Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 944–954
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Pata-type common fixed point results in $b$-metric and $b$-rectangular metric spaces

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**Abstract**

We obtain (common) fixed point results for mappings in $b$-metric and $b$-rectangular metric spaces, under the Pata-type conditions. In particular, we show that the results of paper Balasubramanian, [S. Balasubramanian, Math. Sci. (Springer) 8 (2014), no. 3, 65–69] can be obtained as consequences of more general results and in a much shorter way. We demonstrate these facts by some examples. 

*Keywords:* $b$-metric space, $b$-rectangular metric space, Pata-type condition.
*2010 MSC:* 47H10, 54H25.

## 1. Introduction

There are hundreds of articles dealing with generalization of the basic Banach Contraction Principle. Roughly speaking, they follow two lines of investigation.

The first line is concerned with generalizations of contractive condition. We mention here just the work of Lj. Ćirić (see, e.g., [8, 9]). One of the interesting recent results of this kind was obtained by V. Pata in [32]. Several authors have already used Pata-type conditions to obtain new fixed point results (e.g., [5, 7, 12, 20, 23, 24]).

The other line of investigation deals with various generalizations of metric spaces and the results that can be obtained in new frameworks. Among dozens of such generalizations, we mention the following.

$b$-metric spaces (sometimes called metric-type spaces) were first considered by I. A. Bakhtin in 1989 [4] and S. Czerwik in 1993 [10]. There is a vast literature concerning this type of spaces, we mention just some of them [1, 2, 3, 16, 17, 18, 25, 26, 27, 29, 33, 37, 42].

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*Received 2015-01-14*

Rectangular metric spaces (sometimes called just generalized metric spaces, g.m.s.) were introduced by A. Branciari in 2000 [6]. Some of the papers where the structure of such spaces has been discussed and some fixed point results have been obtained are [11, 14, 21, 22, 28, 30, 31, 38, 39, 41].

As a combination of rectangular and $b$-metric spaces, $b$-rectangular metric spaces were introduced and treated in [13, 34, 40].

In this paper, we obtain (common) fixed point results for mappings in $b$-metric and $b$-rectangular metric spaces, under the Pata-type conditions. In particular, we show that the results of paper [5] can be obtained as consequences of more general results and in a much shorter way. We demonstrate these facts by some examples.

## 2. Preliminaries

In a recent paper, V. Pata obtained the following interesting refinement of the classical Banach Contraction Principle.

**Theorem 2.1** ([32])**.** *Let $(X, d)$ be a metric space, $f : X \to X$, let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants and $\psi : [0, 1] \to [0, \infty)$ be an increasing function, vanishing with continuity at $0$. If the inequality*

$$d(fx, fy) \leq (1 - \varepsilon)d(x, y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta}$$

*is satisfied for every $\varepsilon \in [0, 1]$ and all $x, y \in X$, then $f$ has a unique fixed point $z \in X$. Here, $\|x\| = d(x_0, x)$ for a chosen point $x_0 \in X$.*

It was also shown by an example that the previous theorem is a real generalization of Banach's result. More results of this kind were subsequently obtained by various authors.

$b$-metric spaces were firstly used by I. A. Bakhtin and S. Czerwik.

**Definition 2.2** ([4, 10])**.** Let $X$ be a nonempty set, $s \geq 1$ be a given real number. A mapping $d : X \times X \to X$ is called a $b$-metric with parameter $s$ if

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$ ($b$-triangular inequality).

Then $(X, d)$ is called a $b$-metric space.

The following are some easy and well-known examples of $b$-metric spaces.

**Example 2.3.** Let $(X, \rho)$ be a metric space and $p \geq 1$ be a given real number. Then $d(x, y) = [\rho(x, y)]^p$ is a $b$-metric on $X$ with parameter $s \leq 2^{p-1}$.

**Example 2.4** ([26, 35])**.** Let $(X, \rho)$ be a cone metric space (in the sense of [15]), over a normal cone with normal constant $K$. Then $d(x, y) = \|\rho(x, y)\|$ defines a $b$-metric on $X$ with parameter $s = K$.

