \eta_1 \cdot dist(x_0, Tx_0)\end{aligned}$$

for each $n \in \mathbb{N}$. Put $\kappa_{n+1} = \max\{\eta_1, \eta_2, \cdots, \eta_{n+1}\}$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$d(x_{n+1}, x_{n+2}) \leq \kappa_{n+1}^{n+1} \cdot dist(x_0, Tx_0).$$

Since $\eta_n < 1$ for all $n \in \mathbb{N}$, we obtain that $\kappa_{n+1} < 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, there exists $\kappa \in (0, 1)$ such that

$$\kappa_{n+1} \leq \kappa < 1$$

for all $n \in \mathbb{N} \cup \{0\}$, and we also obtain that

$$d(x_{n+1}, x_{n+2}) \leq \kappa^{n+1} \cdot dist(x_0, Tx_0).$$

By letting $n \to \infty$, we find

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

We will prove that the sequence $\{x_n\}$ is a Cauchy sequence. On account of the discussion above, we have

$$\begin{aligned} d(x_n, x_{n+m}) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m}) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m}) \\ &\vdots \\ &\leq \sum_{i=1}^{m} d(x_{n+i-1}, x_{n+i}) \\ &\leq \sum_{i=1}^{m} \kappa^{n+i-1} d(x_0, x_1) \\ &\leq \frac{\kappa^n}{1-\kappa} d(x_0, x_1). \end{aligned}$$

By letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0.$$

This yields that $\{x_n\}$ is a Cauchy sequence in (Y_∞, d).

By the completeness of (X, d) together with the fact that Y_∞ is closed, the subspace (Y_∞, d) is complete.

Consequently, there exists $p \in Y_\infty$ such that $d(x_n, p) = 0$ as $n \to \infty$. Since T is continuous, we have $\mathcal{H}(Tx_n, Tp) = 0$ as $n \to \infty$. Therefore

$$dist(p, Tp) = \lim_{n\to\infty} dist(x_{n+1}, Tp) \leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tp) = 0.$$

Due to the fact that Tp is a compact subset of Y_∞, we conclude the desired result, that is, $p \in Tp$. $\square$

Definition 2.6 ([9])**.** A function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a weaker Meir-Keeler (in short, $wMKT$), if φ satisfies the following condition:

$$\forall \eta > 0 \ \ \exists \delta > 0 \ \ \forall t \in \mathbb{R}_0^+ \ \ (\eta \leq t < \eta + \delta \Rightarrow \exists n_0 \in \mathbb{N}, \ \ \varphi^{n_0}(t) < \eta).$$

Definition 2.7 ([9])**.** Let Y be a nonempty subset of a metric space (X, d) and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a $wMKT$. A set-valued mapping $T : Y \to 2^Y$ is called a φ-weaker Meir-Keeler Type set contraction with respect to the measure σ (in short, $\varphi - wMKKC_\sigma$) if for each $A \subset Y$ with A is bounded and $T(A)$ is bounded, and for each $\eta > 0$ there exists $\gamma > 0$ such that

$$\eta \leq \varphi(\sigma(A)) < \eta + \gamma \Longrightarrow \sigma(T(A)) < \eta.$$

Remark 2.8 ([9]). Note that if T is $\varphi - wMKKC_\sigma$, then we have that for any bounded subset A of Y

$$\sigma(T(A)) \leq \varphi(\sigma(A)).$$

Theorem 2.9 ([9])**.** *Let Y be a nonempty bounded subadmissible subset of a metric space (X, d), and let $T : Y \to 2^Y$ be $\varphi - wMKKC_\sigma$. If the sequence $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ is decreasing for all $t \in \mathbb{R}_0^+$, then X contains a precompact subadmissible subset $Y_\infty = \cap_{n\in\mathbb{N}\cup\{0\}} Y_n$ with $T(Y_\infty) \subset Y_\infty$, where $x_0 \in Y$, $Y = Y_0$ and $Y_{n+1} = co(T(Y_n) \cup \{x_0\})$ for all $n \in \mathbb{N}$.*

Remark 2.10 ([9]). In the process of the proof of Theorem 2.9, we call the set Y_∞, a $wMKT$ precompact-inducing subadmissible subset of Y.

In this sequel, we let Ω be the class of all nondecreasing functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:

(φ_1) φ is a $wMKT$;

(φ_2) for all $t \in (0, \infty)$, $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(φ_3) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;

(φ_4) for $t > 0$, if $\lim_{n\to\infty} \varphi^n(t) = 0$, then $\lim_{n\to\infty} \sum_{i=n}^{m} \varphi^i(t) = 0$, where $m > n$.

