Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 171–185
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Robust stability analysis of uncertain T-S fuzzy systems with time-varying delay by improved delay-partitioning approach

Jun Yang[a,*], Wen-Pin Luo[b], Kai-Bo Shi[c], Xin Zhao[d]

[a] *College of Computer Science, Civil Aviation Flight University of China, Guanghan, Sichuan 618307, P. R. China.*
[b] *College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, P. R. China.*
[c] *Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.*
[d] *Postgraduate Department, Civil Aviation Flight University of China, Guanghan, Sichuan, 618307, P. R. China.*

Communicated by Xinzhi Liu

Abstract

This paper focuses on the robust stability criteria of uncertain T-S fuzzy systems with time-varying delay by an improved delay-partitioning approach. An appropriate augmented Lyapunov-Krasovskii functional (LKF) is established by partitioning the delay in all integral terms. Since the relationship between each subinterval and time-varying delay has been taken a full consideration, and some tighter bounding inequalities are employed to deal with (time-varying) delay-dependent integral items of the derivative of LKF, less conservative delay-dependent stability criteria can be expected in terms of e_s and LMIs. Finally, two numerical examples are provided to show that the proposed conditions are less conservative than existing ones.

Keywords: T-S fuzzy systems, time-varying delay, delay-partitioning approach, stability, Lyapunov-Krasovskii functional (LKF), linear matrix inequalities (LMIs).
2010 MSC: 93C42, 93D20.

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1. Introduction

Since Takagi-Sugeno (T-S) fuzzy model was first introduced in [22], much effort has been made in the stability analysis and control synthesis of this model during the past three decades, due to the fact that

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*Corresponding author
Email address: yj_uestc@126.com (Jun Yang)

Received 2015-06-25

it can combine the flexibility of fuzzy logic theory and fruitful linear system theory into a unified framework to approximate complex nonlinear systems [23, 24]. On the other hand, as a source of instability and deteriorated performance, time-delays often occur in many dynamic systems such as biological systems, chemical processes,communication networks and so on. Therefore, stability analysis for T-S fuzzy systems with time-delay has received more interest and achieved fruitful results, see, e.g., [7, 10, 14, 19, 29, 30, 31, 32] and references therein.

In recent years, many improved methods, such as free-weighting matrix [6, 8], augmented LKF [9, 12], triple integral form of LKF [8], delay-slope-dependent method [13], reciprocally convex technique [16] and delay-partitioning approach [4, 25] have been developed to reduce the conservatism of stability criteria for time-delay systems. Among the recent techniques adopted in the stability analysis of T-S fuzzy systems with time-varying delay, the most noteworthy is delay-partitioning approach, since it has been proven that less conservative results may be expected with the increasing delay-partitioning segments [17, 33]. Recently, by non-uniformly dividing the whole delay interval into multiple segments and choosing different Lyapunov functionals to different segments, [1] has established less conservative delay-derivative-dependent stability criteria than those in [3, 15] in a convex way for the nominal and uncertain T-S fuzzy systems with interval time-varying delay. Very recently, on the basis of delay-partitioning approach and a tighter bounding inequality established by reciprocally convex technique [16], [17] has developed less conservative stability criteria than those in [14, 25, 27] for the uncertain T-S fuzzy systems with interval time-varying delay. More recently, based on a novel LKF and some new bounding techniques, i.e., Seuret-Wirtinger's integral inequality and Peng-Park's integral inequality, [33] has achieved some less conservative stability criteria than those in [8, 11, 14, 17, 18] by introducing some fuzzy-weighting matrixes to express the relationship of the T-S fuzzy models. However, when revisiting this problem, we find that the aforementioned works still leave plenty of room for improvement on account of the relationship between time-varying delay and each subinterval is almost totally neglected in those works.

Based on the above-mentioned discussion, this paper will develop less conservative stability criteria for uncertain T-S fuzzy systems with time-varying delay by introducing an improved delay-partitioning approach, which partitions the time-varying delay $\tau(t)$ and it's upper bound separately. A modified augmented LKF is established by partitioning the delay in all integral terms, and the $\tau(t)$-dependent / $\rho(t)$-dependent / $[X_{ij}]_{m\times m}$-dependent sub-LKFs are introduced to the augmented LKF, thus, the relationship between each subinterval and time-varying delay and the relationships between the augmented state vectors $[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\delta), \cdots, x^{\mathrm{T}}(t-m\delta)]^{\mathrm{T}}$ have been simultaneously taken a full consideration. Then, some tighter bounding techniques such as Seuret-Wirtinger's integral inequality, Peng-Park's integral inequality and the reciprocally convex approach are employed to deal with (time-varying) delay-dependent integral items, therefore, less conservative stability criteria can be achieved in terms of e_s and LMIs. Finally, two numerical examples are included to show the effectiveness and the benefits of the proposed method.

The rest of this paper is organized as follows. The main problem is formulated in Section 2 and less conservative stability criteria for the uncertain T-S fuzzy systems with time-varying delay are derived in Section 3. In Section 4, two numerical examples are provided; and a concluding remark is given in Section 5.

Notations. Through this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices; the notation $A > (\geq) B$ means that $A - B$ is positive (semi-positive) definite; I (0) is the identity (zero) matrix with appropriate dimension; A^{T} denotes the transpose; He(A) represents the sum of A and A^{T}; $\|\bullet\|$ denotes the Euclidean norm in $\mathbb{R}^n$; "*" denotes the elements below the main diagonal of a symmetric block matrix; $C([-\tau, 0], \mathbb{R}^n)$ is the family of continuous functions ϕ from interval $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|_\tau = \sup_{-\tau\leq\theta\leq 0} \|\phi(\theta)\|$; let $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$.

2. Problem formulation

In this section, a class of uncertain T-S fuzzy system with time-varying delay is concerned. For each $i = 1, \cdots, r$ (r is the number of plant rules), the ith rule of this T-S fuzzy model is represented as follows:

Plant Rule i: **IF** $\theta_1(t)$ **is** $M_{i1}, \theta_2(t)$ **is** $M_{i2}, \cdots, \theta_p(t)$ **is** M_{ip}, **THEN**

$$\begin{cases} \dot{x}(t) = [A_i + \Delta A_i(t)]x(t) + [A_{di} + \Delta A_{di}(t)]x(t-\tau(t)), \quad t \geq 0 \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $\theta_1(t)$, $\theta_2(t)$, $\cdots$, $\theta_p(t)$ are the premise variables, and each $M_{il}(i = 1, \cdots, r; l = 1, \cdots, p)$ is a fuzzy set. $x(t) \in \mathbb{R}^n$ is the state vector; $\phi(t) \in C([-\tau, 0], \mathbb{R}^n)$ is the initial function; A_i and A_{di} are constant real matrices with appropriate dimensions; the delay, $\tau(t)$, is a time-varying functional satisfying

$$0 \leq \tau(t) \leq \tau, \tag{2.2}$$

$$\dot{\tau}(t) < \mu, \tag{2.3}$$

where τ and μ are constants; The matrices $\Delta A_i(t)$ and $\Delta A_{di}(t)$ denote the uncertainties in the system and are defined as

$$[\Delta A_i(t), \Delta A_{di}(t)] = HF(t)[E_i, E_{di}], \tag{2.4}$$

where H, E_i and E_{di} are known constant matrices and $F(t)$ is an unknown matrix function satisfying

$$F^{\mathrm{T}}(t)F(t) \leq I. \tag{2.5}$$

By a center-average defuzzier, product inference and singleton fuzzifier, the dynamic fuzzy model in (2.1) can be represented by

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} h_i(\theta(t))\{[A_i + \Delta A_i(t)]x(t) + [A_{di} + \Delta A_{di}(t)]x(t-\tau(t))\}, \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.6}$$

where

$$h_i(\theta(t)) = \frac{\prod\limits_{l=1}^{p} M_{il}(\theta_l(t))}{\sum\limits_{i=1}^{r} \prod\limits_{l=1}^{p} M_{il}(\theta_l(t))}, \quad i = 1, \cdots, r, \tag{2.7}$$

in which $M_{il}(\theta_l(t))$ is the grade of membership of $\theta_l(t)$ in M_{il}, and $\theta(t) = (\theta_1(t), \cdots, \theta_r(t))$; By definition, the fuzzy weighting functions $h_i(\theta(t))$ satisfy $h_i(\theta(t)) \geq 0$, $\sum\limits_{i=1}^{r} h_i(\theta(t)) = 1$. For notational simplicity, h_i is used to represent $h_i(\theta(t))$ in the following description.