*Remark* 2.5. It is easy to see that $b$-metrics in the previous two examples are (sequentially) continuous functions (in both variables). However, examples were provided [16, 17, 29] showing that, in general, this might not be the case. We present here the following

**Example 2.6.** [17] Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then, considering all possible cases, it can be checked that $(X, d)$ is a $b$-metric space with $s = 5/2$. However, let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

The following definition was given by A. Branciari in 2000.

**Definition 2.7** ([6])**.** Let $X$ be a nonempty set, and let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from $x$ and $y$,

(i) $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ ("rectangular inequality")

hold. Then $(X, d)$ is called a rectangular or generalized metric space.

*Remark* 2.8. Obviously, each metric space is a rectangular metric space, but the converse is not true. Moreover, Sarma et al. [39] and Samet [38] presented examples showing that rectangular spaces might not be Hausdorff and, again, that rectangular metric might be discontinuous. Also, Suzuki showed in [41] that, in general, rectangular spaces do not have a compatible topology. We recall here the following

**Example 2.9** ([39])**.** Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $d : X \times X \to [0, +\infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B \\ y, & x \in A,\ y \in B \\ x, & x \in B,\ y \in A. \end{cases}$$

Then $(X, d)$ is a complete g.m.s. However, it is easy to see that:

- the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ converges to both 0 and 2 and it is not a Cauchy sequence;
- there is no $r > 0$ such that $B_r(0) \cap B_r(2) = \emptyset$; hence, the respective topology is not Hausdorff;
- $B_{2/3}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}$, however there does not exist $r > 0$ such that $B_r(0) \subseteq B_{2/3}(\frac{1}{3})$;
- $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\lim_{n\to\infty} d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$; hence $d$ is not a continuous function.

As a combination of $b$-metric and rectangular metric spaces, $b$-rectangular metric spaces were introduced and used in [13, 34].

**Definition 2.10** ([13, 34])**.** Let $X$ be a nonempty set and $s \geq 1$ be a fixed real number. Let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from $x$ and $y$,

(i) $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ ("$b$-rectangular inequality")

hold. Then $(X, d)$ is called a $b$-rectangular or $b$-generalized metric space with parameter $s$.

**Example 2.11** ([34])**.** If $(X,\rho)$ is a rectangular metric space and $p \geq 1$ is a given real number, then $d(x,y) = [\rho(x,y)]^p$ defines a $b$-rectangular metric on $X$ with parameter $s \leq 3^{p-1}$.

**Example 2.12.** If $(X,\rho)$ is a $b$-metric space with parameter $s'$, then it is also a $b$-rectangular space with parameter $s \leq s'^2$.

*Remark* 2.13. In both of the previous examples, the value of parameter $s$ might be strictly smaller then the given estimate. For example, the $b$-metric $d$ introduced in Example 2.4 is also a $b$-rectangular metric on $X$ with the same value of parameter $s = K$ as for the $b$-metric [26, 35].

## 3. Results in $b$-metric spaces

Throughout the rest of the paper, for a given $b$-(rectangular) metric space $(X,d)$ and a fixed $x_0 \in X$, we will denote $\|x\| = d(x,x_0)$ for $x \in X$. It is easy to see that the obtained results do not depend on the particular choice of point $x_0$. Also, $\psi : [0,1] \to [0,\infty)$ will always be an increasing function, continuous at 0, with $\psi(0) = 0$.

**Theorem 3.1.** *Let $(X,d)$ be a complete $b$-metric space with parameter $s > 1$ and let $f,g : X \to X$ be two self-mappings such that $fX \subseteq gX$. Suppose that for some $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$,*

$$\begin{aligned} d(fx,fy) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(gx,gy)}{2s}, d(gx,fx), d(gy,fy)\right\} \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|gx\| + \|gy\| + \|fx\| + \|fy\|\right]^{\beta}, \end{aligned} \tag{3.1}$$

*holds for all $x,y \in X$ and each $\varepsilon \in [0,1]$. Then $f$ and $g$ have a unique point of coincidence. If, moreover, $f$ and $g$ are weakly compatible, then they have a unique common fixed point.*

*Proof.* For arbitrary $x_0 \in X$, let us form a Jungck sequence $\{y_n\}$ by $y_n = fx_n = gx_{n+1}$ (this is possible since $fX \subseteq gX$). If $y_n = y_{n+1}$ for some $n$, there is nothing to prove. Hence, suppose that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}_0$.

