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Almost periodicity of impulsive Hematopoiesis model with infinite delay

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Abstract

This paper is concerned with almost periodicity of impulsive Hematopoiesis model with infinite delay. By employing the decreasing operator fixed point theorem, we obtain sufficient conditions for the existence of unique almost periodic positive solution. In addition, the exponential stability is derived by Liapunov functional.

Keywords: Impulsive Hematopoiesis model, infinite delay, almost periodic solution, exponential stability, decreasing operator, fixed point theorem.
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1. Introduction

As we know, biological dynamic models are very important and hot research topics. In 1977, Mackey and Glass [16] investigated the Hematopoiesis model

$$x'(t) = -ax(t) + \frac{b}{1 + x^N(t-\tau)},$$

which described the production of blood cells. Gyori and Ladas [6] investigated the global attractivity of positive equilibrium for this model. Moreover, the above Hematopoiesis model and some generalized models have been studied by many authors, see [22, 7, 12, 13, 17, 18, 20].

The assumption that the environment is constant is rarely the case in real life. When the environmental fluctuation is taken into account, a model must be nonautonomous. Due to the various seasonal effects of the environmental factors in real life situation, it is rational and practical to study the biological system

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with periodic coefficients or almost periodic coefficients. In the recent 20 years, the theory of impulsive differential equation has been well developed [1, 2, 4, 14, 19]. In the natural biological systems, there exist many impulsive phenomena, for instance, the human beings harvest or stock species at fixed time, many species are given birth instantaneously and seasonally, and so on. If we incorporate the impulsive factors into biological dynamic models, the models must be governed by impulsive differential equations.

Motivated by the above facts, in this paper, we investigate the following nonautonomous impulsive Hematopoiesis almost periodic model with infinite delay

$$\begin{cases} x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{b_i(t)}{1 + x^{N_i}(t+\tau)} d\tau, & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & t = t_k, \end{cases} \tag{1.1}$$

where $t_k \in R, t_k < t_{k+1}, k \in Z, \lim_{k\to\pm\infty} t_k = \pm\infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$, $a(t), b_i(t) \in PC(R,R)$ and $a(t), b_i(t)$ are positive bounded almost periodic functions, the delay kernel $K_i(\tau) \in C\left((-\infty,0],[0,+\infty)\right)$, $\int_{-\infty}^{0} K_i(\tau)d\tau = 1$, $N_i > 0$ $(i = 1, 2, \cdots, m)$, $I_k(x) : [0,+\infty) \to [0,+\infty)$, $PC(R,R) = \{x(t)|x : R \to R, x(t)$ is continuous for $t \neq t_k, x(t_k^+), x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$.

Denote $BPC\left((-\infty,0],R^+\right) = \{\phi|\phi : (-\infty,0] \to R^+$ is bounded piecewise left continuous function with points of discontinuity of the first kind $\}$. Due to biological significance, we restrict our attention to positive solutions of equation (1.1). The initial condition of equation (1.1) is $x(t) = \phi(t) > 0$ for $t \in (-\infty,0]$, $\phi \in BPC\left((-\infty,0],R^+\right)$.

In the study of biological systems, an important ecological problem is concerned with the existence of positive periodic solutions or positive almost periodic solutions. Recently, many authors investigated the existence of positive periodic solution by using Krasnoselskii cone fixed point theorem and Mawhin's coincidence degree theory [5, 10, 11, 15, 21]. Most of the literatures were concerned with the existence of at least one positive periodic solution.

Almost periodicity is more practical and more close to the reality in biological systems [3, 9]. To our knowledge, few papers study the existence and exponential stability of unique almost periodic positive solution for the above impulsive Hematopoiesis model (1.1) with infinite delay.

For the existence and uniqueness of almost periodic positive solution, the method used in most of the past literatures is the contraction mapping principle.

