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Generalized vector equilibrium problems on Hadamard manifolds

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Abstract

In this paper, we study several types of Generalized Vector Equilibrium Problems (GVEP) on Hadamard manifolds. We prove sufficient conditions under which the solution set of (GVEP)'s is nonempty. As an application, we prove existence theorems for the system of generalized vector variational inequality problems and the system of generalized Pareto optimization problems.

Keywords: Hadamard manifold, variational inequality, equilibrium problem, KKM mapping.
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1. Introduction

Equilibrium problems provide a unifying framework for many important problems, such as optimization problems, variational inequality problems, complementarity problems, minimax inequality problems, fixed point problems and have been widely applied to study real world applications arising in economics, mechanics and engineering science. In recent decades, many results concerned the existence of solutions for equilibrium problems and vector equilibrium problems has been established; please see, Ansari and Yao [1]; Bianchi and Schaible [4]; Blum and Oettli [5]; Fang and Huang [10]; Farajzadeh et al. [11, 12] and the references therein.

On the other hand, recently many researchers are focused on extending some concepts and techniques of nonlinear analysis from Euclidean spaces to Riemannian manifolds. There are some advantages of such generalizations as many nonconvex and nonmonotone functions can be transformed into convex and monotone functions respectively, with the help of proper Riemannian metric. For illustrations, please see Cruz Neto et al. [8] and Pitea et al. [2, 18, 19, 20, 21].

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Nemeth [16] studied geodesic monotone vector fields while Wang et al. [25] studied monotone and accretive vector fields on Riemannian manifolds. Li et al. [15] extended maximal monotone vector fields from Banach space to Hadamard manifold, which is a simply connected complete Riemannian manifold with nonpositive sectional curvature. Nemeth [17] introduced variational inequality problems on Hadamard manifolds. Li et al. [14] studied the variational inequality problems on Riemannian manifolds. Zhou and Huang [26] introduced the notion of the KKM mapping and proved a generalized KKM theorem on the Hadamard manifold.

The first paper dealing with the subject of the existence of solution for equilibrium problems in the Riemannian context was introduced by Colao et al. [7]. Zhou and Huang [27] investigated the relationship between the vector variational inequality and the vector optimization problem by using KKM lemma on Hadamard manifolds. Li and Huang [13], studied the generalized vector quasi equilibrium problems on Hadamard manifolds. Recently, Batista et al. [3] provided a sufficient condition for the existence of a solution for generalized vector equilibrium problems on Hadamard manifolds.

Motivated by above mentioned research works, we introduce different types of generalized vector equilibrium problems (GVEP) and provide sufficient conditions under which the solution sets of the (GVEP)'s are nonempty on Hadamard manifolds. We construct an example to illustrate our results. We also establish the existence of solutions of vector variational inequality problems and vector optimization problems as particular cases.

2. Preliminaries

In this section, we recall fundamental definitions, basic properties and notations which are needed for a comprehensive reading of this paper. This background can be found in classical monographs such as: Sakai [23] or Udrişte [24].

Let M be an n-dimensional connected manifold. We denote by T_xM, the n-dimensional tangent space of M at x and by $TM = \cup_{x\in M} T_xM$, the tangent bundle of M. When M is endowed with a Riemannian metric $< \cdot, \cdot >$ on the tangent space T_xM with corresponding norm denoted by $\| \cdot \|$, then M is a Riemannian manifold. The length of a piecewise smooth curve $\gamma\colon [a, b] \to M$, joining x to y, such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by $L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt$. Then for any $x, y \in M$ the Riemannian distance $d(x, y)$ which induces the original topology on M is defined by minimizing this length over the set of all curves joining x to y. On every Riemannian manifold, there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. Let γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic. A geodesic joining x to y in M is said to be a minimal geodesic if its length equals to $d(x, y)$. A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf-Rinow Theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space, while closed bounded subsets are compact.

Assuming that M is complete, the exponential mapping $\exp_x\colon T_xM \to M$ is defined by $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x.