**Step 1.** Putting $\varepsilon = 0$ in (3.1), we get that

$$d(y_n,y_{n+1}) \leq \frac{1}{s} d(y_{n-1},y_n) \tag{3.2}$$

for each $n \in \mathbb{N}$. It follows that $d(y_n,y_{n+1})$ is a strictly decreasing sequence, tending to 0 as $n \to \infty$.

**Step 2.** We will prove by induction that the sequence $c_n = d(y_n,y_0)$ is bounded by $2sc_1$.

The assertion holds for $n = 1$ and $n = 2$. Suppose that $c_n \leq 2sc_1$ for some $n \in \mathbb{N}$. Then, again taking $\varepsilon = 0$, we obtain that

$$\begin{aligned} c_{n+1} &= d(y_{n+1},y_0) \leq s\left(d(y_{n+1},y_1) + d(y_1,y_0)\right) \\ &= s\left(d(fx_{n+1},fx_1) + d(y_1,y_0)\right) \\ &\leq s\left(\frac{1}{s}\max\left\{\frac{d(gx_{n+1},gx_1)}{2s}, d(gx_{n+1},fx_{n+1}), d(gx_1,fx_1)\right\} + d(y_1,y_0)\right) \\ &= \max\left\{\frac{d(y_n,y_0)}{2s}, d(y_n,y_{n+1}), d(y_0,y_1)\right\} + sd(y_1,y_0) \\ &= d(y_0,y_1) + sd(y_1,y_0) \\ &= (1+s)c_1 \leq 2sc_1, \end{aligned}$$

since both $\frac{d(y_n,y_0)}{2s}$ and $d(y_n,y_{n+1})$ are not greater than $d(y_0,y_1)$. This finishes the inductive proof.

**Step 3.** In order to prove that $\{y_n\}$ is a Cauchy sequence, suppose the contrary. Using [37, Lemma 1.7] (with $\varepsilon$ replaced by $\delta$), we obtain that there exist $\delta > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$,

$$d\left(y_{m(k)}, y_{n(k)}\right) \geq \delta, \quad d\left(y_{m(k)}, y_{n(k)-1}\right) < \delta,$$

and

$$\frac{\delta}{s} \leq \limsup_{n\to\infty} d\left(y_{m(k)+1}, y_{n(k)}\right). \tag{3.3}$$

Replacing $x = x_{m(k)+1}$, $y = x_{n(k)}$ in the condition (3.1) we get

$$\begin{aligned} d\left(y_{m(k)+1}, y_{n(k)}\right) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d\left(y_{m(k)}, y_{n(k)-1}\right)}{2s}, d\left(y_{m(k)}, y_{m(k)+1}\right), d\left(y_{n(k)-1}, y_{n(k)}\right)\right\} \\ + K\varepsilon^{\alpha}\psi(\varepsilon), \end{aligned}$$

for some constant $K$, since the sequence $\{y_n\}$ is bounded. Passing to the upper limit, and using (3.3), we get

$$\begin{aligned} \frac{\delta}{s} &\leq \limsup_{k\to\infty} d\left(y_{m(k)+1}, y_{n(k)}\right) \\ &\leq \frac{1-\varepsilon}{s} \max\left\{\limsup_{k\to\infty} \frac{d\left(y_{m(k)}, y_{n(k)-1}\right)}{2s}, 0, 0\right\} + K\varepsilon^{\alpha}\psi(\varepsilon) \\ &= \frac{1-\varepsilon}{s}\frac{\delta}{2s} + + K\varepsilon^{\alpha}\psi(\varepsilon), \end{aligned}$$

and hence

$$\frac{\delta}{s} \leq \frac{1-\varepsilon}{s}\frac{\delta}{2s} + K\varepsilon^{\alpha}\psi(\varepsilon) < \frac{(1-\varepsilon)\delta}{s} + K\varepsilon^{\alpha}\psi(\varepsilon).$$

Putting $\varepsilon = 0$, we get that $\delta = 0$, a contradiction.