Before proceeding, recall the following lemmas which will be used throughout proofs.

Lemma 2.1 (Peng-Park's integral inequality [16, 17]). *For any matrix* $\begin{bmatrix} Z & S \\ * & Z \end{bmatrix} \geq 0$, *scalars* $\tau > 0, \tau(t) > 0$ *satisfying* $0 < \tau(t) \leq \tau$, *vector function* $\dot{x} : [-\tau, 0] \to \mathbb{R}^n$ *such that the concerned integrations are well defined, then*

$$-\tau \int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)ds \leq \varpi^{\mathrm{T}}(t) \begin{bmatrix} -Z & Z-S & S \\ * & -2Z + \mathrm{He}(S) & -S+Z \\ * & * & -Z \end{bmatrix} \varpi(t),$$

where $\varpi(t) = [x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau(t)), x^{\mathrm{T}}(t-\tau)]^{\mathrm{T}}$.

Lemma 2.2 (Seuret-Wirtinger's integral inequality [21]). *For any positive matrix* Z, *the following inequality holds for all continuously differentiable function* x *in* $[\alpha, \beta] \to \mathbb{R}^n$*:*

$$\int_{\alpha}^{\beta} \dot{x}^{\mathrm{T}}(s)Z\dot{x}(s)ds \geq \frac{1}{\beta - \alpha} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x(s)ds \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 4Z & 2Z & -6Z \\ * & 4Z & -6Z \\ * & * & 12Z \end{bmatrix} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x(s)ds \end{bmatrix}.$$

Lemma 2.3 (Reciprocally convex approach [16])**.** *Let $f_1, f_2, \cdots, f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in an open subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{ij}(t)} \sum_{i\neq j} g_{ij}(t),$$

subject to

$$\left\{ g_{ij} : \mathbb{R}^m \to \mathbb{R}, g_{ji}(t) \triangleq g_{ij}(t), \begin{bmatrix} f_i(t) & g_{ij}(t) \\ g_{ij}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}.$$

Lemma 2.4 ([20])**.** *Let $Q = Q^{\mathrm{T}}$, H, E and $F(t)$ satisfying $F^{\mathrm{T}}(t)F(t) \leq I$ are appropriately dimensional matrices, then the following inequality*

$$Q + \mathrm{He}\{HF(t)E\} < 0$$

is true, if and only if the following inequality holds for any $\varepsilon > 0$,

$$Q + \varepsilon^{-1} HH^{\mathrm{T}} + \varepsilon E^{\mathrm{T}} E < 0.$$

3. Main results

This section aims to develop less conservative stability criteria for uncertain T-S fuzzy systems (2.6) by introducing an improved delay-partitioning approach.

For any integers $m \geq 1$ and $N \geq m$, motivated by [28], define the improved delay-partitioning approach as follows:

$$\delta = \frac{\tau}{m},\ \rho(t) = \frac{\tau(t)}{N}, \tag{3.1}$$

then $[0, \tau]$ can be divided into m segments, i.e., $[0, \tau] = \bigcup_{j=1}^{m} [(j-1)\delta, j\delta) \bigcup \{m\delta\}$.

For any $t \geq 0$, there should exist an integer $k \in \{1, \cdots, m\}$, such that $\tau(t) \in [(k-1)\delta,\ k\delta)$ (in what follows, in the case of $\tau(t) = m\delta$, one can put $\tau(t) \in [(m-1)\delta,\ m\delta]$); noting $0 \leq \rho(t) \leq \delta$, for each subinterval $[(j-1)\delta, j\delta)$ $(j = 1, \cdots, m)$, it is easy to obtain that

$$(j-1)\delta + \rho(t) \in [(j-1)\delta, j\delta],\quad j = 1, \cdots, m. \tag{3.2}$$

Remark 3.1. The improved delay-partitioning approach (3.1), which partitions the time-varying delay $\tau(t)$ and it's upper bound separately, includes the method in [28] as special cases by letting $N = m$ in (3.1). On the other hand, by taking advantage of (3.2), the relationship between the time-varying delay and each subinterval has been taken a full consideration. Furthermore, it is worth mentioning that, when the delay-partitioning number m is fixed, the conservatism is gradually reduced with the increase of another delay-partitioning number N, but without increasing any computing burden, which can be demonstrated later in numerical examples section.

For notational simplification, let

$$\begin{cases} e_s = \Big[\underbrace{0, \cdots, 0}_{s-1}, I, \underbrace{0, \cdots, 0}_{2m-s+3}\Big]^{\mathrm{T}}, \quad s = 1, \cdots, 2m+3, \\ \zeta(t) = \Big[x^{\mathrm{T}}(t-\tau(t)), \zeta_1^{\mathrm{T}}(t), x^{\mathrm{T}}(t-m\delta), \frac{1}{\delta}\int_{t-\delta}^{t} x^{\mathrm{T}}(s)ds, \zeta_2^{\mathrm{T}}(t)\Big]^{\mathrm{T}}, \end{cases} \tag{3.3}$$

where

$$\zeta_1(t) = [x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\delta), \cdots, x^{\mathrm{T}}(t-(m-1)\delta)]^{\mathrm{T}},$$
$$\zeta_2(t) = [x^{\mathrm{T}}(t-\rho(t)), x^{\mathrm{T}}(t-\delta-\rho(t)), \cdots, x^{\mathrm{T}}(t-(m-1)\delta-\rho(t)]^{\mathrm{T}}.$$

Based on the Lyapunov-Krasovskii stability theorem [5], we firstly state the following stability criterion for the nominal system (2.6), i.e. system (2.6) without parameter uncertainties.