In this paper, different from the past literatures, we aim to obtain sufficient conditions that guarantee the existence of unique almost periodic positive solution of model (1.1) by using fixed point theorem of decreasing operator. The technique used in this paper is different from the usual methods employed to solve almost periodic cases such as the contraction mapping principle. Our method is the fixed point theorem of decreasing operator. Particularly, we give iterative sequence which converges to the almost periodic positive solution. Moreover, we investigate exponential stability of the almost periodic positive solution by means of Liapunov functional. The results of this paper complement the past results in [22, 6, 7, 12, 13, 17, 18, 20].

2. Preliminaries

Let $B = \{t_k|t_k \in R, t_k < t_{k+1}, k \in Z, \lim_{k\to\pm\infty} t_k = \pm\infty\}$.

Definition 2.1 ([19])**.** The set of sequences $\{t_k^i|t_k^i = t_{k+i} - t_k, t_k \in B\}$ is said to be uniformly almost periodic if for arbitrary $\varepsilon > 0$ there exists relatively dense set of ε- almost periods common for any sequences.

Definition 2.2 ([19])**.** The function $\psi(t) \in PC(R,R)$ is said to be almost periodic, if :

(i) The set of sequences $\{t_k^i|t_k^i = t_{k+i} - t_k, t_k \in B\}$ is uniformly almost periodic.

(ii) For any $\varepsilon > 0$ there exists real number $\delta > 0$ such that if the point t' and t'' belong to one and the same interval of continuity of $\psi(t)$ and satisfy the inequality $|t' - t''| < \delta$,then $|\psi(t') - \psi(t'')| < \varepsilon$.

(iii) For any $\varepsilon > 0$ there exists relatively dense set T such that if $\sigma \in T$, then $|\psi(t+\sigma)-\psi(t)| < \varepsilon$ for $t \in R$ satisfying $|t-t_k| > \varepsilon$.

The elements of T are called ε-almost periods.

In this paper, we use notations: Let $\tilde{I} = \sup\limits_{k\in Z}\{I_k(0)\}$, $W(t,s) = \exp\left(-\int_s^t a(u)du\right)$.

For any bounded function $f(t)$, we denote $\overline{f} = \sup\limits_{t\in R} f(t)$, $\underline{f} = \inf\limits_{t\in R} f(t)$.

Throughout this paper, we make assumptions:

(C_1) The set of sequences $\{t_k^i | t_k^i = t_{k+i} - t_k, t_k \in B\}$ is uniformly almost periodic, and $\inf\limits_{k\in Z} |t_{k+1}-t_k| = \sigma_0 > 0$.

(C_2) The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in [0,+\infty)$. That is, for any $\varepsilon > 0$ there exists $q \in Z^+$, such that $|I_{k+q}(x) - I_k(x)| < \varepsilon$ for $\forall x \in [0,+\infty), k \in Z$.

(C_3) The bounded almost periodic functions $a(t), b_i(t)$ satisfy $0 < \underline{a} \le a(t) \le \overline{a}$, $0 < \underline{b_i} \le b_i(t) \le \overline{b_i}$, $(i = 1,2,\cdots,m)$.

Lemma 2.3 ([19]). *Let the condition (C_1) be satisfied. Then for each $L > 0$, there exists a positive integer N, such that $i(s,t) \le N(t-s)+N$, where $i(s,t)$ is the number of the points t_k in the interval (s,t) of length L.*

From Lemma 2.3, we get the following Lemma 2.4.

Lemma 2.4. *Let the condition (C_1) be satisfied. Then for $L = 1$, there exists a positive integer H, such that $i(s,t) \le 2H$, where $i(s,t)$ is the number of the points t_k in the interval (s,t) of length 1.*

Definition 2.5. Let X be a Banach space and P be a closed,nonempty subset of X,P is called a cone if (i)$x \in P, \lambda \ge 0$ implies $\lambda x \in P$; (ii)$x \in P, -x \in P$ implies $x = \theta$. (θ is zero element).

Every cone $P \subset X$ induces an ordering in X, we define "$\le$"with respect to P by $x \le y$ if and only if $y - x \in P$.

Definition 2.6. A cone P of X is called normal cone if there exists a positive constant σ, such that $||x+y|| \ge \sigma$ for any $x, y \in P$, $||x|| = ||y|| = 1$.