Hence, $y_n = fx_n = gx_{n+1}$ is a Cauchy sequence, and $gx_n \to gz$, ad $n \to \infty$, for some $z \in X$. We will show that $fz = gz$. We have

$$\begin{aligned} \frac{1}{s} d(fz, gz) &\leq d(fz, fx_n) + d(fx_n, gz) \\ &\leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(gz, gx_n)}{2s}, d(gz, fz), d(gx_n, fx_n)\right\} + K\varepsilon^{\alpha}\psi(\varepsilon) + d(fx_n, gz). \end{aligned}$$

It follows that, for $n$ big enough,

$$\frac{1}{s} d(fz, gz) \leq \frac{1-\varepsilon}{s} d(gz, fz) + K\varepsilon^{\alpha}\psi(\varepsilon),$$

and we easily obtain that $fz = gz$.

The uniqueness of point of coincidence follows easily by taking $\varepsilon = 0$ in (3.1), and that it is a common fixed point of $f$ and $g$ follows in a standard way (see, e.g., [19]). □

Putting $g = i_X$ in the previous theorem, we obtain

**Corollary 3.2.** *Let $(X, d)$ be a complete b-metric space with parameter $s > 1$ and let $f : X \to X$ be a self-mapping. Suppose that for some $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$,*

$$\begin{aligned} d(fx, fy) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(x,y)}{2s}, d(x, fx), d(y, fy)\right\} \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1 + \|x\| + \|y\| + \|fx\| + \|fy\|\right]^{\beta}, \end{aligned} \tag{3.4}$$

*holds for all $x, y \in X$ and each $\varepsilon \in [0, 1]$. Then $f$ has a unique fixed point.*

*Remark* 3.3. All the results of paper [5] are direct consequences of the previous theorem and the proof is much shorter. Moreover, our result is strictly stronger, since it is obtained without the assumption of continuity of the $b$-metric $d$, while the $b$-metric $d(x,y)=\|\rho(x,y\|$ (for a given cone metric $\rho$) that is used in [5], is always continuous (see Example 2.4 and Remark 2.5). We demonstrate this by the following example.

**Example 3.4.** Let $(X,d)$ be the $b$-metric space of Example 2.4 (where the $b$-metric $d$ is not continuous). Consider the following mapping $f:X\to X$.

$$fx=\begin{cases}100, & x\le 100,\\ 4, & \text{otherwise,}\end{cases}$$

and let us check the contractive condition (3.4). The only nontrivial case is when $x\in\{1,2,\dots,100\}$ and $y\in\{101,102,\dots,\infty\}$. Then

$$d\left(fx,fy\right)=d\left(100,4\right)=\left|\frac{1}{100}-\frac{1}{4}\right|=\frac{24}{100},$$

and we will check that

$$\frac{24}{100}\le\frac{2\left(1-\varepsilon\right)}{5}\max\left\{\frac{d\left(x,y\right)}{5},d\left(x,100\right),d\left(y,4\right)\right\}+\varepsilon^2$$

(which is condition (3.4) with $\Lambda=1$, $\psi(\varepsilon)=\varepsilon$, $\alpha=1$, $\beta=0$). We have

$$\max\frac{d\left(x,y\right)}{5}=\max\begin{cases}\frac{1}{5}\left|\frac{1}{x}-\frac{1}{y}\right|, & \text{both } x \text{ and } y \text{ are even or one is even ant the other is } \infty,\\ \frac{1}{5}\cdot 5, & \text{both } x \text{ and } y \text{ are odd or one is odd the other is } \infty,\\ \frac{1}{5}\cdot 2, & \text{otherwise}\end{cases}\le 1,$$

$$\max d\left(x,100\right)=\begin{cases}\left|\frac{1}{x}-\frac{1}{100}\right|, & \text{if } x \text{ is even}\\ 2, & \text{if } x \text{ is odd}\end{cases}\le 2,$$

$$\max d\left(y,4\right)=\begin{cases}\left|\frac{1}{y}-\frac{1}{100}\right|, & \text{if } y \text{ is even or } \infty,\\ 2, & \text{if } y \text{ is odd or } \infty\end{cases}\le 2.$$

Hence, we have to show that

$$\frac{24}{100}\le\frac{2\left(1-\varepsilon\right)}{5}\cdot 2+\varepsilon^2,$$

which is fulfilled for each $\varepsilon\in\mathbb{R}$, a fortiori for $\varepsilon\in[0,1]$.