Recall that a Hadamard manifold is a simply connected and complete Riemannian manifold, with non-positive sectional curvature.

Assumption. *In our subsequent theory, all manifolds M will be considered being Hadamard and finite dimensional.*

Definition 2.1 ([27]). Let K be a nonempty closed geodesic convex subset of the Hadamard manifold M and $G\colon K \to 2^K$ be a set-valued mapping. We say that G is a (KKM) mapping if for any $\{x_1, \dots, x_m\} \subset K$, we have

$$\mathrm{co}(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^{m} G(x_i).$$

Lemma 2.2 ([7]). *Let K be a nonempty closed convex subset of the Hadamard manifold M and $G\colon K \to 2^K$ be a set-valued mapping such that for each $x \in K$, $G(x)$ is closed. Suppose that,*

(i) there exists $x_0 \in K$, such that $G(x_0)$ is compact;

(ii) $\forall x_1, \dots, x_m \in K$, $\mathrm{co}(\{x_1, \dots, x_m\}) \subset \bigcup_{i=1}^m G(x_i)$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Definition 2.3 ([22]). A subset K of a Hadamard manifold M is said to be convex (geodesic convex) if and only if for any two points $x, y \in K$, the geodesic joining x to y is contained in K. That is, if $\gamma\colon [0,1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K$ for $t \in [0,1]$.

Definition 2.4 ([22]). Let M be a Hadamard manifold. A real-valued function $f\colon M \to \mathbb{R}$ defined on a convex set (geodesic convex set) K is said to be convex (geodesic convex) if and only if for $0 \leq t \leq 1$,

$$f(\gamma(t)) \leq (1-t) f(\gamma(0)) + t f(\gamma(1)).$$

Definition 2.5 ([13]). Let M be a Hadamard manifold and $S\colon M \to 2^{TM}$ be a set-valued mapping. Then S is said to be lower semicontinuous at a point $x_0 \in M$, if for any open set $V \subseteq M$ satisfying $S(x_0) \cap V \neq \emptyset$, there exists an open neighborhood $U(x_0)$ of x_0 such that $S(x) \cap V \neq \emptyset$ for all $x \in U(x_0)$. S is said to be lower semicontinuous on M, if S is lower semicontinuous at every point $x \in M$.

3. Existence results

Let M be a finite dimensional Hadamard manifold and K be a nonempty convex subset of M. Let Y be a topological vector space and 2^Y denotes all subsets of Y. Let $C\colon K \to 2^Y$ is a pointed closed convex cone, that is $C + C \subseteq C$, $tC \subseteq C$ for $t \geq 0$ and $C \cap -C = \{0\}$. Finally, let $F\colon K \times K \to 2^Y$ be a set valued map.

We consider the following generalized vector equilibrium problems (GVEPs in short) in the setting of Hadamard manifold

$$(GVEP_1) \text{ find } x \in K \text{ such that } F(x,y) \nsubseteq -\mathrm{int}C(x)\ \forall y \in K,$$

$$(GVEP_2) \text{ find } x \in K \text{ such that } F(x,y) \nsubseteq -C(x) \setminus \{0\}\ \forall y \in K.$$

It is clear that a solution of $(GVEP_2)$ is also a solution of $(GVEP_1)$.

Next we define the following concepts that are needed in this sequel.

Definition 3.1. The set valued map $F\colon K \times K \to 2^Y$ is called strongly C-pseudomonotone if for any given x and y in K,

$$F(x,y) \nsubseteq -\mathrm{int}C(x) \Rightarrow F(y,x) \subseteq -C(y). \tag{3.1}$$

Definition 3.2. The set valued map $F\colon K \times K \to 2^Y$ is called C-pseudomonotone if for any given x and y in K,

$$F(x,y) \nsubseteq -C(x) \setminus \{0\} \Rightarrow F(y,x) \subseteq -C(y). \tag{3.2}$$

Note that every strongly C-pseudomonotone function is C-pseudomonotone.