Definition 2.11. Let (X, d) be a metric space, Y be a nonempty bounded subadmissible subset of X, and $\varphi \in \Omega$. A set-valued mapping $T : Y \to N(Y)$ is called a $(\varphi, \mathcal{L}(x, y))$-weaker Meir-Keeler-Khan type contraction with respect to the measure σ (in short, $(\varphi, \mathcal{L}(x, y)) - wMKKTC_\sigma$) if

1. T is $\varphi - wMKKC_\sigma$;
2. T fulfills
$$\mathcal{H}(Tx, Ty) \neq 0 \implies \alpha(x, y)\mathcal{H}(Tx, Ty) \leq \varphi(\mathcal{L}(x, y)), \tag{2.1}$$
where
$$\mathcal{L}(x, y) = \frac{dist(x, Tx)dist(x, Ty) + dist(y, Ty)dist(y, Tx)}{dist(x, Ty) + dist(y, Tx)}.$$

Theorem 2.12. *Let (X, d) be a complete metric space and let Y be a nonempty bounded subadmissible subset of (X, d). If $T : Y \to CL(Y)$ is continuous and $(\varphi, \mathcal{L}(x, y)) - wMKKTC_\sigma$ and $\overline{T(Y)} \subset Y$, then T has a fixed point in X.*

Proof. By taking Theorem 2.9 and Remark 2.10 into account, we get a weaker Meir-Keeler-inducing precompact subadmissible subset Y_∞ of Y. By regarding $\overline{T(Y)} \subset Y$ and $T(Y_{n+1}) \subset T(Y_n) \subset T(Y)$, we have that $\overline{T(Y_{n+1})} \subset \overline{T(Y_n)} \subset Y$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \sigma(\overline{T(Y_n)}) = 0$, we get that Y_∞ is a nonempty compact subset of X. By owing to the fact that Tx is closed, we derive that Tx is compact for each $x \in Y_\infty$.

Take $x_0 \in Y_\infty$. If $x_0 \in Tx_0$, then x_0 is a fixed point of T, and this proof is complete. Suppose that $x_0 \notin Tx_0$. Since Tx_0 is a compact subset of Y_∞, we have that $dist(x_0, Tx_0) > 0$. Let $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T, and subsequently, this proof is complete. Suppose that $x_1 \notin Tx_1$. Since Tx_1 is a compact subset of Y_∞, we have that $dist(x_1, Tx_1) > 0$. Since T is $(\varphi, \mathcal{L}(x, y)) - wMKKTC_\sigma$, we also have

$$\begin{aligned}\mathcal{H}(Tx_0, Tx_1) &\leq \varphi\left(\frac{dist(x_0, Tx_0)dist(x_0, Tx_1) + dist(x_1, Tx_1)dist(x_1, Tx_0)}{dist(x_0, Tx_1) + dist(x_1, Tx_0)}\right)\\ &= \varphi(dist(x_0, Tx_0)).\end{aligned}$$

By the definition of the Hausdorff metric and above inequality, we obtain that

$$dist(x_1, Tx_1) \leq \mathcal{H}(Tx_0, Tx_1) \leq \varphi(dist(x_0, Tx_0)).$$

Since Tx_1 is a compact subset of Y_∞, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) = dist(x_1, Tx_1).$$

Thus, we have

$$d(x_1, x_2) \leq \varphi(dist(x_0, Tx_0)).$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T, and this proof is complete. Suppose that $x_2 \notin Tx_2$. Since T is $(\varphi, \mathcal{L}(x, y)) - wMKKTC_\sigma$, by taking $x = x_1$ and $y = x_2$ in (2.1), we have

$$\begin{aligned}\mathcal{H}(Tx_1, Tx_2) &\leq \varphi\left(\frac{dist(x_1, Tx_1)dist(x_1, Tx_2) + dist(x_2, Tx_2)dist(x_2, Tx_1)}{dist(x_1, Tx_2) + dist(x_2, Tx_1)}\right)\\ &= \varphi(dist(x_1, Tx_1)),\end{aligned}$$

and by the definition of the Hausdorff metric, we obtain that

$$dist(x_2, Tx_2) \leq \mathcal{H}(Tx_1, Tx_2) \leq \varphi(dist(x_1, Tx_1)).$$

Since Tx_2 is a compact subset of X, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) = dist(x_2, Tx_2).$$

Thus, we also have

$$\begin{aligned} d(x_2, x_3) &\leq \varphi(dist(x_1, Tx_1)) \\ &\leq \varphi^2(dist(x_0, Tx_0)). \end{aligned}$$

By the induction, we can obtain a sequence $\{x_n\}$ of Y_∞ satisfying

$$x_{n+1} \in Tx_n, \quad x_{n+1} \notin Tx_{n+1}, \quad \alpha(x_n, x_{n+1}) \geq 1,$$

and for each $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \varphi^n(dist(x_0, Tx_0)).$$

