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# Fixed point theorems for $(\alpha, \beta)$-$(\psi, \varphi)$-contractive mappings in $b$-metric spaces with some numerical results and applications

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**Abstract**

In this paper, we introduce the concept of $(\alpha, \beta)$-$(\psi, \varphi)$-contractive mapping in $b$-metric spaces. We establish some fixed point theorems for such mappings and also give an example supporting our results. Finally, we apply our main results to prove a fixed point theorem involving a cyclic mapping. 

*Keywords:* $b$-metric space, cyclic $(\alpha, \beta)$-admissible mapping, $(\alpha, \beta)$-$(\psi, \varphi)$-contractive mapping.
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## 1. Introduction

The Banach contraction principle is one of the most important results in mathematical analysis. It is the most widely applied fixed point result in many branches of mathematics and it was generalized in many different directions.

In 1993, Czerwik [5] introduced the concept of $b$-metric spaces as a generalization of metric spaces and also proved the Banach contraction mapping principle in this setting. Afterwards, many mathematicians studied fixed point theorems for single-valued and multi-valued mappings in $b$-metric spaces (see [3, 4, 6]).

Recently, Alizadeh et al. [2] introduced the notion of cyclic $(\alpha, \beta)$-admissible mapping and proved some new fixed point results for such mappings in the setting of complete metric spaces.

In this paper, we consider $(\alpha, \beta)$-$(\psi, \varphi)$-contractive mappings in $b$-metric spaces and establish some fixed point theorems for this class of mappings. We also provide an illustrative example. Finally, we use our results to prove a fixed point theorem involving a cyclic mapping.

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## 2. Preliminaries

In this section, we recall some essential notations, definitions and primary results known in the literature. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

In 1984, Khan et al. [7] introduced altering distance functions as follows:

**Definition 2.1** ([7]). The function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties hold:

i) $\varphi$ is continuous and non-decreasing;

ii) $\varphi(t) = 0$ if and only if $t = 0$.

**Example 2.2.** Let $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = \sinh^{-1}(t)$ for all $t \in [0,\infty)$. Then $\varphi$ is an altering distance function.

**Example 2.3.** Let $\varphi : [0,\infty) \to [0,\infty)$ defined by

$$\varphi(t) = \begin{cases} \frac{t}{7}, & t < 3, \\ \frac{t^2+3}{t^2+4t+7}, & t \geq 3. \end{cases}$$

Then $\varphi$ is an altering distance function.

The concept of $b$-metric space was introduced by Czerwik in 1993 in the paper [5].

**Definition 2.4** ([5]). Let $X$ be a nonempty set and $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions:

i) $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z, \in X$.

Then $(X, d)$ is called a $b$-metric space with coefficient $s$.

Every metric space is a $b$-metric space with $s = 1$, but, in general, a $b$-metric space need not necessarily be a metric space. Thus, the class of $b$-metric spaces is larger than the class of metric spaces. Some known examples in this respect are provided below.

**Example 2.5.** Let $X = \mathbb{R}$ and let the mapping $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = |x - y|^2 \quad \text{for all } x, y \in X.$$

Then $(X, d)$ is a $b$-metric space with coefficient $s = 2$.

**Example 2.6.** The set $l_p(\mathbb{R})$ with $0 < p < 1$, where

$$l_p(\mathbb{R}) := \{\{x_n\} \subseteq \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with the mapping $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$d(x, y) := \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

for each $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$, is a $b$-metric space with coefficient $s = 2^{\frac{1}{p}} > 1$. The above result also holds for the general case $l_p(X)$ with $0 < p < 1$, where $X$ is a Banach space.

**Example 2.7.** Let $p$ be a given real number in the interval $(0, 1)$. The space

$$L_p[0,1] := \left\{ x : [0,1] \to \mathbb{R} | \int_0^1 |x(t)|^p dt < 1 \right\}$$

together with the mapping $d : L_p[0,1] \times L_p[0,1] \to \mathbb{R}_+$ defined by

$$d(x,y) := \left( \int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}$$

for each $x, y \in L_p[0,1]$, is a $b$-metric space with constant $s = 2^{\frac{1}{p}} > 1$.