Theorem 3.2. *Given positive integers* m *and* $N \geq m$, *scalars* $\tau \geq 0$, μ, $\delta = \frac{\tau}{m}$ *and* $\alpha_k \in (0,1)$ $(k = 1,\cdots,m)$, *then the nominal system* (2.6) *with a time-delay* $\tau(t)$ *satisfying* (2.2) *and* (2.3) *is asymptotically stable if there exist symmetric positive matrices* $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}$, $X = [X_{ij}]_{m\times m} \triangleq \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & \ddots & \vdots \\ * & \cdots & X_{mm} \end{bmatrix}$, $Q_j,\ W_j,\ Z_0,\ Z_j,\ R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix}$ *and any matrices* S_{ij}, U_{ij} *and* V_{ij} $(i = 1,\cdots,r; j = 1,\cdots,m; l = 1,\cdots,m-1)$ *with appropriate dimensions, such that the following LMIs hold for* $i = 1,\cdots,r$ *and* $k = 1,\cdots,m$:

$$\Lambda(i,k) = \begin{bmatrix} Z_j & S_{ij} \\ * & Z_j \end{bmatrix} \geq 0,\ j = 1,\cdots,m, j \neq k, \tag{3.4}$$

$$\Upsilon(i,k) = \begin{cases} \begin{bmatrix} NZ_1 & U_{i1} \\ * & \alpha_1 Z_1 \end{bmatrix} \geq 0, \begin{bmatrix} \frac{N}{N-1}Z_1 & V_{i1} \\ * & (1-\alpha_1)Z_1 \end{bmatrix} \geq 0, \quad k = 1 \\ \begin{cases} \begin{bmatrix} \frac{N}{N-k+1}Z_k & U_{ik} \\ * & \frac{\alpha_k}{(N-k+1)}Z_k \end{bmatrix} \geq 0, \begin{bmatrix} \frac{N}{k-1}Z_k & V_{ik} \\ * & \frac{(1-\alpha_k)(N-1)}{(k-1)}Z_k \end{bmatrix} \geq 0, \\ \text{or} \qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq k \leq m, \\ \begin{bmatrix} \frac{\alpha_k(N-1)}{N-k}Z_k & U_{ik} \\ * & \frac{N}{N-k}Z_k \end{bmatrix} \geq 0, \begin{bmatrix} \frac{(1-\alpha_k)}{k}Z_k & V_{ik} \\ * & \frac{N}{k}Z_k \end{bmatrix} \geq 0, \end{cases} \end{cases} \tag{3.5}$$

$$\begin{bmatrix} \Xi(i,k) & \delta\Gamma_i^{\mathrm{T}}\bar{Z} \\ * & -\bar{Z} \end{bmatrix} < 0, \tag{3.6}$$

where

$$\Gamma_i = A_i e_2^{\mathrm{T}} + A_{di}e_1^{\mathrm{T}}, \quad \bar{Z} = \sum_{j=0}^{m} Z_j,\ \Xi(i,k) = \sum_{j=0}^{3}\Xi_j + \Xi_4(k) + \Xi_5(i,k) - \Xi_6(i,k)$$

with

$$\Xi_0 = \begin{bmatrix} e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} \\ e_{m+3}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -4Z_0 & -2Z_0 & 6Z_0 \\ * & -4Z_0 & 6Z_0 \\ * & * & -12Z_0 \end{bmatrix} \begin{bmatrix} e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} \\ e_{m+3}^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_1 = \mathrm{He}\left\{ \begin{bmatrix} e_2^{\mathrm{T}} \\ \delta e_{m+3}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \begin{bmatrix} \Gamma_i \\ e_2^{\mathrm{T}} - e_3^{\mathrm{T}} \end{bmatrix} \right\},$$

$$\Xi_2 = \begin{bmatrix} e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} \\ \vdots \\ e_{m+1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} X \begin{bmatrix} e_2^{\mathrm{T}} \\ e_3^{\mathrm{T}} \\ \vdots \\ e_{m+1}^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} e_3^{\mathrm{T}} \\ e_4^{\mathrm{T}} \\ \vdots \\ e_{m+2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} X \begin{bmatrix} e_3^{\mathrm{T}} \\ e_4^{\mathrm{T}} \\ \vdots \\ e_{m+2}^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_3 = \sum_{j=1}^{m-1}\left(\begin{bmatrix} e_{j+1}^{\mathrm{T}} \\ e_{j+2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} R_j \begin{bmatrix} e_{j+1}^{\mathrm{T}} \\ e_{j+2}^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} e_{j+2}^{\mathrm{T}} \\ e_{j+3}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} R_j \begin{bmatrix} e_{j+2}^{\mathrm{T}} \\ e_{j+3}^{\mathrm{T}} \end{bmatrix} \right),$$

$$\begin{aligned} \Xi_4(k) = &\sum_{j=1}^{k-1}\left[e_{j+1}Q_j e_{j+1}^{\mathrm{T}} - e_{j+2}Q_j e_{j+2}^{\mathrm{T}}\right] + e_{k+1}Q_k e_{k+1}^{\mathrm{T}} - (1-\mu)e_1 Q_k e_1^{\mathrm{T}} \\ &+ \sum_{j=1}^{m}\left[e_{j+1}W_j e_{j+1}^{\mathrm{T}} - (1-\frac{\mu}{N})e_{m+j+3}W_j e_{m+j+3}^{\mathrm{T}}\right], \end{aligned}$$

$$\Xi_5(i,k)=\sum_{j=1,j\neq k}^{m}\left(\begin{bmatrix} e_{j+1}^{\mathrm{T}}\\ e_{m+j+3}^{\mathrm{T}}\\ e_{j+2}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -Z_j & Z_j-S_{ij} & S_{ij}\\ * & -2Z_j+\mathrm{He}(S_{ij}) & Z_j-S_{ij}\\ * & * & -Z_j\end{bmatrix}\begin{bmatrix} e_{j+1}^{\mathrm{T}}\\ e_{m+j+3}^{\mathrm{T}}\\ e_{j+2}^{\mathrm{T}}\end{bmatrix}\right),$$

$$\Xi_6(i,k)=\begin{cases}\begin{pmatrix}[e_2-e_{m+4}](NZ_1)[e_2^{\mathrm{T}}-e_{m+4}^{\mathrm{T}}]\\ +[e_1-e_3]Z_1[e_1^{\mathrm{T}}-e_3^{\mathrm{T}}]\\ +\mathrm{He}([e_2-e_{m+4}]U_{i1}[e_1^{\mathrm{T}}-e_3^{\mathrm{T}}])\\ +[e_{m+4}-e_1]\dfrac{N}{N-1}Z_1[e_{m+4}^{\mathrm{T}}-e_1^{\mathrm{T}}]\\ +\mathrm{He}([e_{m+4}-e_1]V_{i1}[e_1^{\mathrm{T}}-e_3^{\mathrm{T}}])\end{pmatrix}, \quad k=1;\\[2ex] \begin{cases}\begin{pmatrix}[e_{m+k+3}-e_{k+2}]\dfrac{N}{N-k+1}Z_k[e_{m+k+3}^{\mathrm{T}}-e_{k+2}^{\mathrm{T}}]\\ +[e_{k+1}-e_1]\dfrac{\alpha_k}{N-k+1}Z_k[e_{k+1}^{\mathrm{T}}-e_1^{\mathrm{T}}]\\ +\mathrm{He}([e_{m+k+3}-e_{k+2}]U_{ik}[e_{k+1}^{\mathrm{T}}-e_1^{\mathrm{T}}])\\ +[e_{k+1}-e_1]\dfrac{(1-\alpha_k)(N-1)}{k-1}Z_k[e_{k+1}^{\mathrm{T}}-e_1^{\mathrm{T}}]\\ +[e_1-e_{m+k+3}]\dfrac{N}{k-1}Z_k[e_1^{\mathrm{T}}-e_{m+k+3}^{\mathrm{T}}]\\ +\mathrm{He}([e_{k+1}-e_1]V_{ik}[e_1^{\mathrm{T}}-e_{m+k+3}^{\mathrm{T}}])\end{pmatrix}\\ \triangleq \Xi_6(1,i,k), \quad 2\le k\le m,\\ or\\ \begin{pmatrix}[e_1-e_{k+2}]\dfrac{\alpha_k(N-1)}{N-k}Z_k[e_1^{\mathrm{T}}-e_{k+2}^{\mathrm{T}}]\\ +[e_{m+k+3}-e_1]\dfrac{N}{N-k}Z_k[e_{m+k+3}^{\mathrm{T}}-e_1^{\mathrm{T}}]\\ +\mathrm{He}([e_1-e_{k+2}]U_{ik}[e_{m+k+3}^{\mathrm{T}}-e_1^{\mathrm{T}}])\\ +[e_1-e_{k+2}]\dfrac{(1-\alpha_k)}{k}Z_k[e_1^{\mathrm{T}}-e_{k+2}^{\mathrm{T}}]\\ +[e_{k+1}-e_{m+k+3}]\dfrac{N}{k}Z_k[e_{k+1}^{\mathrm{T}}-e_{m+k+3}^{\mathrm{T}}]\\ +\mathrm{He}([e_1-e_{k+2}]V_{ik}[e_{k+1}^{\mathrm{T}}-e_{m+k+3}^{\mathrm{T}}])\end{pmatrix}\\ \triangleq \Xi_6(2,i,k), \quad 2\le k\le m.\end{cases}\end{cases}$$