## 4. Results in $b$-rectangular metric spaces

The method of proof based on the approach as in [36, Lemma 2.1] was used in many articles. We give here a version (adapted from [11, 31, 34]) that can be used in $b$-rectangular metric spaces.

**Lemma 4.1.** *Let $(X,d)$ be a $b$-rectangular metric space with $s\ge 1$ and let $\{x_n\}$ be a sequence in $X$ such that*

$$x_n\ne x_m \text{ whenever } n\ne m \text{ and } \lim_{n\to\infty} d\left(x_n,x_{n+1}\right)=0. \tag{4.1}$$

*If $\{x_n\}$ is not a $b$-rectangular-Cauchy sequence, then there exist $\delta>0$ and two sequences $\{m\left(k\right)\}$ and $\{n\left(k\right)\}$ of positive integers such that the following hold:*

$$d\left(x_{m(k)},x_{n(k)}\right)\ge\delta,\quad \frac{\delta}{s}\le\limsup_{k\to\infty}d\left(x_{m(k)},x_{n(k)-2}\right)\le\delta$$
$$\text{and } \frac{\delta}{s}\le\limsup_{k\to\infty}d\left(x_{m(k)+1},x_{n(k)-1}\right).$$

*Proof.* If $\{x_n\}$ is not a $b$-rectangular-Cauchy sequence, then there exists $\delta > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) - 3 \geq m(k) > k \text{ and } d\left(x_{m(k)}, x_{n(k)}\right) \geq \delta. \tag{4.2}$$

This means that

$$d\left(x_{m(k)}, x_{n(k)-2}\right) < \delta.$$

Now taking the upper limit as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)-2}\right) \leq \delta.$$

On the other hand, we have

$$\frac{1}{s} d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1,} x_{n(k)-1}\right) + d\left(x_{n(k)-1,} x_{n(k)}\right).$$

Using (4.1), (4.2) and taking the upper limit as $k \to \infty$, we get

$$\frac{\delta}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)+1,} x_{n(k)-1}\right).$$

Using the $b$-rectangular inequality once again we have the following inequality

$$\frac{1}{s} d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-2}\right) + d\left(x_{n(k)-2,} x_{n(k)-1}\right) + d\left(x_{n(k)-1,} x_{n(k)}\right).$$

Using (4.1), (4.2) and taking the upper limit as $k \to \infty$, we now get

$$\frac{\delta}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k),} x_{n(k)-2}\right).$$

The proof of Lemma 4.1 is complete. □

We present now the main result of this section.

**Theorem 4.2.** *Let $(X, d)$ be a complete $b$-rectangular metric space with parameter $s > 1$ and let $f, g : X \to X$ be such that $fX \subseteq gX$. Suppose that for some $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0, \alpha]$,*

$$\begin{aligned} d(fx, fy) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(gx, gy)}{2s}, d(gx, fx), d(gy, fy)\right\} \\ + \Lambda \varepsilon^{\alpha} \psi(\varepsilon) \left[1 + \|gx\| + \|gy\|\right]^{\beta}, \end{aligned} \tag{4.3}$$

*holds for all $x, y \in X$ and each $\varepsilon \in [0, 1]$. Then $f$ and $g$ have a unique point of coincidence. If, moreover, $f$ and $g$ are weakly compatible, then they have a unique common fixed point.*

*Proof.* Similarly as in the proof of Theorem 3.1, it can be proved that

$$d(y_n, y_{n+1}) \leq \frac{1}{s} d(y_{n-1}, y_n)$$

and then that $d(y_n, y_{n+1}) \to 0$, as $n \to \infty$. Moreover, similarly as in [21, 23], $y_n \neq y_m$ whenever $n \neq m$. Let us prove that the sequence $c_n = d(y_0, y_n)$ is bounded.

Using that $y_n \neq y_m$ for $n \neq m$, we get

$$\begin{aligned} \frac{1}{s} c_n = \frac{1}{s} d(y_n, y_0) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_1) + d(y_1, y_0) \\ &\leq 2c_1 + d(fx_{n+1}, fx_1) \end{aligned}$$

$$\begin{aligned}
&\leq 2c_1 + \frac{1-\varepsilon}{s}\max\left\{\frac{d(y_n,y_0)}{2s}, d(y_n,y_{n+1}), d(y_0,y_1)\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|y_n\|+\|y_0\|\right]^{\beta}\\
&\leq 2c_1 + \frac{1-\varepsilon}{s}\max\left\{\frac{d(y_n,y_0)}{2s}, d(y_n,y_{n+1}), d(y_0,y_1)\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+\|y_n\|+\|y_0\|\right]^{\beta}\\
&\leq 2c_1 + \frac{1-\varepsilon}{s}\max\left\{\frac{c_n}{2s}, c_1\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+c_n\right]^{\alpha}.
\end{aligned}$$

Consider now the following two cases.