Definition 2.7. Let P be a cone of X and $A : P \to P$ an operator. A is called decreasing if $\theta \le x \le y$ implies $Ax \ge Ay$.

The following fixed point theorem of decreasing operator (see [8]) will be used in our proofs.

Lemma 2.8 ([8]). *Suppose that*

(i) P is normal cone of Banach space X, operator $A : P \to P$ is decreasing;

(ii)$A\theta > \theta, A^2\theta \ge \varepsilon_0 A\theta$, where $\varepsilon_0 > 0$;

(iii) For $\forall 0 < a < c < 1$, there exists $\eta = \eta(a,c) > 0$ such that

$$A(\lambda x) \le [\lambda(1+\eta)]^{-1} Ax \quad \text{for} \quad \forall a \le \lambda \le c \quad \text{and} \quad \theta < x \le A\theta.$$

Then, the operator A has a unique positive fixed point $x^* > \theta$. Moreover,$\|x_k - x^*\| \to 0, \quad (k \to \infty)$,where $x_k = Ax_{k-1} \quad (k = 1,2,\cdots)$ for any initial $x_0 \in P$.

Remark 2.9. In Lemma 2.8, the operator A does not need continuity and compactness.

Let $X = \{x(t)|x \in PC(R,R), x(t)\text{is almost periodic function}\}$. For $x \in X$, we define $\|x\| = \sup_{t\in R} |x(t)|$, then X is Banach space.

It is easy to verify that $x(t)$ is the solution of equation (1.1) if and only if $x(t)$ is the solution of the following integral equation

$$x(t) = \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{b_i(s)}{1 + x^{N_i}(s+\tau)} d\tau ds + \sum_{t_k<t} W(t,t_k) I_k(x(t_k)).$$

We define operator $A : X \to X$,

$$(Ax)(t) = \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{b_i(s)}{1 + x^{N_i}(s+\tau)} d\tau ds + \sum_{t_k<t} W(t,t_k) I_k(x(t_k)).$$

It is obvious that $x(t) \in PC(R,R)$ is the almost periodic solution of equation (1.1) if and only if x is the fixed point of operator A.

Define a cone

$$\Omega = \{x|x \in X, x(t) \geq 0, t \in R\}.$$

For $\forall x \in \Omega$, we can check that $Ax \in \Omega$. Hence we have $A\Omega \subset \Omega$.

3. Existence of the unique almost periodic positive solution

Let $M = \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{b_i} + \frac{2H\tilde{I}}{1 - e^{-\underline{a}}}$, we make the following assumptions:

(S_1) $I_k(x)$ is decreasing with respect to x ;

(S_2) there exists $0 < \mu < 1$, such that $I_k(M) \geq \mu I_k(0)$;

(S_3) $I_k(\lambda x) \leq \frac{1}{\sqrt{\lambda}} I_k(x)$ for $\forall 0 < \lambda < 1$ and $0 < x \leq M$;

(S_4) $0 < N_i \leq 1$;

(S_5) $N_i > 1, \quad (N_i - 1)M^{N_i} \leq 1$.

Theorem 3.1. *Assume that $(C_1) - (C_3)$ and $(S_1) - (S_3)$ are satisfied. For any $i \in \{1, 2, \cdots, m\}$, if (S_4) or (S_5) holds, then equation (1.1) has a unique almost periodic positive solution $x^*(t)$. Moreover, $\|x_k - x^*\| \to 0, \quad (k \to \infty)$, where $x_k = Ax_{k-1} \quad (k = 1, 2, \cdots)$ for any initial $x_0 \in \Omega$.*

Proof. It is clear that Ω is normal cone, $A : \Omega \to \Omega$ is decreasing operator.

Now, we will show that condition (ii) of Lemma 2.8 is satisfied.