Definition 3.3. The mapping $F\colon K \times K \to 2^Y$ is C-uppersign continuous if for all $x, y \in K$,

$$F(\gamma(t), y) \cap C(\gamma(t)) \neq \emptyset, \forall t \in (0,1) \Rightarrow F(x,y) \cap C(x) \neq \emptyset, \tag{3.3}$$

where $\gamma(t)$ is a geodesic joining x, y, with $\gamma(0) = x$.

Definition 3.4. $F\colon K \times K \to 2^Y$ is C-convex in the second variable if for all x, z_1, $z_2 \in K$ and $t \in (0,1)$, the following holds

$$F(x, \gamma(t)) \subseteq (1-t) F(x, z_1) + t F(x, z_2) - C(x),$$

where $\gamma(t)$ is a geodesic joining z_1 and z_2.

Next we prove the following lemma which is required to prove the main existence result.

Lemma 3.5. *Assume that the map $F\colon K \times K \to 2^Y$ satisfies the following conditions:*

(i) F is C-pseudomonotone,

(ii) $F(x,x) \cap C(x) \neq \emptyset$ for each $x \in K$,

(iii) F is C-uppersign continuous,

(iv) for each fixed $x \in K$, the mapping $z \to F(x,z)$ is C-convex.

Then for any given $y \in K$, the following are equivalent

(I) $F(y,z) \nsubseteq -C(y) \setminus \{0\}, \ \forall z \in K$.

(II) $F(z,y) \subseteq -C(z), \ \forall z \in K$.

Proof. From the definition of C-pseudomonotonicity of F it is obvious that (I) $\Rightarrow$ (II).

Suppose that (II) holds. That is for any given $y \in K$, we have

$$F(z,y) \subseteq -C(z, \ \forall z \in K. \tag{3.4}$$

Let $\gamma(t)$ be a geodesic joining y and z, where $t \in [0,1]$. Since K is convex we then have by (3.4),

$$F(\gamma(t),y) \subseteq -C(\gamma(t)) \ \forall t \in (0,1). \tag{3.5}$$

Next we claim that $F(\gamma(t),z) \cap C(\gamma(t)) \neq \emptyset$.

If possible let $F(\gamma(t),z) \cap C(\gamma(t)) = \emptyset$ for some $t \in (0,1)$. Then

$$F(\gamma(t),z) \subseteq Y \setminus C(\gamma(t)) \quad \text{for this } t \in (0,1). \tag{3.6}$$

Therefore, by (iv) and using (3.5) and (3.6)

$$\begin{aligned} F(\gamma(t),\gamma(t)) &\subseteq (1-t)F(\gamma(t),y) + tF(\gamma(t),z) - C(\gamma(t)) \\ &\subseteq -C(\gamma(t)) + (Y \setminus -C(\gamma(t))) - C(\gamma(t))) \\ &\subseteq Y \setminus -C(\gamma(t)), \end{aligned}$$

which contradicts to (ii).

Hence for all $t \in (0,1)$, the set $F(\gamma(t),z) \cap C(\gamma(t))$ is nonempty. Thus by (iii) there is an element $u \in F(y,z) \cap C(y)$. Since $C(y) \cap -C(y) \setminus \{0\} = \emptyset$, then $u \notin (-C(y) \setminus \{0\})$. Consequently $F(y,z) \nsubseteq -C(y) \setminus \{0\}$. This completes the proof. □

By a similar method as in Lemma 3.5 we can prove the following result.

Lemma 3.6. *Suppose that the map $F\colon K \times K \to 2^Y$ satisfies the following conditions:*

(i) F is strongly C-pseudomonotone,

(ii) $F(x,x) \nsubseteq -intC(x)$ for each $x \in K$,

(iii) F is C-uppersign continuous,

(iv) for each fixed $x \in K$, the mapping $z \to F(x,z)$ is convex.