Proof. For any $t\ge 0$, there should exist an integer $k\in\{1,\cdots,m\}$, such that $\tau(t)\in[(k-1)\delta,\ k\delta)$. Then, choose the following augmented LKF candidate for the nominal system (2.6):

$$V(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\ k\delta)\}}=\sum_{i=1}^{5}V_i(x_t), \tag{3.7}$$

where

$$V_1(x_t)=\eta_0^{\mathrm{T}}(t)P\eta_0(t),$$

$$V_2(x_t)=\int_{t-\delta}^{t}\zeta_1^{\mathrm{T}}(s)X\zeta_1(s)ds,$$

$$V_3(x_t)=\sum_{j=1}^{m-1}\int_{t-j\delta}^{t-(j-1)\delta}\eta_1^{\mathrm{T}}(s)R_j\eta_1(s)ds,$$

$$V_4(x_t)=\sum_{j=1}^{k-1}\int_{t-j\delta}^{t-(j-1)\delta}x^{\mathrm{T}}(s)Q_jx(s)ds+\int_{t-\tau(t)}^{t-(k-1)\delta}x^{\mathrm{T}}(s)Q_kx(s)ds+\sum_{j=1}^{m}\int_{t-(j-1)\delta-\rho(t)}^{t-(j-1)\delta}x^{\mathrm{T}}(s)W_jx(s)ds,$$

$$V_5(x_t) = \sum_{j=1}^{m} \delta \int_{-j\delta}^{-(j-1)\delta} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Z_j \dot{x}(s) ds d\theta + \delta \int_{-\delta}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Z_0 \dot{x}(s) ds d\theta,$$

with $\eta_0(t) = [x^{\mathrm{T}}(t), \int_{t-\delta}^{t} x^{\mathrm{T}}(s)ds]^{\mathrm{T}}$, $\eta_1(s) = [x^{\mathrm{T}}(s), x^{\mathrm{T}}(s-\delta)]^{\mathrm{T}}$. Taking the derivative of $V(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\, k\delta)\}}$ along the trajectory of the nominal system (2.6) yields:

$$\dot{V}(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\, k\delta)\}} = \sum_{i=1}^{5} \dot{V}_i(x_t). \tag{3.8}$$

where

$$\dot{V}_1(x_t) = 2\eta_0^{\mathrm{T}}(t) P \dot{\eta}_0(t) = \zeta^{\mathrm{T}}(t)\Xi_1\zeta(t),$$

$$\dot{V}_2(x_t) = \zeta_1^{\mathrm{T}}(t) X \zeta_1(t) - \zeta_1^{\mathrm{T}}(t-\delta) X \zeta_1(t-\delta) = \zeta^{\mathrm{T}}(t)\Xi_2\zeta(t),$$

$$\dot{V}_3(x_t) = \sum_{j=1}^{m-1} [\eta_1^{\mathrm{T}}(t-(j-1)\delta) R_j \eta_1(t-(j-1)\delta) - \eta_1^{\mathrm{T}}(t-j\delta) R_j \eta_1(t-j\delta)] = \zeta^{\mathrm{T}}(t)\Xi_3\zeta(t),$$

$$\begin{aligned}\dot{V}_4(x_t) \le &\sum_{j=1}^{k-1}[x^{\mathrm{T}}(t-(j-1)\delta)Q_j x(t-(j-1)\delta) - x^{\mathrm{T}}(t-j\delta)Q_j x(t-j\delta)] \\ &+x^{\mathrm{T}}(t-(k-1)\delta)Q_k x(t-(k-1)\delta) - (1-\mu)x^{\mathrm{T}}(t-\tau(t))Q_k x(t-\tau(t)) \\ &+\sum_{j=1}^{m}[x^{\mathrm{T}}(t-(j-1)\delta)W_j x(t-(j-1)\delta)] \\ &-\sum_{j=1}^{m}[(1-\frac{\mu}{N})x^{\mathrm{T}}(t-(j-1)\delta-\rho(t))W_j x(t-(j-1)\delta-\rho(t))] = \zeta^{\mathrm{T}}(t)\Xi_4(k)\zeta(t),\end{aligned}$$

$$\dot{V}_5(x_t) = \dot{x}^{\mathrm{T}}(t)\left(\delta^2\sum_{j=0}^{m} Z_j\right)\dot{x}(t) - \delta\sum_{j=1}^{m}\int_{t-j\delta}^{t-(j-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_j\dot{x}(s)ds - \delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_0\dot{x}(s)ds. \tag{3.9}$$

For the case of $\tau(t) \notin [(k-1)\delta, k\delta]$: consider $-\delta \sum_{j=1,j\neq k}^{m} \int_{t-j\delta}^{t-(j-1)\delta} \dot{x}^{\mathrm{T}}(s)Z_j\dot{x}(s)ds$ in (3.9). Noting (3.2), (3.3) and applying Lemma 2.1 (Peng-Park's integral inequality), it can be deduced for $\begin{bmatrix} Z_j & \widehat{S}_j \\ * & Z_j \end{bmatrix} \ge 0$ $(j = 1,\cdots,m, j\neq k)$ (where $\widehat{S}_j = \sum_{i=1}^{r} h_i S_{ij}$) that

$$\begin{aligned}-\delta\sum_{j=1,j\neq k}^{m}\int_{t-j\delta}^{t-(j-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_j\dot{x}(s)ds &\le \sum_{j=1,j\neq k}^{m}\varpi_1^{\mathrm{T}}(t)\begin{bmatrix} -Z_j & Z_j-\widehat{S}_j & \widehat{S}_j \\ * & -2Z_j+\mathrm{He}(\widehat{S}_j) & Z_j-\widehat{S}_j \\ * & * & -Z_j\end{bmatrix}\varpi_1(t) \\ &= \sum_{i=1}^{r} h_i\zeta^{\mathrm{T}}(t)\Xi_5(i,k)\zeta(t),\end{aligned} \tag{3.10}$$

where $\varpi_1(t) = [x^{\mathrm{T}}(t-(j-1)\delta), x^{\mathrm{T}}(t-(j-1)\delta-\rho(t)), x^{\mathrm{T}}(t-j\delta)]^{\mathrm{T}}$.