Notice that

$$\int_{-\infty}^{t} W(t,s)ds = \int_{-\infty}^{t} e^{-\int_s^t a(u)du} ds \leq \int_{-\infty}^{t} e^{-\int_s^t \underline{a} du} ds = \int_{-\infty}^{t} e^{-\underline{a}(t-s)} ds = \frac{1}{\underline{a}},$$

$$\int_{-\infty}^{t} W(t,s)ds = \int_{-\infty}^{t} e^{-\int_s^t a(u)du} ds \geq \int_{-\infty}^{t} e^{-\int_s^t \bar{a} du} ds = \int_{-\infty}^{t} e^{-\bar{a}(t-s)} ds = \frac{1}{\bar{a}}.$$

We also have

$$\sum_{t_k<t} W(t,t_k) = \sum_{t_k<t} e^{-\int_{t_k}^t a(u)du} \leq \sum_{t_k<t} e^{-\underline{a}(t-t_k)} = \sum_{j=0}^{+\infty} \left(\sum_{t-j-1 \leq t_k < t-j} e^{-\underline{a}(t-t_k)} \right)$$

$$\leq \sum_{j=0}^{+\infty}\left(\sum_{t-j-1\leq t_k<t-j} e^{-\underline{a}j}\right) \leq \sum_{j=0}^{+\infty}\left(2He^{-\underline{a}j}\right) = \frac{2H}{1-e^{-\underline{a}}}.$$

Thus we get

$$\begin{aligned}(A\theta)(t) &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m} b_i(s)ds + \sum_{t_k<t} W(t,t_k)I_k(0)\\ &\leq \sum_{i=1}^{m}\overline{b_i}\int_{-\infty}^{t} W(t,s)ds + \tilde{I}\sum_{t_k<t} W(t,t_k)\\ &\leq \frac{1}{\underline{a}}\sum_{i=1}^{m}\overline{b_i} + \frac{2H\tilde{I}}{1-e^{-\underline{a}}} = M.\end{aligned}$$

In addition, we have

$$\begin{aligned}(A\theta)(t) &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m} b_i(s)ds + \sum_{t_k<t} W(t,t_k)I_k(0)\\ &\geq \int_{-\infty}^{t} W(t,s)ds\sum_{i=1}^{m}\underline{b_i}\\ &\geq \int_{-\infty}^{t} e^{-\bar{a}(t-s)}ds\sum_{i=1}^{m}\underline{b_i} = \frac{1}{\bar{a}}\sum_{i=1}^{m}\underline{b_i} > 0,\end{aligned}$$

which implies $A\theta > \theta$.

On the other hand, we have

$$\begin{aligned}(A^2\theta)(t) &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+(A\theta)^{N_i}(s+\tau)}d\tau ds\\ &\quad + \sum_{t_k<t} W(t,t_k)I_k\left((A\theta)(t_k)\right)\\ &\geq \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+M^{N_i}}d\tau ds + \sum_{t_k<t} W(t,t_k)I_k\left(M\right)\\ &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\frac{b_i(s)}{1+M^{N_i}}ds + \sum_{t_k<t} W(t,t_k)I_k\left(M\right)\\ &\geq \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\frac{b_i(s)}{1+\gamma}ds + \mu\sum_{t_k<t} W(t,t_k)I_k\left(0\right)\\ &= \frac{1}{1+\gamma}\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m} b_i(s)ds + \mu\sum_{t_k<t} W(t,t_k)I_k\left(0\right)\\ &\geq \varepsilon_0\left(\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m} b_i(s)ds + \sum_{t_k<t} W(t,t_k)I_k(0)\right) = \varepsilon_0(A\theta)(t),\end{aligned}$$

this implies $A^2\theta \geq \varepsilon_0 A\theta$, here $\varepsilon_0 = \min\left\{\mu, \frac{1}{1+\gamma}\right\}$, $\gamma = \max_{1\leq i\leq m}\left\{M^{N_i}\right\}$.

Finally, we show that condition (iii) of Lemma 2.8 is satisfied.