Then for any given $y \in K$, the following are equivalent

(I) $F(y,z) \nsubseteq -intC(y),\ \forall z \in K.$

(II) $F(z,y) \subseteq -C(z),\ \forall z \in K.$

Lemma 3.7. *Under the assumptions of* Lemma 3.5*, the solution set of* $(GVEP_2)$ *is convex.*

Proof. Let x_1 and x_2 be solutions of $(GVEP_2)$. Then by Lemma 3.5, we have

$$F(z,x_i) \subseteq -C(z) \quad \forall z \in K, \quad i=1,2. \tag{3.7}$$

Let $\gamma(t)$ be a geodesic joining x_1 and x_2, $t \in (0,1)$.

By condition (iv) of Lemma 3.5, $\forall t \in (0,1)$, we have

$$F(z,\gamma(t)) \subseteq (1-t)F(z,x_1) + tF(z,x_2) - C(z) \subseteq -C(z) \ \ \forall z \in K.$$

Hence from Lemma 3.5, we get

$$F(\gamma(t),z) \nsubseteq -C(\gamma(t)) \setminus \{0\} \ \ \forall z \in K.$$

This shows that $\gamma(t)$ is a solution of $(GVEP_2)$. Therefore the solution set of $(GVEP_2)$ is convex. □

Similarly we can easily prove the following.

Lemma 3.8. *Under the assumptions of* Lemma 3.6*, the solution set of* $(GVEP_1)$ *is convex.*

We are now in a position to prove the existence theorem.

Theorem 3.9. *Let all assumptions of* Lemma 3.5 *hold and for fixed* $x \in K$ *and the mapping* $y \to F(x,y)$ *is lower semicontinuous, where* $y \in K$*. If there exists a nonempty compact subset* B *of* K *such that for each* $x \in K \setminus B$*, there exists* $y_0 \in K$ *such that* $F(y_0,x) \nsubseteq -C(y_0)$*, then the solution set of the generalized vector equilibrium problem* $(GVEP_2)$ *is nonempty.*

Proof. We first define two set valued mappings $\Gamma_1,\ \Gamma_2 \colon K \to 2^K$, respectively by the formula

$$\Gamma_1(y) = \{x \in K : F(x,y) \nsubseteq -C(x) \setminus \{0\}\}, \qquad \Gamma_2(y) = \{x \in K : F(y,x) \subseteq -C(y)\}.$$

We claim that Γ_1 is a KKM mapping.

We prove that for any choice of $y_1, \ldots, y_m \in K$,

$$co(\{y_1,\ldots,y_m\}) \subset \bigcup_{i=1}^{m} \Gamma_1(y_i). \tag{3.8}$$

Suppose on the contrary that there exists a point $z_0 \in K$, such that $z_0 \in co(\{y_1,\ldots,y_m\})$ but $z_0 \notin \bigcup_{i=1}^{m} \Gamma_1(y_i)$. That is

$$F(z_0,y_i) \subseteq -C(z_0) \setminus \{0\} \ \forall i \in \{1,\ldots,m\}. \tag{3.9}$$

This implies that for any $i \in \{1,\ldots,m\}$, $y_i \in \{y \in K : F(z_0,y) \subseteq -C(z_0) \setminus \{0\}\}$. Since the function $y \mapsto F(z_0,y)$ is convex, the set $\{y \in K : F(z_0,y) \subseteq -C(z_0) \setminus \{0\}\}$ is a convex set. Then

$$z_0 \in co(\{y_1,\ldots,y_m\}) \subseteq \{y \in K : F(z_0,y) \subseteq -C(z_0) \setminus \{0\}\}.$$

Therefore $F(z_0,z_0) \subseteq -C(z_0) \setminus \{0\}$, which is a contradiction to condition (ii) of Lemma 3.5. Hence Γ_1 is a (KKM) mapping.

By Lemma 3.5 we have Γ_2 is also a (KKM) mapping. Also as for fixed $x \in K$, the mapping $y \to F(x,y)$ is lower semicontinuous, then $\Gamma_2(y)$ is closed for all $y \in K$.