By (φ_2) and since $\{\varphi^n(dist(x_0, Tx_0))\}_{n\in\mathbb{N}}$ is decreasing, it converges to $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of the $wMKT$, there exists $\delta > 0$ such that for $x_0 \in X$ with $\eta \leq dist(x_0, Tx_0) < \delta + \eta$ and $\varphi^{n_0}(dist(x_0, Tx_0)) < \eta$, for some $n_0 \in \mathbb{N}$. Due to the limit $\lim_{n\to\infty} \varphi^n(dist(x_0, Tx_0)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \varphi^m(d(x_0, x_1)) < \delta + \eta$, for all $m \geq m_0$. As a result, we have $\varphi^{m_0+n_0}(dist(x_0, Tx_0)) < \eta$, a contradiction. Hence, we find

$$\lim_{n\to\infty} \varphi^n(dist(x_0, Tx_0)) = 0,$$

that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

We shall prove that the sequence $\{x_n\}$ is a Cauchy sequence. By regarding the discussion above, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m}) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m}) \\ &\vdots \\ &\leq \sum_{i=1}^{m} d(x_{n+i-1}, x_{n+i}) \\ &\leq \sum_{i=1}^{m} \varphi^{n+i-1} d(x_0, x_1). \end{aligned}$$

On account of the condition (φ_4), by letting $n \to \infty$, we derive that

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0.$$

This yields that $\{x_n\}$ is a Cauchy sequence in (Y_∞, d).

By regarding that (X, d) is complete and Y_∞ is closed, we conclude that the subspace (Y_∞, d) is complete.

Consequently, there exists $p \in Y_\infty$ such that $d(x_n, p) = 0$ as $n \to \infty$. Since T is continuous, we have $\mathcal{H}(Tx_n, Tp) = 0$ as $n \to \infty$. So, we find

$$dist(p, Tp) = \lim_{n\to\infty} dist(x_{n+1}, Tp) \leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tp) = 0.$$

Since Tp is a compact subset of Y_∞, we conclude that $p \in Tp$. □

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References

[1] A. Aghajani, R. Allahyari, M. Mursaleen, *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math., **260** (2014), 68–77. 1

[2] A. Aghajani, J. Banaś, N. Sabzali, *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math. Soc. Simon Stevin, **20** (2013), 345–358.

[3] A. Aghajani, M. Mursaleen, A. Shole Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci. Ser. B Engl. Ed., **35** (2015), 552–566.

[4] A. Aghajani, N. Sabzali, *Existence of coupled fixed points via measure of noncompactness and applications*, J. Nonlinear Convex Anal., **15** (2014), 941–952.

[5] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiĭ, *Measures of noncompactness and condensing operators*, Translated from the 1986 Russian original by A. Iacob, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, (1992). 1.2

[6] M. U. Ali, T. Kamram, E. Karapınar, *An approach to existence of fixed points of generalized contractive multi-valued mappings of integral type via admissible mapping*, Abstr. Appl. Anal., **2014** (2014), 7 pages. 1

[7] A. Amini, M. Fakhar, J. Zafarani, *KKM mappings in metric spaces*, Nonlinear Anal., **60** (2005), 1045–1052. 1

[8] J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, (1980). 1, 1.4

[9] C. M. Chen, T. H. Chang, *Fixed point theorems for a weaker Meir-Keeler type ψ-set contraction in metric spaces*, Fixed Point Theory Appl., **2009** (2009), 8 pages. 2.6, 2.7, 2.8, 2.9, 2.10

[10] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, (Italian) Rend. Sem. Mat. Univ. Padova, **24** (1955), 84–92. 1, 1.3

[11] B. Fisher, *On a theorem of Khan*, Riv. Mat. Univ. Parma, **4** (1978), 135–137.

[12] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329. 1

[13] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1

[14] R. Nussbaum, *The fixed point index and fixed point theorems for k-set contractions*, Thesis (Ph.D.)–The University of Chicago, ProQuest LLC, Ann Arbor, MI, (1969). 1.1

[15] N. Redjel, A. Dehici, *Some results in fixed point theory and application to the convergence of some iterative processes*, Fixed Point Theory Appl., **2015** (2015), 17 pages. 1, 1.6, 1

[16] B. Samet, C. Vetro, P. Vetro, *Some results on fixed points of α-ψ-Ciric generalized multifunctions*, Fixed Point Theory Appl., **2013** (2013), 10 pages. 1.7

[17] Z. Wang, H. Li, *Fixed point theorems and endpoint theorems for (α, ψ)-Meir-Keeler-Khan multivalued mappings*, Fixed Point Theory Appl., **2016** (2016), 18 pages. 1, 1.8