Next, we give the concepts of convergence, Cauchy sequence, $b$-continuity, completeness and closedness in $b$-metric spaces.

**Definition 2.8** ([3])**.** Let $(X, d)$ be a $b$-metric space. Then a sequence $\{x_n\}$ in $X$ is called:

i) $b$-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

ii) $b$-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

**Proposition 2.9** ([3])**.** *In a $b$-metric space $(X, d)$, the following assertions hold:*

$(p_1)$ *a $b$-convergent sequence has a unique limit;*

$(p_2)$ *each $b$-convergent sequence is $b$-Cauchy;*

$(p_3)$ *in general, a $b$-metric is not continuous.*

Because of $(p_3)$, we need the following lemma about $b$-convergent sequences in the proof of our results.

**Lemma 2.10** ([1])**.** *Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be $b$-convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

*In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have*

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

**Definition 2.11** ([3])**.** Let $(X, d)$ and $(X', d')$ be two $b$-metric spaces.

i) The space $(X, d)$ is $b$-complete if every $b$-Cauchy sequence in $X$ is $b$-convergent.

ii) A function $f : X \to X'$ is $b$-continuous at a point $x \in X$ if it is $b$-sequentially continuous at $x$, that is, whenever $\{x_n\}$ is $b$-convergent to $x$, $\{fx_n\}$ is $b$-convergent to $fx$.

**Definition 2.12** ([3])**.** Let $Y$ be a nonempty subset of a $b$-metric space $(X, d)$. The closure $\overline{Y}$ of $Y$ is the set of limits of all $b$-convergent sequences of points in $Y$, i.e.,

$$\overline{Y} = \{x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } \lim_{n\to\infty} x_n = x\}.$$

**Definition 2.13** ([3])**.** Let $(X, d)$ be a $b$-metric space. Then a subset $Y \subseteq X$ is called closed if and only if for each sequence $\{x_n\}$ in $Y$ which $b$-converges to an element $x$, we have $x \in Y$ (i.e. $\overline{Y} = Y$).

Recently, Alizadeh et al. [2] introduced the notion of cyclic $(\alpha, \beta)$-admissible mapping as follows:

**Definition 2.14** ([2])**.** Let $X$ be a nonempty set, $f$ be a self-mapping on $X$ and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that $f$ is a cyclic $(\alpha, \beta)$-admissible mapping if

$$x \in X \text{ with } \alpha(x) \geq 1 \Rightarrow \beta(fx) \geq 1$$

and

$$x \in X \text{ with } \beta(x) \geq 1 \Rightarrow \alpha(fx) \geq 1.$$

## 3. Main results

Let $(X,d)$ be a $b$-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a self-mapping. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_s(x,y) := \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\right\}.$$

Note that $M_s(x,y) = M_s(y,x)$. If $s = 1$, we write $M(x,y)$ instead $M_s(x,y)$, that is,

$$M(x,y) := \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}.$$

**Definition 3.1.** Let $(X,d)$ be a $b$-metric space with coefficient $s \geq 1$, and let $\alpha, \beta : X \to [0,\infty)$ be two given mappings. We say that $f : X \to X$ is an $(\alpha,\beta)$-$(\psi,\varphi)$-contractive mapping if the following condition holds:

$$x, y \in X \quad \text{with} \quad \alpha(x)\beta(y) \geq 1 \implies \psi(s^3 d(fx,fy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y)), \tag{3.1}$$

where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions.

**Theorem 3.2.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s \geq 1$, $\alpha, \beta : X \to [0,\infty)$ be two mappings and $f : X \to X$ be an $(\alpha,\beta)$-$(\psi,\varphi)$-contractive mapping. Suppose that*

(1) *one of the following condition holds:*

  (1.1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$;*

  (1.2) *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$;*

(2) *$f$ is continuous;*

(3) *$f$ is a cyclic $(\alpha,\beta)$-admissible mapping.*

*Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0$ is an initial point in condition (1.1) and the sequence $\{y_n\}$ in $X$ defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that $y_0$ is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of $f$.*

*Proof.* **Case I:** Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$. We will construct the iterative sequence $\{x_n\}$, where $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{\tilde{n}} = x_{\tilde{n}+1}$ for some $\tilde{n} \in \mathbb{N} \cup \{0\}$, then $x_{\tilde{n}}$ is a fixed point of $f$, and the proof is finished. Hence, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $d(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$.