1) If $\max\left\{\frac{c_n}{2s}, c_1\right\} = c_1$ then $c_n \leq 2sc_1$ and the proof is over.

2) $\max\left\{\frac{c_n}{2s}, c_1\right\} = \frac{c_n}{2s}$. Then we have

$$\frac{1}{s}c_n \leq 2c_1 + \frac{1-\varepsilon}{s}\frac{c_n}{2s} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left[1+c_n\right]^{\alpha},$$

i.e.

$$c_n \leq 2sc_1 + (1-\varepsilon)c_n + a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha}.$$

Hence we have obtained

$$\varepsilon c_n \leq b + a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha}$$

where $a, b$ are positive constants. Now it follows that the sequence $\{c_n\}$ is bounded in the same way as in [32, Lemma 3].

Now we apply Lemma 4.1. If $\{y_n\}$ were not a $b$-rectangular-Cauchy then, putting $\varepsilon = 0$, $x = x_{m(k)+1}$ and $y = x_{n(k)-1}$ in (4.3), we would obtain

$$d\left(y_{m(k)+1}, y_{n(k)-1}\right) \leq \frac{1}{s}\max\left\{\frac{d\left(y_{m(k)}, y_{n(k)-1}\right)}{2s}, d\left(y_{m(k)}, y_{m(k)+1}\right), d\left(y_{n(k)-1}, y_{n(k)}\right)\right\}.$$

Taking now the upper limit as $k \to \infty$ and using Lemma 4.1, it would follow that

$$\frac{\delta}{s} \leq \limsup_{k\to\infty} d\left(y_{m(k)+1}, y_{n(k)-1}\right) \leq \frac{1}{2s^2}\limsup_{k\to\infty} d\left(y_{m(k)}, y_{n(k)-1}\right) \leq \frac{\delta}{2s^2},$$

a contradiction since $s > 1$, $\delta > 0$. Hence, $\{y_n\}$ is a $b$-rectangular Cauchy sequence.

Moreover, we have

$$\begin{aligned}
\frac{1}{s}d(fz,gz) &\leq d(fz,fx_n) + d(fx_n,gx_n) + d(gx_n,gz)\\
&\leq \frac{1-\varepsilon}{s}\max\left\{\frac{d(gz,gx_n)}{2s}, d(gz,fz), d(gx_n,fx_n)\right\}\\
&\quad + K\varepsilon^{\alpha}\psi(\varepsilon) + d(fx_n,gx_n) + d(gx_n,gz).
\end{aligned}$$

It follows that for $n$ big enough,

$$\frac{1}{s}d(fz,gz) \leq \frac{1-\varepsilon}{s}d(gz,fz) + K\varepsilon^{\alpha}\psi(\varepsilon),$$

i.e.

$$\frac{\varepsilon}{s}d(fz,gz) \leq K\varepsilon^{\alpha}\psi(\varepsilon)$$

and hence $fz = gz$.

The rest of proof is standard. □

Specifying $g = i_X$ in the previous theorem, we get the extension of Pata's basic result (Theorem 2.1) to the framework of $b$-rectangular metric spaces.

Also, the main result (Theorem 1) of [5] is thus extended from cone metric spaces to the framework of rectangular cone metric spaces:

**Corollary 4.3.** *Let $(X,d)$ be a complete rectangular cone metric space with normal constant $K$, $x_0 \in X$, $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$ be fixed constants. If for all $\varepsilon \in [0,1]$, $x,y \in X$, the map $f : X \to X$ satisfies*

$$\|d(fx,fy)\| \leq \frac{1-\varepsilon}{K}\, M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|d(x,x_0)\| + \|d(y,x_0)\|]^{\beta},$$

*where $M(x,y) = \max\{\|d(x,fx)\|, \|d(y,fy)\|, \frac{1}{2K}\|d(x,y)\|\}$, then $f$ has a unique fixed point.*

*Proof.* It follows directly from Theorem 4.2 since under these assumptions, $(X,\|d\|)$ is a $b$-rectangular metric space with parameter $K$. □

The following example is inspired by [13, Example 3.2].