Let $\forall 0 < a < c < 1$, for $\forall a \leq \lambda \leq c$ and $\theta < x \leq A\theta$, we have $0 \leq ||x|| \leq ||A\theta|| \leq M$.

$$\begin{aligned} A(\lambda x)(t) &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+\lambda^{N_i}x^{N_i}(s+\tau)}d\tau ds + \sum_{t_k<t} W(t,t_k)I_k(\lambda x(t_k)) \\ &= \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}\frac{1+x^{N_i}(s+\tau)}{1+\lambda^{N_i}x^{N_i}(s+\tau)}d\tau ds \\ &\quad + \sum_{t_k<t} W(t,t_k)I_k(\lambda x(t_k)). \end{aligned} \tag{3.1}$$

Note that

$$\frac{1+x^{N_i}(s+\tau)}{1+\lambda^{N_i}x^{N_i}(s+\tau)} = \lambda^{-N_i}\left(1+\frac{\lambda^{N_i}-1}{1+\lambda^{N_i}x^{N_i}(s+\tau)}\right) \le \lambda^{-N_i}\left(1+\frac{\lambda^{N_i}-1}{1+\lambda^{N_i}M^{N_i}}\right) = \frac{1+M^{N_i}}{1+\lambda^{N_i}M^{N_i}}.$$

It follows from (3.1) that

$$\begin{aligned} A(\lambda x)(t) &\le \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}\frac{1+M^{N_i}}{1+\lambda^{N_i}M^{N_i}}d\tau ds + \sum_{t_k<t} W(t,t_k)I_k(\lambda x(t_k)) \\ &\le \int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}\frac{1+M^{N_i}}{1+\lambda^{N_i}M^{N_i}}d\tau ds + \sum_{t_k<t} W(t,t_k)\frac{1}{\sqrt{\lambda}}I_k(x(t_k)) \\ &= \frac{1}{\lambda}\left(\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}\frac{(1+M^{N_i})\lambda}{1+\lambda^{N_i}M^{N_i}}d\tau ds \right. \\ &\quad \left. + \sqrt{\lambda}\sum_{t_k<t} W(t,t_k)I_k(x(t_k))\right). \end{aligned} \tag{3.2}$$

Let $f_i(t) = \dfrac{(1+M^{N_i})t}{1+t^{N_i}M^{N_i}}$, it is easy to show that

$$f_i'(t) = \frac{(1+M^{N_i})\left[1+(1-N_i)t^{N_i}M^{N_i}\right]}{(1+t^{N_i}M^{N_i})^2}.$$

Since $(S_4) 0 < N_i \le 1$ or $(S_5) N_i > 1, (N_i-1)M^{N_i} \le 1$ $(i = 1,2,\cdots,m)$ holds, then we know $f_i'(t) > 0$ for $0 < t < 1$, so we have $0 = f_i(0) < f_i(a) \le f_i(\lambda) \le f_i(c) < f_i(1) = 1$.

Let $h(c) = \max\limits_{1\le i\le m}\{f_i(c)\}$, $\max\{h(c),\sqrt{c}\} = g(c)$, then $0 < g(c) < 1$.
Hence, by (3.2) we obtain

$$\begin{aligned} A(\lambda x)(t) &\le \frac{1}{\lambda}\left(\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}f_i(\lambda)d\tau ds + \sqrt{\lambda}\sum_{t_k<t} W(t,t_k)I_k(x(t_k))\right) \\ &\le \frac{1}{\lambda}\left(\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}f_i(c)d\tau ds + \sqrt{c}\sum_{t_k<t} W(t,t_k)I_k(x(t_k))\right) \\ &\le \frac{1}{\lambda}\left(h(c)\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}d\tau ds + \sqrt{c}\sum_{t_k<t} W(t,t_k)I_k(x(t_k))\right) \\ &\le \frac{1}{\lambda}g(c)\left(\int_{-\infty}^{t} W(t,s)\sum_{i=1}^{m}\int_{-\infty}^{0} K_i(\tau)\frac{b_i(s)}{1+x^{N_i}(s+\tau)}d\tau ds + \sum_{t_k<t} W(t,t_k)I_k(x(t_k))\right) \\ &= \frac{1}{\lambda}g(c)(Ax)(t) = \frac{1}{\lambda}\cdot\frac{1}{1+\left(\frac{1}{g(c)}-1\right)}(Ax)(t) = \frac{1}{\lambda}\cdot\frac{1}{1+\eta(c)}(Ax)(t) \end{aligned}$$