We will complete the proof in three steps.

**Step I**. We prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Since $f$ is a cyclic $(\alpha,\beta)$-admissible mapping, we have

$$\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) = \beta(fx_0) \geq 1 \Rightarrow \alpha(x_2) = \alpha(fx_1) \geq 1.$$

By continuing this process, we obtain that

$$\alpha(x_{2k}) \geq 1 \quad \text{and} \quad \beta(x_{2k+1}) \geq 1 \tag{3.2}$$

for all $k \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_0)\beta(x_1) \geq 1$, we get

$$\begin{aligned}\psi(d(fx_0, fx_1)) &\leq \psi(s^3 d(fx_0, fx_1)) \\ &\leq \psi(M_s(x_0,x_1)) - \varphi(M_s(x_0,x_1)).\end{aligned}$$

Again, since $\alpha(x_2)\beta(x_1) \geq 1$, we have

$$\begin{aligned}\psi(d(fx_1, fx_2)) &= \psi(d(fx_2, fx_1)) \\ &\leq \psi(s^3 d(fx_2, fx_1)) \\ &\leq \psi(M_s(x_2, x_1)) - \varphi(M_s(x_2, x_1)) \\ &= \psi(M_s(x_1, x_2)) - \varphi(M_s(x_1, x_2)).\end{aligned}$$

Proceeding in the same way, we obtain

$$\begin{aligned}\psi(d(fx_n, fx_{n+1})) &\leq \psi(s^3 d(fx_n, fx_{n+1})) \\ &\leq \psi(M_s(x_n, x_{n+1})) - \varphi(M_s(x_n, x_{n+1})) \end{aligned} \tag{3.3}$$

for each $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} M_s(x_n, x_{n+1}) &= \max\left\{ d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s} \right\} \\ &= \max\left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s} \right\} \\ &= \max\left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \right\}. \end{aligned}$$

From (3.3) and the properties of $\psi$ and $\varphi$, it follows that

$$\begin{aligned}\psi(d(fx_n, fx_{n+1})) &\leq \psi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \\ &\quad - \varphi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \\ &< \psi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} = d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$, then by (3.4) we have,

$$\begin{aligned}\psi(d(fx_n, fx_{n+1})) &\leq \psi\left(d(x_{n+1}, x_{n+2})\right) - \varphi\left(d(x_{n+1}, x_{n+2})\right) \\ &< \psi\left(d(x_{n+1}, x_{n+2})\right), \end{aligned}$$

a contradiction. Therefore,

$$\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} = d(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. By (3.4), we get

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \psi\left(d(x_n, x_{n+1})\right) - \varphi\left(d(x_n, x_{n+1})\right) \\ &< \psi(d(x_n, x_{n+1})) \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $\psi$ is a non-decreasing mapping, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Letting $n \to \infty$ in (3.5), we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r).$$

This implies that $\varphi(r) = 0$ and thus $r = 0$. Consequently,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

This completes the first step of the proof.

**Step II**. We now claim that $\{x_n\}$ is a $b$-Cauchy sequence in $X$, that is, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$.

Assume, to the contrary, that there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$, $m(k)$ is even and $n(k)$ is odd,

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{3.7}$$

and $n(k)$ is the smallest number such that (3.7) holds. From (3.7), we get

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{3.8}$$

By the triangle inequality, (3.7) and (3.8), we obtain that

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq s[d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})] \\ &< s[\epsilon + d(x_{n(k)-1}, x_{n(k)})]. \end{aligned} \tag{3.9}$$

Taking limit supremum as $k \to \infty$ in (3.9), by using (3.6) we get

$$\epsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq s\epsilon. \tag{3.10}$$

From the triangle inequality, we get

$$d(x_{m(k)}, x_{n(k)}) \leq s[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})] \tag{3.11}$$

and

$$d(x_{m(k)}, x_{n(k)+1}) \leq s[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})]. \tag{3.12}$$

Taking limit supremum as $k \to \infty$ in (3.11) and (3.12), from (3.6) and (3.10), we obtain that

$$\epsilon \leq s\left(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})\right)$$

and

$$\limsup_{k\to\infty}(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon.$$