Again by the assumption there exists a nonempty compact subset B of K such that, for each $x \in K \setminus B$, there exists $y_0 \in K$ such that $F(y_0,x) \nsubseteq -C(y_0)$.

That is $\Gamma_2(y_0) = \{x \in B : F(y_0, x) \subseteq -C(y_0)\}$ is compact. Hence by Lemma 2.2, it follows that $\bigcap_{y\in K} \Gamma_2(y) \neq \emptyset$. Also by Lemma 3.5, $\bigcap_{y\in K} \Gamma_1(y) \neq \emptyset$. So there exists $z \in K$ such that

$$F(z, y) \nsubseteq -C(z) \setminus \{0\}. \tag{3.10}$$

Hence z is a solution of $(GVEP_2)$, and this completes the proof. □

Example 3.10. Let $H^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = -1,\ x_2 > 0\}$ be the hyperbolic 1-space which forms a Hadamard manifold [6], endowed with the metric

$$\langle x, y\rangle = x_1y_1 - x_2y_2, \quad x = (x_1, x_2), \quad y = (y_1, y_2) \in \mathbb{R}^2.$$

Let K be a subset of H^1 defined by $K = \{x = (x_1, x_2) \in H^1 : -1 \leq x_1 \leq 1\}$. Remark that K is a compact subset of H^1. We take the topological vector space $Y = \mathbb{R}$ and $C(x) = [0, \infty)$ and define the bifunction

$$F : K \times K \to 2^{\mathbb{R}}, \quad F(x, y) = [x_2(x_1 - y_1), (x_1 - y_1)]. \tag{3.11}$$

It is easy to see that $F(x, x) \cap C(x) = \{0\}$ for all $x \in K$. Also F is C-pseudomonotone on K as

$$F(x, y) \nsubseteq [0, \infty) \Rightarrow F(y, x) \subseteq (-\infty, 0].$$

F is C-uppersign continuous and for each fixed $x \in K$, the mapping $z \to F(x, z)$ is C-convex and lower semicontinuous. Hence, by Theorem 3.9, there exists a point $x \in K$ such that

$$F(x, y) \nsubseteq -C(x) \setminus \{0\}\ \forall y \in K.$$

Similarly we can deduce the following

Theorem 3.11. *Let all assumptions of* Lemma 3.6 *hold and for fixed $x \in K$, the mapping $y \to F(x, y)$ is lower semicontinuous, where $y \in K$. If there exists a nonempty compact subset B of K such that, for each $x \in K \setminus B$, there exists $y_0 \in K$ such that $F(y_0, x) \nsubseteq -C(y_0)$. Then the solution set of the generalized vector equilibrium problem $(GVEP_1)$ is nonempty.*

4. Consequences of our main result

In order to clarify the interest of generalized vector equilibrium problems, we consider some problems whose existence of solutions can be ensured by dealing with the (GVEP)'s.

4.1. Generalized vector variational inequality problems

Let $V : K \to TM$ be a vector field, that is, $V_x \in T_xM$ for each $x \in K$ and $\exp^{-1}$ denotex the inverse of the exponential map. Then the generalized vector variational inequality problems (GVVIPs in short) can be considered by the following ways:

$$(GVVIP_1) \text{ find } x \in K \text{ such that } \langle V(x), \exp_x^{-1} y\rangle \nsubseteq -\operatorname{int} C(x)\ \forall y \in K,$$

$$(GVVIP_2) \text{ find } x \in K \text{ such that } \langle V(x), \exp_x^{-1} y\rangle \nsubseteq -C(x) \setminus \{0\}\ \forall y \in K.$$

Clearly a solution of $(GVVIP_2)$ is a solution of $(GVVIP_1)$.

Lemma 4.1 ([15])**.** *Let M be a Hadamard manifold. Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold:*

(i) For any $y \in M$, $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.

(ii) If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.