here $\eta = \eta(c) = \frac{1}{g(c)} - 1 > 0$.
By Lemma 2.8 , we know operator A has a unique positive fixed point $x^* > \theta$, which means equation (1.1) has a unique almost periodic positive solution $x^*(t)$. Moreover, $\|x_k - x^*\| \to 0, \quad (k \to \infty)$, $x_k = Ax_{k-1} \quad (k = 1, 2, \cdots)$ for any initial $x_0 \in \Omega$. The proof of Theorem 3.1 is complete. □

Remark 3.2. We can derive the upper and lower bounds. In fact, from the above proof, we have

$$\begin{aligned} x^*(t) = (Ax^*)(t) &= \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{b_i(s)}{1 + x^{*N_i}(s+\tau)} d\tau ds + \sum_{t_k < t} W(t,t_k) I_k(x^*(t_k)) \\ &\le \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} b_i(s) ds + \sum_{t_k < t} W(t,t_k) I_k(0) \le \sum_{i=1}^{m} \overline{b_i} \int_{-\infty}^{t} W(t,s) ds + \tilde{I} \sum_{t_k < t} W(t,t_k) \\ &\le \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{b_i} + \frac{2H\tilde{I}}{1 - e^{-\underline{a}}} = M. \end{aligned}$$

We also have

$$\begin{aligned} x^*(t) = (Ax^*)(t) &= \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{b_i(s)}{1 + x^{*N_i}(s+\tau)} d\tau ds + \sum_{t_k < t} W(t,t_k) I_k(x^*(t_k)) \\ &\ge \int_{-\infty}^{t} W(t,s) \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \frac{\underline{b_i}}{1 + M^{N_i}} d\tau ds \\ &= \sum_{i=1}^{m} \frac{\underline{b_i}}{1 + M^{N_i}} \int_{-\infty}^{t} W(t,s) ds \ge \frac{1}{\bar{a}} \sum_{i=1}^{m} \frac{\underline{b_i}}{1 + M^{N_i}}. \end{aligned}$$

So we get the upper and lower bounds of $x^*(t)$, that is

$$\frac{1}{\bar{a}} \sum_{i=1}^{m} \frac{\underline{b_i}}{1 + M^{N_i}} \le x^*(t) \le M, \qquad \text{here} \quad M = \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{b_i} + \frac{2H\tilde{I}}{1 - e^{-\underline{a}}}.$$

4. Exponential stability

We make further assumptions:

(S_6) $\int_{-\infty}^{0} K_i(\tau) e^{-\tau} d\tau < +\infty, (i = 1, 2, \cdots, m)$.

(S_7) there exists constant $0 < r < 1$, such that $|I_k(u_1) - I_k(u_2)| \le r\,|u_1 - u_2|$ for $\forall u_1, u_2 \in R^+, \forall k \in Z$.

(S_8) $N_i \ge 1, \quad (N_i - 1) M^{N_i} \le 1$, $(i = 1, 2, \cdots, m)$.

(S_9) $\underline{a} > \sum_{i=1}^{m} N_i \overline{b_i}$.

Theorem 4.1. *Assume that* $(C_1) - (C_3)$ *hold. If* $(S_1) - (S_3)$ *and* $(S_6) - (S_9)$ *are satisfied, then equation* (1.1) *has a unique exponentially stable almost periodic positive solution.*

Proof. By $(S_1) - (S_3)(S_8)$ and Theorem 3.1, we know equation (1.1) has a unique almost periodic positive solution $x^*(t)$, and $\frac{1}{\bar{a}} \sum_{i=1}^{m} \frac{\underline{b_i}}{1 + M^{N_i}} \le x^*(t) \le M$. Now we prove that $x^*(t)$ is exponentially stable.