**Example 4.4.** Let $X = A \cup B$, where $A = \left\{\frac{1}{n} : n \in \{2,3,4,5\}\right\}$ and $B = [1,2]$. Define $d : X \times X \to [0,\infty)$ so that $d(x,y) = d(y,x)$ for all $x,y \in X$, and

$$d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.03; \quad d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = 0.02;$$
$$d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = 0.06; \quad d(x,y) = (x-y)^2 \ \text{ otherwise.}$$

Then $(X,d)$ is a $b$-rectangular metric space with coefficient $s = 3$ (which follows from Example 2.4; in [13], the value $s = 4$ was used). But $(X,d)$ is neither a metric space nor a rectangular metric space. Let $f,g : X \to X$ be defined as:

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in A \\ \frac{1}{5}, & \text{if } x \in B, \end{cases} \qquad g(x) = x.$$

We will check the condition

$$d(fx,fy) \leq \frac{1-\varepsilon}{3}\max\left\{\frac{d(x,y)}{6}, d(x,fx), d(y,fy)\right\} + \varepsilon^2\left[1 + \|x\| + \|y\|\right],$$

with $x_0 = \frac{1}{5}$ (i.e., $\|x\| = d(x,\frac{1}{5})$). It is enough to consider the case when $x \in A$, $y \in B$. Then,
$d(fx,fy) = d\left(\frac{1}{3},\frac{1}{5}\right) = 0.06$,
$\max\frac{d(x,y)}{6} = \max\frac{|x-y|^2}{6} = \frac{\left(2-\frac{1}{5}\right)^2}{6} = \frac{27}{50}$,
$\max d(x,fx) = \max d\left(x,\frac{1}{3}\right) = 0.06$,
$\max d(y,fy) = \max d\left(y,\frac{1}{5}\right) = \left(2-\frac{1}{5}\right)^2 = \frac{81}{25}$.
Hence, if $x \in A$, $y \in B$ we have that $\max\left\{\frac{d(x,y)}{6}, d(x,fx), d(y,fy)\right\} = \frac{81}{25}$, and we have to check that

$$0.06 \leq \frac{1-\varepsilon}{3}\cdot\frac{81}{25} + \varepsilon^2\left(1 + \|x\| + \|y\|\right).$$

Since $\min\{1 + \|x\| + \|y\|\} = 1 + 0 + \left(1-\frac{1}{5}\right)^2 = 1 + \frac{16}{25} = \frac{41}{25}$, we have the inequality

$$0.06 \leq \frac{1-\varepsilon}{3}\cdot\frac{81}{25} + \frac{41}{25}\varepsilon^2,$$

which is satisfied for all $\varepsilon \in \mathbb{R}$, a fortiori for $\varepsilon \in [0,1]$.

Thus, we have proved that all the conditions of Theorem 4.2 are fulfilled and $f$ and $g$ have a unique common fixed point (which is $z = \frac{1}{3}$).

Note that the result could also be obtained using Theorem 3.1, even in an easier way, since in this case the parameter that have to be used is $s = 2$.

The following would be a Pata-version of the well-known Ćirić's result on quasicontractions [8] in the framework of $b$-metric or $b$-rectangular metric space.

**Question 4.5.** *Prove or disprove the following. Let $(X,d)$ be a $b$-metric or a $b$-rectangular metric space with parameter $s$, let $f : X \to X$ and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0,\alpha]$ be fixed constants. If the inequality*

$$d(fx,fy) \leq \frac{1-\varepsilon}{s} \max\left\{\frac{d(x,y)}{2s}, d(x,fx), d(y,fy), d(x,fy), d(y,fx)\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x\|+\|y\|]^{\beta}$$

*is satisfied for every $\varepsilon \in [0,1]$ and all $x,y \in X$, then $f$ has a unique fixed point $z \in X$.*

**Acknowledgements:**

The first author is grateful to the Ministry of Education, Science and Technological Development of Serbia, Grant No. 174002.

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