For the case of $\tau(t) \in [(k-1)\delta, k\delta]$: (i) when $k=1, -\delta\int_{t-k\delta}^{t-(k-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds = -\delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds$, noting $\rho(t) = \tau(t)/N$ and $\alpha_1 \in (0,1)$, then it follows from Jensen's inequality that

$$-\delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds = -\delta\left(\int_{t-\rho(t)}^{t} + \int_{t-\tau(t)}^{t-\rho(t)} + \int_{t-\delta}^{t-\tau(t)}\right)\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds$$

$$
\begin{aligned}
&\le -\frac{\delta}{\tau(t)}[x(t)-x(t-\rho(t))]^{\mathrm{T}}(NZ_1)[x(t)-x(t-\rho(t))]\\
&-\frac{\delta}{\delta-\tau(t)}[x(t-\tau(t))-x(t-\delta)]^{\mathrm{T}}\alpha_1 Z_1[x(t-\tau(t))-x(t-\delta)]\\
&-\frac{\delta}{\tau(t)}[x(t-\rho(t))-x(t-\tau(t))]^{\mathrm{T}}\frac{N}{N-1}Z_1[x(t-\rho(t))-x(t-\tau(t))]\\
&-\frac{\delta}{\delta-\tau(t)}[x(t-\tau(t))-x(t-\delta)]^{\mathrm{T}}(1-\alpha_1)Z_1[x(t-\tau(t))-x(t-\delta)].
\end{aligned}
\tag{3.11}
$$

Next, LMIs (3.5) give that $\begin{bmatrix} NZ_1 & \tilde{U}_1 \\ * & \alpha_1 Z_1 \end{bmatrix} \ge 0, \begin{bmatrix} \frac{N}{N-1}Z_1 & \tilde{V}_1 \\ * & (1-\alpha_1)Z_1 \end{bmatrix} \ge 0$(where $\tilde{U}_1 = \sum_{i=1}^{r} h_i U_{i1}$, $\tilde{V}_1 = \sum_{i=1}^{r} h_i V_{i1}$), then it follows from (3.11) and Lemma 2.3 (Reciprocally convex approach) that

$$
\begin{aligned}
-\delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds \le\ &-[x(t)-x(t-\rho(t))]^{\mathrm{T}}(NZ_1)[x(t)-x(t-\rho(t))]\\
&-[x(t-\tau(t))-x(t-\delta)]^{\mathrm{T}}\alpha_1 Z_1[x(t-\tau(t))-x(t-\delta)]\\
&-\mathrm{He}\{[x(t)-x(t-\rho(t))]^{\mathrm{T}}\tilde{U}_1[x(t-\tau(t))-x(t-\delta)]\}\\
&-[x(t-\rho(t))-x(t-\tau(t))]^{\mathrm{T}}\frac{N}{N-1}Z_1[x(t-\rho(t))-x(t-\tau(t))]\\
&-[x(t-\tau(t))-x(t-\delta)]^{\mathrm{T}}(1-\alpha_1))Z_1[x(t-\tau(t))-x(t-\delta)]\\
&-\mathrm{He}\{[x(t-\rho(t))-x(t-\tau(t))]^{\mathrm{T}}\tilde{V}_1[x(t-\tau(t))-x(t-\delta)]\}\\
=&-\sum_{i=1}^{r}h_i\zeta^{\mathrm{T}}(t)\Xi_6(i,k)\zeta(t),\quad (k=1).
\end{aligned}
\tag{3.12}
$$

(ii) When $2 \le k \le m$:
(a) When $\tau(t) < (k-1)\delta + \rho(t)$, noting $\rho(t) = \tau(t)/N$ and $\alpha_k \in (0,1)$, then it follows from Jensen inequality that

$$
\begin{aligned}
&-\delta\int_{t-k\delta}^{t-(k-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds\\
=&-\delta\left(\int_{t-k\delta}^{t-(k-1)\delta-\rho(t)}+\int_{t-(k-1)\delta-\rho(t)}^{t-\tau(t)}+\int_{t-\tau(t)}^{t-(k-1)\delta}\right)\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds\\
&-\frac{(N-k+1)\delta}{N\delta-\tau(t)}[x(t-(k-1)\delta-\rho(t))-x(t-k\delta)]^{\mathrm{T}}\frac{NZ_k}{N-k+1}[x(t-(k-1)\delta-\rho(t))-x(t-k\delta)]\\
&-\frac{(N-k+1)\delta}{\tau(t)-(k-1)\delta}[x(t-(k-1)\delta)-x(t-\tau(t))]^{\mathrm{T}}\frac{\alpha_k Z_k}{N-k+1}[x(t-(k-1)\delta)-x(t-\tau(t))]\\
&-\frac{\frac{k-1}{N-1}\delta}{\frac{N(k-1)}{N-1}\delta-\tau(t)}[x(t-\tau(t))-x(t-(k-1)\delta-\rho(t))]^{\mathrm{T}}\frac{NZ_k}{k-1}[x(t-\tau(t))-x(t-(k-1)\delta-\rho(t))]\\
&-\frac{\frac{k-1}{N-1}\delta}{\tau(t)-(k-1)\delta}[x(t-(k-1)\delta)-x(t-\tau(t))]^{\mathrm{T}}\frac{(1-\alpha_k)(N-1)Z_k}{k-1}[x(t-(k-1)\delta)-x(t-\tau(t))].
\end{aligned}
\tag{3.13}
$$

By (3.5), it gives that $\begin{bmatrix} \frac{N}{N-k+1}Z_k & \tilde{U}_k \\ * & \frac{\alpha_k}{N-k+1}Z_k \end{bmatrix} \ge 0$, $\begin{bmatrix} \frac{N}{k-1}Z_k & \tilde{V}_k \\ * & \frac{(1-\alpha_k)(N-1)}{k-1}Z_k \end{bmatrix} \ge 0$, where $\tilde{U}_k = \sum_{i=1}^{r} h_i U_{ik}$, $\tilde{V}_k = \sum_{i=k}^{r} h_i V_{ik}$. Then it follows from (3.13) and Lemma 2.3 that

$$
\begin{aligned}
&-\delta\int_{t-k\delta}^{t-(k-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds\\
&\le -[x(t-(k-1)\delta-\rho(t))-x(t-k\delta)]^{\mathrm{T}}\frac{NZ_k}{N-k+1}[x(t-(k-1)\delta-\rho(t))-x(t-k\delta)]
\end{aligned}
$$

$$
\begin{aligned}
&-[x(t-(k-1)\delta)-x(t-\tau(t))]^{\mathrm{T}}\frac{\alpha_k Z_k}{N-k+1}[x(t-(k-1)\delta)-x(t-\tau(t))]\\
&-\mathrm{He}\{[x(t-(k-1)\delta-\rho(t))-x(t-k\delta)]^{\mathrm{T}}\tilde{U}_k[x(t-(k-1)\delta)-x(t-\tau(t))]\}\\
&-[x(t-\tau(t))-x(t-(k-1)\delta-\rho(t))]^{\mathrm{T}}\frac{NZ_k}{k-1}[x(t-\tau(t))-x(t-(k-1)\delta-\rho(t))]\\
&-[x(t-(k-1)\delta)-x(t-\tau(t))]^{\mathrm{T}}\frac{(1-\alpha_k)(N-1)Z_k}{k-1}[x(t-(k-1)\delta)-x(t-\tau(t))]\\
&-\mathrm{He}\{[x(t-\tau(t))-x(t-(k-1)\delta-\rho(t))]^{\mathrm{T}}\tilde{V}_k[x(t-(k-1)\delta)-x(t-\tau(t))]\}\\
=&-\sum_{i=1}^{r}h_i\zeta^{\mathrm{T}}(t)\Xi_6(1,i,k)\zeta(t),\quad (2\le k\le m).
\end{aligned}
\tag{3.14}
$$