Suppose $x(t)$ is arbitrary positive solution of equation (1.1) with initial function $x(t) = \phi(t) > 0$ for $-\infty < t \le 0$. Assume the initial function of the almost periodic positive solution $x^*(t)$ is $x^*(t) = \psi(t) > 0$ for $-\infty < t \le 0$.
Since $\displaystyle\int_{-\infty}^{0} K_i(\tau) e^{-\tau} d\tau < +\infty$, $\left| K_i(\tau) e^{-\tau x} \right| \le K_i(\tau) e^{-\tau}$ for $x \in [0, 1]$, $\tau \in (-\infty, 0]$, then we know the

improper integral $\int_{-\infty}^{0} K_i(\tau)e^{-\tau x}d\tau$ converges uniformly on $x \in [0,1]$.

Consider function $G(x) = x - \underline{a} + \sum_{i=1}^{m} N_i\overline{b_i} \int_{-\infty}^{0} K_i(\tau)e^{-\tau x}d\tau, \;\; x \in [0,1]$.

Since $G(0) = -\underline{a} + \sum_{i=1}^{m} N_i\overline{b_i} < 0$, then there exists a constant $\lambda \in (0,1)$ such that $G(\lambda) < 0$.

That is

$$\lambda - \underline{a} + \sum_{i=1}^{m} N_i\overline{b_i} \int_{-\infty}^{0} K_i(\tau)e^{-\lambda\tau}d\tau < 0. \tag{4.1}$$

Let $y(t) = x(t) - x^*(t)$, we define $V(t) = |y(t)|\, e^{\lambda t}$.
For $t \neq t_k$, we have

$$D^+V(t) \leq -a(t)\,|y(t)|\, e^{\lambda t} + \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \left| \frac{b_i(t)}{1+x^{N_i}(t+\tau)} - \frac{b_i(t)}{1+x^{*N_i}(t+\tau)} \right| d\tau e^{\lambda t} + \lambda\,|y(t)|\, e^{\lambda t}. \tag{4.2}$$

For $t = t_k$,

$$\begin{aligned} V(t_k^+) &= \left|y(t_k^+)\right| e^{\lambda t_k} = \left|x(t_k^+) - x^*(t_k^+)\right| e^{\lambda t_k} \\ &= \left|x(t_k) + I_k\left(x(t_k)\right) - \left[x^*(t_k) + I_k\left(x^*(t_k)\right)\right]\right| e^{\lambda t_k} \\ &= \left|x(t_k) - x^*(t_k) + \left[I_k\left(x(t_k)\right) - I_k\left(x^*(t_k)\right)\right]\right| e^{\lambda t_k}, \end{aligned}$$

$$V(t_k) = |y(t_k)|\, e^{\lambda t_k} = |x(t_k) - x^*(t_k)|\, e^{\lambda t_k}.$$

By the conditions (S_1), (S_7) , we know

$$\left|x(t_k) - x^*(t_k) + \left[I_k\left(x(t_k)\right) - I_k\left(x^*(t_k)\right)\right]\right| < |x(t_k) - x^*(t_k)|\,,$$

hence $V(t_k^+) < V(t_k)$.
Let $Q = M + \sup_{-\infty<t\leq 0} |\phi(t) - \psi(t)|$.
For $\forall t \in (-\infty, 0]$, $V(t) = |y(t)|\, e^{\lambda t} \leq |y(t)| = |x(t) - x^*(t)| = |\phi(t) - \psi(t)|$

$$\leq \sup_{-\infty<t\leq 0} |\phi(t) - \psi(t)| < M + \sup_{-\infty<t\leq 0} |\phi(t) - \psi(t)| = Q.$$