This implies that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon. \tag{3.13}$$

Again, using above process, we get

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\epsilon. \tag{3.14}$$

Finally, we obtain that

$$d(x_{m(k)}, x_{n(k)+1}) \leq s[d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1})]. \tag{3.15}$$

Taking limit supremum as $k \to \infty$ in (3.15), from (3.6) and (3.13), we obtain that

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{3.16}$$

Similarly, we have

$$\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{3.17}$$

By (3.16) and (3.17) we get

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\epsilon. \tag{3.18}$$

Relation (3.2) implies that

$$\alpha(x_{m(k)})\beta(x_{n(k)}) \geq 1.$$

From (3.1), we have

$$\begin{aligned}\psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) &= \psi(s^3 d(fx_{m(k)}, fx_{n(k)}))\\ &\leq \psi(M_s(x_{m(k)}, x_{n(k)})) - \varphi(M_s(x_{m(k)}, x_{n(k)})),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}M_s(x_{m(k)}, x_{n(k)}) &= \max\Big\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, fx_{m(k)}), d(x_{n(k)}, fx_{n(k)}),\\ &\qquad \frac{d(x_{m(k)}, fx_{n(k)}) + d(x_{n(k)}, fx_{m(k)})}{2s}\Big\}\\ &= \max\Big\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}),\\ &\qquad \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\Big\}.\end{aligned}$$

Taking limit supremum as $k \to \infty$ in above equation and using (3.6), (3.10), (3.13) and (3.14), we obtain

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Also, we can show that

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Taking limit supremum as $k \to \infty$ in (3.19) and using (3.18), it follows that

$$\begin{aligned}\psi(s\epsilon) &= \psi\left(s^3\left(\frac{\epsilon}{s^2}\right)\right)\\ &\leq \psi\left(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})\right)\\ &\leq \psi\left(\limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\right) - \varphi\left(\liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\right)\\ &\leq \psi(s\epsilon) - \varphi(\epsilon).\end{aligned}$$

This implies that $\varphi(\epsilon) = 0$ and then $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a $b$-Cauchy sequence.

**Step III**. We show that $f$ has a fixed point. From Step II, $\{x_n\}$ is a $b$-Cauchy sequence in $X$. By the completeness of the $b$-metric space $X$, there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and hence

$$\lim_{n\to\infty} d(fx_n, x) = \lim_{n\to\infty} d(x_{n+1}, x) = 0. \tag{3.20}$$

By the continuity of $f$, we get

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

From the triangle inequality, we have

$$d(x, fx) \leq s[d(x, fx_n) + d(fx_n, fx)] \tag{3.21}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality, we obtain that $d(x, fx) = 0$, that is, $x$ is a fixed point of $f$.

**Case II:** Assume that there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$. Proceeding in a similar manner as above, we obtain the conclusion. □

**Example 3.3.** Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Then $(X, d)$ is a complete $b$-metric space with $s = 2$.

Define mappings $\alpha, \beta : X \to [0,\infty)$ and $f : X \to X$ as follows:

$$\alpha(x) = \begin{cases} \frac{x+5}{4}, & x \in [0, 0.5]; \\ 0, & \text{otherwise}, \end{cases} \tag{3.22}$$

$$\beta(x) = \begin{cases} (x+1)^2, & x \in [0, 0.5]; \\ 0, & \text{otherwise}, \end{cases} \tag{3.23}$$

and

$$fx = \begin{cases} \frac{x}{3}, & x \in [0, 0.5]; \\ \frac{212x-105}{6}, & x \in (0.5, 0.51]; \\ x + 0.01, & \text{otherwise}. \end{cases} \tag{3.24}$$

First, we will show that $f$ is a cyclic $(\alpha, \beta)$-admissible mapping.

For $x \in X$, we have

$$\alpha(x) \geq 1 \Rightarrow x \in [0, 0.5] \Rightarrow \beta(fx) = \beta\left(\frac{x}{3}\right) = \left(\frac{x}{3} + 1\right)^2 \geq 1$$

and

$$\beta(x) \geq 1 \Rightarrow x \in [0, 0.5] \Rightarrow \alpha(fx) = \alpha\left(\frac{x}{3}\right) = \frac{x + 15}{12} \geq 1.$$

Therefore, our claim is proved.