(iii) Given the sequence $\{u_n\}$ *and* $\{v_n\}$ *with* $u_n, v_n \in T_{x_n}M$, *if* $u_n \to u_0$ *and* $v_n \to v_0$ *with* $u_0, v_0 \in T_{x_0}M$, *then* $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.

Proposition 4.2 ([9])**.** *Let* M *be a Hadamard manifold with null sectional curvature,* $y \in M$ *and* $u \in T_yM$ *nonzero. Then* $g\colon M \to \mathbb{R}$, $g(x) = \left\langle \exp_y^{-1} x, u \right\rangle$, *is an affine linear function.*

If we consider $F(x,y) = \langle V(x), \exp_x^{-1} y \rangle$, then the following result follows.

Theorem 4.3. *Let* M *be a Hadamard manifold with null sectional curvature. Assume that the following conditions hold:*

(i) V *is* C*-pseudomonotone,*

(ii) for each fixed y, *the mapping* $x \to \langle V(x), \exp_x^{-1} y \rangle$ *is* C*-uppersign continuous,*

(iii) for fixed $x \in K$, *the mapping* $y \to \langle V(x), \exp_x^{-1} y \rangle$ *is lower semicontinuous, where* $y \in K$,

(iv) there exists a nonempty compact subset B *of* K *such that for each* $x \in K \setminus B$, *there exists* $y_0 \in K$ *such that* $\langle V(y_0), \eta(y_0, x) \rangle \nsubseteq -C(y_0)$.

Then the solution set of $(GVVIP_2)$ *is nonempty.*

Similarly, one can get

Theorem 4.4. *Let* M *be a Hadamard manifold with null sectional curvature. Assume that the following conditions hold:*

(i) V *is strongly* C*-pseudomonotone,*

(ii) for each fixed y, *the mapping* $x \to \langle V(x), \exp_x^{-1} y \rangle$ *is* C*-uppersign continuous,*

(iii) for fixed $x \in K$, *the mapping* $y \to \langle V(x), \exp_x^{-1} y \rangle$ *is lower semicontinuous, where* $y \in K$,

(iv) there exists a nonempty compact subset B *of* K *such that for each* $x \in K \setminus B$, *there exists* $y_0 \in K$ *such that* $\langle V(y_0), \eta(y_0, x) \rangle \nsubseteq -C(y_0)$.

Then the solution set of $(GVVIP_1)$ *is nonempty.*

4.2. Generalized Pareto optimization problem

We consider the following weak Pareto optimization problem

$$(GWPOP) \quad \text{find } \bar{x} \in K \text{ such that } f(y) - f(\bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+ \text{ for each } y \in \ K,$$

where $f\colon M \to \mathbb{R}^m$ is a vector function. Taking $F(x,y) = f(y) - f(x)$, we obtain as an immediate consequence the following corollary.

Corollary 4.5. *Suppose that*

(i) f *is continuous and* C*-convex;*

(ii) there exists a nonempty compact subset B *of* K *such that for each* $x \in K \setminus B$, *there exists* $y_0 \in K$ *such that* $f(y_0) - f(x) \in int\mathbb{R}^m_+$.

Then the solution set of the generalized Pareto optimization problem $(GWPOP)$ *is nonempty.*

4.3. Equilibrium problem and variational inequality problem

Let us take $Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+$. Then the generalized vector equilibrium problems $(GVEP_1)$ reduce to the equilibrium problem introduced by Colao et al. [7]:

$$(EP) \text{ find } x \in K \text{ such that } F(x,y) \geq 0 \text{ for all } y \in K.$$

In this case, if we take $F(x,y) = \langle V(x), \exp_x^{-1} y \rangle$, where $V\colon K \to TM$ is a vector field, that is, $V_x \in T_xM$ for each $x \in K$ and $\exp^{-1}$ denote the inverse of the exponential map, then we get the scalar variational inequality problem introduced by Nemeth [17]:

$$(VIP) \text{ find } x \in K \text{ such that } \langle V(x), \exp_x^{-1} y \rangle \geq 0 \text{ for all } y \in K.$$

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