We claim that

$$V(t) < Q \qquad \text{for all} \;\; t > 0. \tag{4.3}$$

Suppose the claim (4.3) is not true , then there must exist $t^* > 0$, $K^* \in Z^+$ and $t^* \in (t_{K^*-1}, t_{K^*}]$, such that $V(t^*) = Q$, $V(t) < Q$ for $t < t^*$, and $D^+V(t)|_{t=t^*} \geq 0$.
It follows from (4.2) that

$$\begin{aligned} 0 &\leq D^+V(t)|_{t=t^*} \\ &\leq -a(t^*)\,|y(t^*)|\, e^{\lambda t^*} + \sum_{i=1}^{m} \int_{-\infty}^{0} K_i(\tau) \left| \frac{b_i(t^*)}{1+x^{N_i}(t^*+\tau)} - \frac{b_i(t^*)}{1+x^{*N_i}(t^*+\tau)} \right| d\tau e^{\lambda t^*} + \lambda\,|y(t^*)|\, e^{\lambda t^*} \\ &= -a(t^*)V(t^*) + \sum_{i=1}^{m} b_i(t^*) \int_{-\infty}^{0} K_i(\tau) \left| \frac{1}{1+x^{N_i}(t^*+\tau)} - \frac{1}{1+x^{*N_i}(t^*+\tau)} \right| d\tau e^{\lambda t^*} + \lambda V(t^*). \end{aligned} \tag{4.4}$$

By the mean value theorem, we have

$$\begin{aligned}\left|\frac{1}{1+x^{N_i}(t^*+\tau)}-\frac{1}{1+x^{*N_i}(t^*+\tau)}\right| &= \left|-\frac{N_i\xi^{N_i-1}}{\left(1+\xi^{N_i}\right)^2}\left[x(t^*+\tau)-x^*(t^*+\tau)\right]\right| \\ &= \frac{N_i\xi^{N_i-1}}{\left(1+\xi^{N_i}\right)^2}\left|x(t^*+\tau)-x^*(t^*+\tau)\right|,\end{aligned} \tag{4.5}$$

in which ξ lies between $x(t^*+\tau)$ and $x^*(t^*+\tau)$. Note that the function $g_i(x)=\dfrac{N_i x^{N_i-1}}{\left(1+x^{N_i}\right)^2}<N_i$ for $\forall x\in(0,+\infty)$ and $N_i\geq 1$, $(i=1,2,\cdots,m)$.
Thus we have

$$\frac{N_i\xi^{N_i-1}}{\left(1+\xi^{N_i}\right)^2}<N_i \quad \text{for} \quad N_i\geq 1.$$

From (4.5) , we get

$$\left|\frac{1}{1+x^{N_i}(t^*+\tau)}-\frac{1}{1+x^{*N_i}(t^*+\tau)}\right|<N_i\left|x(t^*+\tau)-x^*(t^*+\tau)\right|. \tag{4.6}$$

Hence, from (4.4) and (4.6), we obtain

$$\begin{aligned}0\leq D^+V(t)|_{t=t^*} &\leq -a(t^*)V(t^*)+\sum_{i=1}^m b_i(t^*)\int_{-\infty}^0 K_i(\tau)N_i\left|x(t^*+\tau)-x^*(t^*+\tau)\right|d\tau e^{\lambda t^*}+\lambda V(t^*)\\ &\leq -\underline{a}V(t^*)+\lambda V(t^*)+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)\left|x(t^*+\tau)-x^*(t^*+\tau)\right|d\tau e^{\lambda t^*}\\ &= -\underline{a}Q+\lambda Q+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)\left|y(t^*+\tau)\right|e^{\lambda(t^*+\tau)}e^{-\lambda\tau}d\tau\\ &= -\underline{a}Q+\lambda Q+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)V(t^*+\tau)e^{-\lambda\tau}d\tau\\ &< -\underline{a}Q+\lambda Q+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)Qe^{-\lambda\tau}d\tau=\left[-\underline{a}+\lambda+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)e^{-\lambda\tau}d\tau\right]Q.\end{aligned}$$

Thus, $-\underline{a}+\lambda+\sum_{i=1}^m N_i\overline{b_i}\int_{-\infty}^0 K_i(\tau)e^{-\lambda\tau}d\tau>0$, which contradicts (4.1). So the claim (4.3) is true. Hence, $V(t)=|y(t)|e^{\lambda t}<Q$ for all $t>0$. That is $|x(t)-x^*(t)|<Qe^{-\lambda t}$ for all $t>0$, which means $x^*(t)$ is exponentially stable. The proof of Theorem 4.1 is complete. □

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