(b) When $\tau(t)=(k-1)\delta+\rho(t)$, one has $\zeta^{\mathrm{T}}(t)(e_1-e_{m+k+3})=0$, so (3.14) still holds.
(c) When $\tau(t)>(k-1)\delta+\rho(t)$, in the same manner, by

$$
\begin{bmatrix} \frac{\alpha_k(N-1)}{N-k}Z_k & \tilde{U}_k \\ * & \frac{N}{N-k}Z_k \end{bmatrix}\ge 0,\ \begin{bmatrix} \frac{(1-\alpha_k)}{k}Z_k & \tilde{V}_{ik} \\ * & \frac{N}{k}Z_k \end{bmatrix}\ge 0,
$$

one has

$$
\begin{aligned}
-\delta\int_{t-k\delta}^{t-(k-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds&=-\delta\left(\int_{t-k\delta}^{t-\tau(t)}+\int_{t-\tau(t)}^{t-(k-1)\delta-\rho(t)}+\int_{t-(k-1)\delta-\rho(t)}^{t-(k-1)\delta}\right)\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds\\
&\le-\sum_{i=1}^{r}h_i\zeta^{\mathrm{T}}(t)\Xi_6(2,i,k)\zeta(t),\quad (2\le k\le m).
\end{aligned}
\tag{3.15}
$$

In what follows, by Lemma 2.2 (Seuret-Wirtinger's integral inequality), it gives

$$
-\delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds\le\begin{bmatrix} x(t)\\ x(t-\delta)\\ \frac{1}{\delta}\int_{t-\delta}^{t}x(s)ds\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -4Z_0 & -2Z_0 & 6Z_0\\ * & -4Z_0 & 6Z_0\\ * & * & -12Z_0\end{bmatrix}\begin{bmatrix} x(t)\\ x(t-\delta)\\ \frac{1}{\delta}\int_{t-\delta}^{t}x(s)ds\end{bmatrix}=\zeta^{\mathrm{T}}(t)\Xi_0\zeta(t).
\tag{3.16}
$$

Hence, by (3.8)-(3.16), the following inequality holds

$$
\dot{V}(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\,k\delta)\}}\le\sum_{i=1}^{r}h_i\zeta^{\mathrm{T}}(t)[\Xi(i,k)+\delta^2\Gamma_i^{\mathrm{T}}\bar{Z}\Gamma_i]\zeta(t),
\tag{3.17}
$$

where $\Xi(i,k)$, Γ_i, $\bar{Z}$ are defined in Theorem 3.2.
On the other hand, by Schur complement, LMIs (3.6) give that $\Xi(i,k)+\delta^2\Gamma_i^{\mathrm{T}}\bar{Z}\Gamma_i<0$, which implies $\dot{V}(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\,k\delta)\}}<0$ by (3.17). This means $\dot{V}(t,x_t)|_{\{\tau(t)\in[(k-1)\delta,\,k\delta)\}}<-\gamma\|x(t)\|^2$ for a sufficiently small $\gamma>0$. Therefore, according to Lyapunov-Krasovskii stability theorem [5], the nominal system (2.6) with time-varying delay $\tau(t)$ satisfying (2.2) and (2.3) is globally asymptotically stable. This completes the proof. □

For the uncertain T-S fuzzy system (2.6), replacing A_i and A_{di} with $A_i+HF(t)E_i$ and $A_{di}+HF(t)E_{di}$ in (3.6), the following result can be easily derived by applying Lemma 2.4 and Schur complement [2]. Thus, it is omitted here.

Theorem 3.3. *Given positive integers m and $N\ge m$, scalars $\tau\ge 0$, μ, $\delta=\frac{\tau}{m}$ and $\alpha_k\in(0,1)$ $(k=1,\cdots,m)$, then the uncertain T-S system (2.6) with the time-delay $\tau(t)$ satisfying (2.2) and (2.3) is asymptotically stable if there exist scalars $\varepsilon_{ik}>0$ $(i=1,\cdots,r;k=1,\cdots,m)$, symmetric positive matrices $P=\begin{bmatrix} P_1 & P_2\\ * & P_3\end{bmatrix}$, $X=[X_{ij}]_{m\times m}$, Q_j, W_j, Z_0, Z_j, $R_l=\begin{bmatrix} R_{1l} & R_{2l}\\ * & R_{3l}\end{bmatrix}$ and any matrices S_{ij}, U_{ij} and*

V_{ij} $(i = 1, \cdots, r; j = 1, \cdots, m; l = 1, \cdots, m-1)$ *with appropriate dimensions, such that the following LMIs hold for* $i = 1, \cdots, r$ *and* $k = 1, \cdots, m$*:*

$$\Lambda(i,k) \geq 0, \ \Upsilon(i,k) \geq 0,$$

$$\begin{bmatrix} \Xi(i,k) & \delta\Gamma_i^{\mathrm{T}}\bar{Z} & (e_2P_1 + \delta e_{m+3}P_2)H & \varepsilon_{ik}(e_2E_i^{\mathrm{T}} + e_1E_{di}^{\mathrm{T}}) \\ * & -\bar{Z} & \delta\bar{Z}H & 0 \\ * & * & -\varepsilon_{ik}I & 0 \\ * & * & * & -\varepsilon_{ik}I \end{bmatrix} < 0, \tag{3.18}$$

where $\Lambda(i,k)$, $\Upsilon(i,k)$, $\Xi(i,k)$, Γ_i *and* $\bar{Z}$ *are defined in Theorem 3.2.*

Remark 3.4. Based on the improved delay-partitioning approach (3.1), the LKF (3.7) is quite different from those in [8, 11, 14, 15, 18, 26, 33] in the following aspects: (a) the modified augmented LKF (3.7) is established by partitioning time delay in all integral terms; (b) the time-varying delay $\tau(t)$-dependent / $\rho(t)$-dependent sub-LKFs are included in the LKF (3.7), so the relationship between each subinterval and time-varying delay has benn taken a full consideration; (c) the $[X_{ij}]_{m\times m}$-dependent sub-LKF is also included in the LKF (3.7), as a result, the relationships between the augmented state vectors $[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\delta), \cdots, x^{\mathrm{T}}(t-(m-1)\delta)]^{\mathrm{T}}$ have been fully taken into account. With these differences and advantages above-mentioned, less conservative results than those in [8, 11, 14, 15, 18, 26, 33] can be achieved, which will be demonstrated later by numerical example.

Remark 3.5. For the case of $\tau(t) \in [(k-1)\delta, k\delta)$, in order to estimate $-\delta\int_{t-k\delta}^{t-(k-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)ds$, the subinterval $[(k-1)\delta, k\delta)$ is only decomposed into two segments, i.e., $[(k-1)\delta, \tau(t)]$ and $[\tau(t), k\delta)$ in [17, 33] and references therein. In this paper, the subinterval $[(k-1)\delta, k\delta)$ is not only decomposed into two segments $[(k-1)\delta, \tau(t)]$ and $[\tau(t), k\delta)$, but also into another two segments, i.e., $[(k-1)\delta, (k-1)\delta + \rho(t))$ and $[(k-1)\delta + \rho(t), k\delta)$. Then, by combining the reciprocally convex approach with the this improved delay-partitioning method, less conservative conditions have achieved, which will be demonstrated through two numerical examples later.