Next, we will show that $f$ is an $(\alpha, \beta)$-$(\psi, \varphi)$-contractive mapping with altering distance functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = bt \quad \text{and} \quad \varphi(t) = (b - 1)t$$

for all $t \in [0,\infty)$, where $b \in \left(1, \frac{9}{8}\right]$.

Assume that $x, y \in X$ are such that $\alpha(x)\beta(y) \geq 1$. Then we have $x, y \in [0, 0.5]$ and hence

$$\begin{aligned} \psi(s^3 d(fx, fy)) &= 8b|fx - fy|^2 = 8b\left|\frac{x}{3} - \frac{y}{3}\right|^2 \\ &= \frac{8}{9}b|x - y|^2 = \frac{8}{9}bd(x, y) \\ &\leq \frac{8}{9}bM_s(x, y) \leq M_s(x, y) \\ &= \psi(M_s(x, y)) - \varphi(M_s(x, y)). \end{aligned}$$

Also, we note that $f$ is continuous and there exists $x_0 = 0.5 \in X$ such that $\alpha(x_0) = \alpha(0.5) = 1.375 \geq 1$ and $\beta(x_0) = \beta(0.5) = 2.25 \geq 1$, so (3.1) is satisfied. Hence, all conditions of Theorem 3.2 hold, implying that $f$ has at least one fixed point. In this case, 0 and $\frac{105}{206}$ are fixed points of $f$. For the initial point $x_0 = 0.4, 0.5, 0.6, 0.7$, results of the iteration process $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ are given in Table 1.

| **Step** | $x_0 = 0.4$ | $x_0 = 0.5$ | $x_0 = 0.6$ | $x_0 = 0.7$ |
|---|---|---|---|---|
| 1 | 0.133333333333 | 0.166666666667 | 0.610000000000 | 0.710000000000 |
| 2 | 0.044444444444 | 0.055555555556 | 0.620000000000 | 0.720000000000 |
| 3 | 0.014814814815 | 0.018518518519 | 0.630000000000 | 0.730000000000 |
| 4 | 0.004938271605 | 0.006172839506 | 0.640000000000 | 0.740000000000 |
| 5 | 0.001646090535 | 0.002057613169 | 0.650000000000 | 0.750000000000 |
| 6 | 0.000548696845 | 0.000685871056 | 0.660000000000 | 0.760000000000 |
| 7 | 0.000182898948 | 0.000228623685 | 0.670000000000 | 0.770000000000 |
| 8 | 0.000060966316 | 0.000076207895 | 0.680000000000 | 0.780000000000 |
| 9 | 0.000020322105 | 0.000025402632 | 0.690000000000 | 0.790000000000 |
| 10 | 0.000006774035 | 0.000008467544 | 0.700000000000 | 0.800000000000 |
| 11 | 0.000002258012 | 0.000002822515 | 0.710000000000 | 0.810000000000 |
| 12 | 0.000000752671 | 0.000000940838 | 0.720000000000 | 0.820000000000 |
| 13 | 0.000000250890 | 0.000000313613 | 0.730000000000 | 0.830000000000 |
| 14 | 0.000000083630 | 0.000000104538 | 0.740000000000 | 0.840000000000 |
| 15 | 0.000000027877 | 0.000000034846 | 0.750000000000 | 0.850000000000 |
| 16 | 0.000000009292 | 0.000000011615 | 0.760000000000 | 0.860000000000 |
| 17 | 0.000000003097 | 0.000000003872 | 0.770000000000 | 0.870000000000 |
| 18 | 0.000000001032 | 0.000000001291 | 0.780000000000 | 0.880000000000 |
| 19 | 0.000000000344 | 0.000000000430 | 0.790000000000 | 0.890000000000 |
| 20 | 0.000000000115 | 0.000000000143 | 0.800000000000 | 0.900000000000 |
| 21 | 0.000000000038 | 0.000000000048 | 0.810000000000 | 0.910000000000 |
| 22 | 0.000000000013 | 0.000000000016 | 0.820000000000 | 0.920000000000 |
| 23 | 0.000000000004 | 0.000000000005 | 0.830000000000 | 0.930000000000 |
| 24 | 0.000000000001 | 0.000000000002 | 0.840000000000 | 0.940000000000 |
| 25 | 0.000000000000 | 0.000000000001 | 0.850000000000 | 0.950000000000 |
| 26 | 0.000000000000 | 0.000000000000 | 0.860000000000 | 0.960000000000 |
| 27 | 0.000000000000 | 0.000000000000 | 0.870000000000 | 0.970000000000 |
| 28 | 0.000000000000 | 0.000000000000 | 0.880000000000 | 0.980000000000 |
| 29 | 0.000000000000 | 0.000000000000 | 0.890000000000 | 0.990000000000 |
| 30 | 0.000000000000 | 0.000000000000 | 0.900000000000 | 1.000000000000 |