Remark 3.6. A tighter bounding inequality, i.e., Peng-Park's integral inequality (Lemma 2.1), is employed to effectively estimate the time-varying delay-dependent integral items $-\delta\sum_{j=1,j\neq k}^{m}\int_{t-j\delta}^{t-(j-1)\delta}\dot{x}^{\mathrm{T}}(s)Z_j\dot{x}(s)ds$ by means of introducing the variable $(j-1)\delta + \rho(t) \in [(j-1)\delta, j\delta]$ in (3.2), therefore, less conservative results can be expected since none of any useful time-varying items are arbitrarily ignored [17]. On the other hand, the Seuret-Wirtinger's integral inequality (Lemma 2.2), that is shown less conservative than previous inequalities often based on Jensen's theorem, is adopted to estimate the integral term $-\delta\int_{t-\delta}^{t}\dot{x}^{\mathrm{T}}(s)Z_0\dot{x}(s)ds$, which will also lead to less conservative conditions.

Remark 3.7. When dealing with the inequalities (3.11) and (3.13), the positive scalars α_k $(k = 1, \cdots, m)$ are arbitrarily set as 0.5 in [28], so our method is theoretically better than [28].

Remark 3.8. The vector e_s defined in (3.3) plays a key role in representing the derivative of LKF (3.7) in a concise and unified framework of the state vector augmentation $\zeta(t)$, without listing out each elements of the ultra-large-scale symmetric block-matrix (3.6) one by one. It's worth mentioning that, the LMIs-based stability criteria in terms of vector e_s can be directly implemented by Matlab LMI Toolbox, for example, the term $\delta e_{m+3}P_2^{\mathrm{T}}e_2^{\mathrm{T}}$ in (3.6) directly shows that one of $(m+3, 2)$'s elements in LMI (3.6) is δP_2^{T}.

Finally, in the case of the time-varying delay $\tau(t)$ being non-differentiable or unknown $\dot{\tau}(t)$, setting $Q_k = 0$ $(Q_j \neq 0, j = 1, \cdots, k-1)$ and $W_j = 0$ $(j = 1, \cdots, m)$ in Theorem 3.3, we have the following corollary.

Corollary 3.9. *Given positive integers* m *and* $N \geq m$, *scalars* $\tau \geq 0,$, $\delta = \frac{\tau}{m}$ *and* $\alpha_k \in (0,1)$ $(k = 1, \cdots, m)$, *then the uncertain T-S system* (2.6) *with a time-delay* $\tau(t)$ *satisfying* (2.2) *is asymptotically stable if there exist scalars* $\varepsilon_{ik} > 0$ $(i = 1, \cdots, r; k = 1, \cdots, m)$, *symmetric positive matrices* $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}$,

$X = [X_{ij}]_{m\times m}$, Q_j, Z_0, Z_j, $R_l = \begin{bmatrix} R_{1l} & R_{2l} \\ * & R_{3l} \end{bmatrix}$ *and any matrices* S_{ij}, U_{ij} *and* V_{ij} $(i = 1,\cdots,r;$ $j = 1,\cdots,m; l = 1,\cdots,m-1)$ *with appropriate dimensions, such that the following LMIs hold for* $i = 1,\cdots,r$ *and* $k = 1,\cdots,m$:

$$\Lambda(i,k) \geq 0,\ \Upsilon(i,k) \geq 0,$$

$$\begin{bmatrix} \widetilde{\Xi}(i,k) & \delta\Gamma_i^{\mathrm{T}}\bar{Z} & (e_2P_1 + \delta e_{m+3}P_2)H & \varepsilon_{ik}(e_2E_i^{\mathrm{T}} + e_1E_{di}^{\mathrm{T}}) \\ * & -\bar{Z} & \delta\bar{Z}H & 0 \\ * & * & -\varepsilon_{ik}I & 0 \\ * & * & * & -\varepsilon_{ik}I \end{bmatrix} < 0, \tag{3.19}$$

where $\widetilde{\Xi}(i,k)$ *is obtained from* $\Xi(i,k)$ *by substituting* $\Xi_4(k)$ *with* $\sum_{j=1}^{k-1}\left[e_{j+1}Q_je_{j+1}^{\mathrm{T}} - e_{j+2}Q_je_{j+2}^{\mathrm{T}}\right]$, *and* $\Lambda(i,k)$, $\Upsilon(i,k)$, Γ_i *and* $\bar{Z}$ *are defined in Theorem 3.2.*

4. Numerical examples

This section gives two examples to demonstrate the effectiveness of the proposed approach. For comparisons, the T-S fuzzy system (2.6) with fuzzy rules investigated in recent publications [8, 11, 15, 17, 18, 33] has been studied.

Example 1. Consider the T-S fuzzy systems (2.6) with time-varying delay and plant rules as follows [8, 11, 15, 17, 18, 33]:

$$\begin{aligned} &R^1: \ \text{If } \theta(t) \text{ is} \pm \pi/2, \text{ then } x(t) = A_1x(t) + A_{d1}x(t-\tau(t)); \\ &R^2: \ \text{If } \theta(t) \text{ is } 0, \text{ then } x(t) = A_2x(t) + A_{d2}x(t-\tau(t)). \end{aligned} \tag{4.1}$$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

The membership functions for above rules 1 and 2 are $h_1(\theta(t)) = \frac{1}{1+\exp(-2\theta(t))}$, $h_2(\theta(t)) = 1 - h_1(\theta(t))$, where the premise variable $\theta(t) = x_1(t)$.

For the convenience of computing, set $\alpha_1 = \cdots = \alpha_m = 0.5$. For different known μ, the Maximum allowable delay bounds of the time-varying delay computed by Theorem 3.2 are listed in Table 1. For comparison, the upper bounds obtained by the conditions in [8, 11, 14, 15, 18, 26, 33] are also tabulated in Table 1, where "−" denotes that the results are not provided in these papers. It is clear that the method proposed in this paper is less conservative than those in [8, 11, 14, 15, 18, 26, 33]. With initial state condition $[1,-1]^{\mathrm{T}}$, Fig. 1 shows the simulation results of the state responses of the system (4.1) with $\mu = 1$ and $0 \leq \tau(t) \leq 1.774$ listed in Table 1; and the phase portrait of the system (4.1) is given in Fig. 2. It shows from the simulation results (Figs.1 and 2) that the maximum allowable delay bounds of τ listed in Table 1 are capable of guaranteeing asymptotical stability of the given system (4.1).

Example 2. Consider the following uncertain T-S fuzzy system [14, 15, 33]

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(\theta(t))[A_i + \Delta A_{di}(t)]x(t) + [A_{di} + \Delta A_{di}(t)]x(t-\tau(t)) \tag{4.2}$$

where

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0.5 & -1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -1.6 & 0 \\ 0 & -1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix},\ E_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, E_2 = \begin{bmatrix} 1.6 & 0 \\ 0 & -0.05 \end{bmatrix},\ E_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.03 & 0 \\ 0 & -0.03 \end{bmatrix},$$

and the membership functions for rules 1 and 2 are $h_1(\theta(t)) = \left(1 - \frac{1}{1+\exp(-3(\theta(t)-0.5\pi))}\right)$, $h_2(\theta(t)) = 1 - h_1(\theta(t))$, where the premise variable $\theta(t) = x_1(t)$. Once again, we set $\alpha_1 = \cdots = \alpha_m = 0.5$ for the convenience of computing. Then, for different known/unknown μ, by Theorem 3.3, Corollary 3.9 and the conditions in [14, 15, 33], the upper bounds that guarantee the robust stability of system (4.2) are summarized in Table 3, where "$-$" denotes that the results are not provided in these papers. It can be concluded that the result proposed in this paper is significantly less conservative than those in [14, 15, 33]. With initial state conditions $[1, -1]^{\mathrm{T}}$ and the unknown matrix function $F(t) = \mathrm{diag}\{\sin t, \cos t\}$, Fig. 3 shows the simulation results of the state responses of the system (4.2) with $\mu = 0.5$ and $0 \le \tau(t) \le 1.558$ listed in Table 3; and the phase portrait of the system (4.2) is given in Fig. 4. It shows from the simulation results (Figs.3 and 4) that the maximum allowable delay bounds of τ listed in Table 3 are capable of guaranteeing robust asymptotical stability of the given system (4.2).

Meanwhile, it is also concluded from Tables 1-2 that the conservatism is gradually reduced with the increase of delay-partitioning numbers m and N. It's worth mentioning that, when the delay-partitioning number m is fixed, less conservatism can be achieved with increase of another delay-partitioning number N, but without increasing any computing burden. However, as m increases, testing the proposed results is much time-consuming since the more numbers of LMIs and LMI scalar decision variables are included in the corresponding criterion. So, one can choose the appropriate m for a tradeoff between the better results and the computational efficiency.

Methods \ μ	0	0.1	≥ 1
[26]	1.597	–	0.721
[14]	1.597	1.484	0.831
[11]	1.597	1.484	0.982
[15]	1.597	1.495	1.264
[18]	1.803	–	0.990
[8]	1.661	1.533	1.269
[33] ($m = 2$)	1.967	1.787	1.344
[33] ($m = 3$)	2.000	1.809	1.363
Th. 3.2 ($m = 2,\ N = m^2$)	2.343	2.144	1.538
Th. 3.2 ($m = 3,\ N = m^2$)	2.453	2.225	1.579
Th. 3.2 ($m = 3,\ N = m^3$)	2.754	2.489	1.774
[33] improved by ($m = 3$)	> 37.70 %	> 37.60 %	> 30.15 %

Table 1: Maximum allowable delay bounds of τ for different known μ (Example 1)

Methods \ μ	0	0.1	0.5	Unknown
[14]	1.168	1.122	0.934	0.499
[15]	1.192	1.155	1.100	1.050
[33] ($m = 2$)	1.390	1.318	1.132	1.127
Th. 3.3 / Cor. 3.9 ($m = 2,\ N = m^2$)	1.634	1.556	1.345	1.313
Th. 3.3 / Cor. 3.9 ($m = 2,\ N = m^3$)	1.908	1.817	1.558	1.501
[33] improved by ($m = 2$)	> 37.26 %	> 37.86 %	> 37.63 %	> 33.18 %

Table 2: Maximum allowable delay bounds of τ for different known/unknown μ (Example 2)

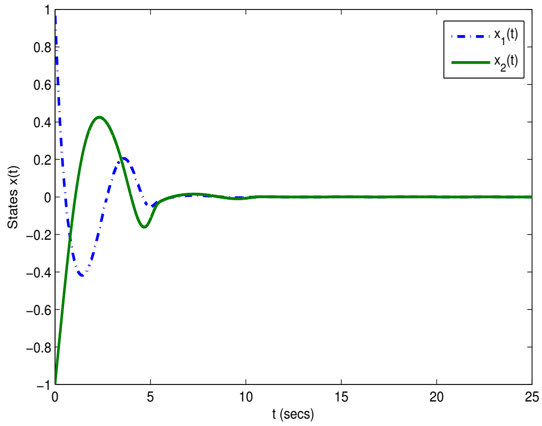

Figure 1: The state responses of the nominal system (4.1).

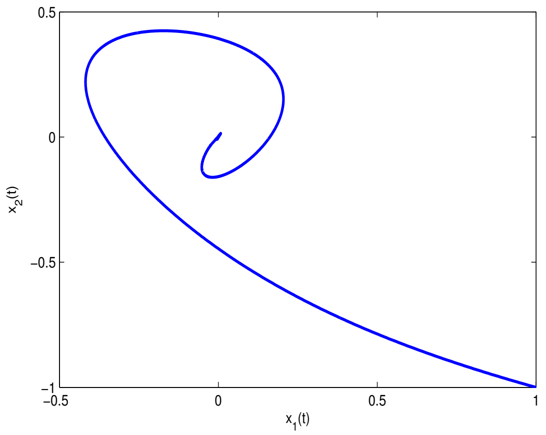

Figure 2: The phase portrait of the nominal system (4.1).

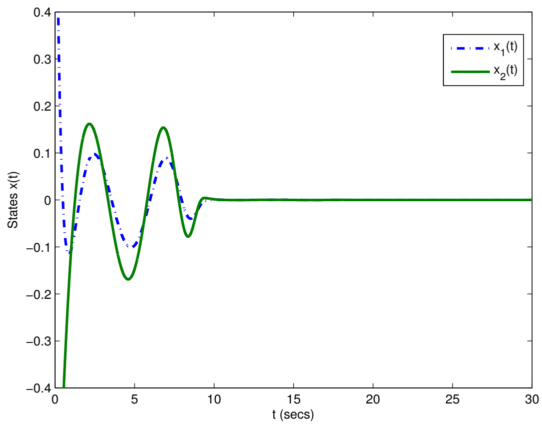

Figure 3: The state responses of the uncertain system (4.2).

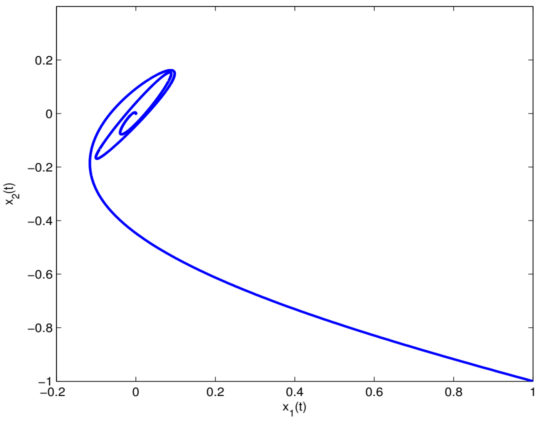

Figure 4: The phase portrait of the uncertain system (4.2).

5. Conclusion

By means of an improved delay-partitioning approach and the reciprocally convex technique, this paper is mainly concerned with the new stability criteria for uncertain T-S fuzzy systems with time-varying delay. A modified augmented LKF is established by partitioning the delay in all integral, and the time-varying delay $\tau(t)$-dependent and $[X_{ij}]_{m\times m}$-dependent sub-LKFs are also introduced to the augmented LKF, which make the LKF encompass more useful state information. Then, some tighter bounding inequalities such as Seuret-Wirtinger's integral inequality and Peng-Park's integral inequality have been employed to bound the derivative of LKF, therefore, less conservative LMI-based results can be expected since none of any useful time-varying items are arbitrarily ignored. Finally, two numerical examples are included to show the merits of the proposed results.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality of the paper. The authors are also deeply indebted to Professor Chen Peng (School of Electrical and Automation Engineering, Nanjing Normal University), Professor Hong-Bing Zeng (School of Electrical and Information Engineering, Hunan University of Technology) for their kindhearted help. This work was supported by the scientific research foundation of CAFUC (Grant No. Q2010-75). The material in this paper was not presented at any conference.

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