Table 1: Comparative results of Example 3.3

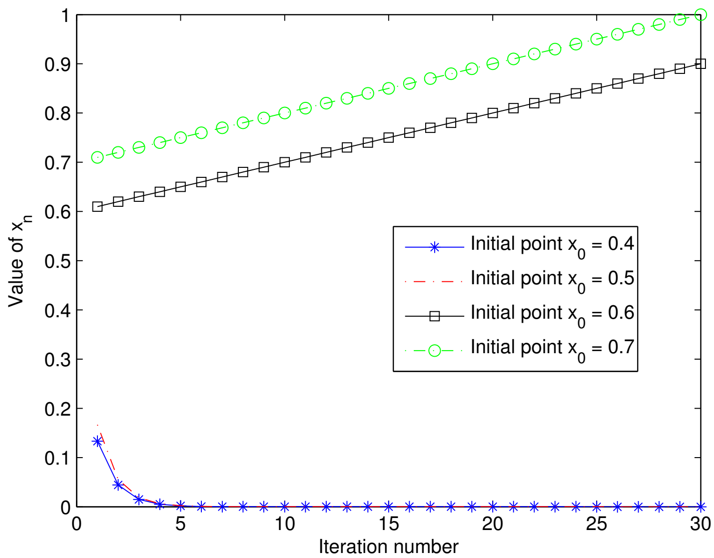


Figure 1: Behavior of the iteration process with initial point $x_0 = 0.4, 0.5, 0.6, 0.7$ for the function given in Example 3.3.

**Corollary 3.4.** *Let $(X,d)$ be a complete b-metric space with coefficient $s \geq 1$, and let $\alpha, \beta : X \to [0,\infty)$ and $f : X \to X$ be three mappings such that*

$$x, y \in X \quad \text{with} \quad \alpha(x)\beta(y) \geq 1 \implies s^3 d(fx, fy) \leq kM_s(x,y),$$

*where $k \in [0,1)$. Suppose that*

(1) *one of the following condition holds:*

  (1.1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$;*

  (1.2) *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$;*

(2) *$f$ is continuous;*

(3) *$f$ is a cyclic $(\alpha,\beta)$-admissible mapping.*

*Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0$ is an initial point in condition (1.1) and the sequence $\{y_n\}$ in $X$ defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that $y_0$ is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of $f$.*

*Proof.* The result follows from Theorem 3.2 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$. □

**Corollary 3.5.** *Let $(X,d)$ be a complete b-metric space with coefficient $s \geq 1$, and $f : X \to X$ be a continuous mapping such that*

$$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y))$$

*for all $x, y \in X$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of $f$.*

*Proof.* The result follows from Theorem 3.2 by taking $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$. □

**Corollary 3.6.** *Let $(X,d)$ be a complete b-metric space with coefficient $s \geq 1$, and let $f : X \to X$ be a continuous mapping such that*

$$s^3 d(fx, fy) \leq kM_s(x,y)$$

*for all $x, y \in X$, where $k \in [0,1)$. Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of $f$.*

*Proof.* It follows from Theorem 3.2 by taking $\psi(t) = t$, $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$ and $\alpha(x) = 1$, $\beta(x) = 1$ for all $x \in X$. □

Taking $s = 1$ we obtain the following fixed point results in the framework of classical metric spaces:

**Corollary 3.7.** *Let $(X,d)$ be a complete metric space, and $\alpha, \beta : X \to [0,\infty)$ and $f : X \to X$ be three mappings such that*

$$x, y \in X \quad \text{with} \quad \alpha(x)\beta(y) \geq 1 \implies \psi(d(fx, fy)) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

*where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Suppose that*

(1) *one of the following condition holds:*

  (1.1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$;*

  (1.2) *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$;*

(2) *$f$ is continuous;*

(3) *$f$ is a cyclic $(\alpha, \beta)$-admissible mapping.*

*Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0$ is an initial point in condition (1.1) and the sequence $\{y_n\}$ in $X$ defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that $y_0$ is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of $f$.*

**Corollary 3.8.** *Let $(X, d)$ be a complete metric space, and $\alpha, \beta : X \to [0, \infty)$ and $f : X \to X$ be three mappings such that*

$$x, y \in X \quad \text{with} \quad \alpha(x)\beta(y) \geq 1 \implies d(fx, fy) \leq kM(x, y),$$

*where $k \in [0, 1)$. Suppose that*

(1) *one of the following condition holds:*

  (1.1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$;*
  (1.2) *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$;*

(2) *$f$ is continuous;*

(3) *$f$ is a cyclic $(\alpha, \beta)$-admissible mapping.*

*Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0$ is an initial point in condition (1.1) and the sequence $\{y_n\}$ in $X$ defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that $y_0$ is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of $f$.*

**Corollary 3.9.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a continuous mapping such that*

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

*for all $x, y \in X$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of $f$.*

**Corollary 3.10.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a continuous mapping such that*

$$d(fx, fy) \leq kM(x, y)$$

*for all $x, y \in X$, where $k \in [0, 1)$. Then $f$ has a fixed point. Moreover, if the sequence $\{x_n\}$ in $X$ defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of $f$.*

## 4. Applications

In this section, we apply our main results to prove a fixed point theorem involving a cyclic mapping.

**Definition 4.1** ([8])**.** Let $A$ and $B$ be nonempty subsets of a set $X$. A mapping $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.

**Definition 4.2.** Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$. We say that a mapping $f : A \cup B \to A \cup B$ is a $(A, B)$-$(\psi, \varphi)$-contractive mapping if

$$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) \tag{4.1}$$

for all $x \in A$ and $y \in B$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.

**Theorem 4.3.** *Let $A$ and $B$ be two nonempty subsets of the complete b-metric space $(X,d)$ with coefficient $s \geq 1$ and $f : A \cup B \to A \cup B$ be a b-continuous cyclic mapping which is an $(A,B)$-$(\psi,\varphi)$-contractive mapping. Then $f$ has a fixed point in $A \cap B$.*

*Proof.* Define mappings $\alpha, \beta : A \cup B \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & x \in B; \\ 0, & \text{otherwise}. \end{cases}$$

For $x, y \in A \cup B$ such that $\alpha(x)\beta(y) \geq 1$, we get $x \in A$ and $y \in B$. Then we have

$$\psi(s^3 d(fx,fy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y))$$

and thus the condition (3.1) holds. Therefore, $f$ is an $(\alpha,\beta)$-$(\psi,\varphi)$-contractive mapping. It is easy to see that $f$ is a cyclic $(\alpha,\beta)$-admissible mapping. Since $A$ and $B$ are nonempty subsets, there exists $x_0 \in A$ such that $\alpha(x_0) \geq 1$ and there exists $y_0 \in B$ such that $\beta(y_0) \geq 1$. Now, all conditions of Theorem 3.2 hold, so $f$ has a fixed point in $A \cup B$, say $z$. If $z \in A$, then $z = fz \in B$. Similarly, if $z \in B$, then we have $z \in A$. Hence, $z \in A \cap B$. This completes the proof. □

**Corollary 4.4.** *Let $A$ and $B$ be two nonempty subsets of the complete b-metric space $(X,d)$ with coefficient $s \geq 1$ and $f : A \cup B \to A \cup B$ be a b-continuous cyclic mapping. Assume that*

$$s^3 d(fx,fy) \leq kM_s(x,y)$$

*for all $x \in A$ and $y \in B$, where $k \in [0,1)$. Then $f$ has a fixed point in $A \cap B$.*

*Proof.* It follows from Theorem 4.3 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$. □

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