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On some new fixed point results for rational Geraghty contractive mappings in ordered b-metric spaces

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Abstract

In this paper we prove some new fixed point results in the context of ordered b-metric spaces for rational Geraghty contractive mappings. Thus our results in the new context generalize, extend, unify, enrich and complement fixed point theorems of contractive mappings in several aspects. One example is given to show the validity of our results. In addition, we obtain the periodic property of these mappings.

Keywords: Fixed point, b-metric space, rational Geraghty contraction, ordered b-metric space.
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1. Introduction and Preliminaries

Fixed point theory is one of the most important topics in development of nonlinear and mathematical analysis in general. Also, fixed point theory has been used effectively in many other branches of science, such as computer science, engineering, chemistry, biology, economics....It is well known that Banach's contraction principle [5] is one of the pivotal results of nonlinear analysis. A mapping $f : X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq kd(x, y).$$

If the metric space (X, d) is complete, then the mapping f satisfying the above inequality has a unique fixed point.

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Otherwise, there are many generalizations of metric spaces: Menger spaces, fuzzy metric spaces, generalized metric spaces, abstract metric spaces, b-metric spaces or metric type spaces called by some authors.... Czerwik in [6] introduced the concept of b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see, e.g., [7, 9, 12, 13, 17, 19, 21]).

Definition 1.1. Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric if the following conditions are satisfied:
(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$
for all $x, y, z \in X$. In this case, the pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces since a b-metric is a metric when $s = 1$, and there are b-metric spaces which are not metric spaces. For some useful examples of b-metric spaces, the reader may refer to [3, 7, 9, 12, 14, 17, 19, 21]. Also, the notions such as b-convergent sequence, b-Cauchy sequence and b-complete space are defined by an obvious way.

Use different forms of contractive conditions in various generalized metric spaces, there is a large number of extensions of the Banach's contraction principle ([8]). Some of such generalizations are obtained via rational contractive conditions (see [4, 10]).

On the other hand, fixed points of monotone mappings in ordered metric spaces have been a matter of investigation ever since the first result was given by Ran and Reurings [18]. They gave many useful results in matrix equations. Recently, many researchers have focused on different contractive conditions in complete metric and b-metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results in ordered metric and ordered b -metric spaces, we refer the reader to [1, 3, 15, 16, 17, 19, 20, 21].

2. Main results

Let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies that } t_n \to 0 \text{ as } n \to \infty.$$

Definition 2.1. Let $(X, d, \preceq)$ be an ordered b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$s^{\varepsilon} d(fx, fy) \leq \beta\left(M_I(x, y)\right) M_I(x, y) \tag{2.1}$$

for all comparable elements $x, y \in X$, where

$$M_I(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)\, d(y, fy)}{1 + d(fx, fy)}\right\}. \tag{2.2}$$

Definition 2.2. Let $(X, d, \preceq)$ be an ordered b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$s^{\varepsilon} d(fx, fy) \leq \beta\left(M_{II}(x, y)\right) M_{II}(x, y) \tag{2.3}$$

for all comparable elements $x, y \in X$, where

$$M_{II}(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s\left[d(x, fx) + d(y, fy)\right]}, \right.$$

$$\left. \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s\left[d(x, fy) + d(y, fx)\right]}\right\}. \tag{2.4}$$

Definition 2.3. Let $(X, d, \preceq)$ be an ordered b-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$s^{\varepsilon} d(fx, fy) \le \beta \left(M_{III} (x, y)\right) M_{III} (x, y) \tag{2.5}$$

for all comparable elements $x, y \in X$, where

$$M_{III} (x, y) = \max \left\{ d (x, y) , \frac{d (x, fx) d (y, fy)}{1 + s [d (x, y) + d (x, fy) + d (y, fx)]}, \frac{d (x, fy) d (x, y)}{1 + sd (x, fx) + s^3 [d (y, fx) + d (y, fy)]} \right\}. \tag{2.6}$$

The following theorem unifies all results from [21] but with complete new approach. Also, our proof is without using recent lemma of Aghajani et al. [3] about the b-convergent sequences. Because, our proof is much shorter.

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f is a rational Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ (resp. type $\mathbf{II}_{\varepsilon,\beta}$; resp type $\mathbf{III}_{\varepsilon,\beta}$). If*

(1) f is continuous, or,

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Assume that $x_{n+1} \neq x_n$ for all $n = 0, 1, 2, ...,$ where $x_n = f^n (x_0)$. In this case we have

$$x_0 \prec f (x_0) \prec f^2 (x_0) \prec \cdots \prec f^n (x_0) \prec f^{n+1} (x_0) \prec \cdots .$$

Putting $x_{n+1} = fx_n$ $(n = 0, 1, 2, \ldots)$, we shall prove that

$$d (x_{n+1}, x_n) \le \frac{1}{s^{1+\varepsilon}} d (x_n, x_{n-1}) \tag{2.7}$$

for all $n \in \mathbb{N}$ as well as for all three types ($\mathbf{I}_{\varepsilon,\beta}$, $\mathbf{II}_{\varepsilon,\beta}$ and $\mathbf{III}_{\varepsilon,\beta}$) of function f. Since $\beta \in \mathcal{F}_s$, according to (2.1), (2.3) and (2.5), we obtain that

$$d(fx, fy) \le \frac{1}{s^{1+\varepsilon}} M_I (x, y) , \quad d(fx, fy) \le \frac{1}{s^{1+\varepsilon}} M_{II} (x, y) , \quad d(fx, fy) \le \frac{1}{s^{1+\varepsilon}} M_{III} (x, y) ,$$

for all comparable elements $x, y \in X$, where $M_I (x, y) , M_{II} (x, y)$ and $M_{III} (x, y)$ are given by (2.2), (2.4) and (2.6), respectively.

Firstly, let f be a rational Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$. Then

$$d (x_{n+1}, x_n) = d (fx_n, fx_{n-1}) \le \frac{1}{s^{1+\varepsilon}} M_I (x_n, x_{n-1}) , \tag{2.8}$$

where

$$\begin{aligned} M_I (x_n, x_{n-1}) &= \max \left\{ d (x_n, x_{n-1}) , \frac{d (x_n, x_{n+1}) d (x_{n-1}, x_n)}{1 + d (x_n, x_{n-1})}, \frac{d (x_n, x_{n+1}) d (x_{n-1}, x_n)}{1 + d (x_{n+1}, x_n)} \right\} \\ &\le \max \left\{ d (x_n, x_{n-1}) , \frac{d (x_n, x_{n+1}) d (x_{n-1}, x_n)}{d (x_n, x_{n-1})}, \frac{d (x_n, x_{n+1}) d (x_{n-1}, x_n)}{d (x_{n+1}, x_n)} \right\} \\ &= \max \left\{ d (x_n, x_{n-1}) , d (x_n, x_{n+1}) \right\} . \end{aligned}$$

If $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, then by (2.8) it follows that $d(x_{n+1}, x_n) \leq \frac{1}{s^{1+\varepsilon}} d(x_n, x_{n+1}) < d(x_n, x_{n+1})$. A contradiction. Hence, $M_I(x_n, x_{n-1}) \leq d(x_n, x_{n-1})$, that is, (2.7) holds. Therefore, by [12, Lemma 3.1] and [17, Lemma 2.3], $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

(i) Now, let (1) hold, accordingly, we arrive at

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = fu,$$

that is, u is a fixed point of f.

(ii) Further, let (2) hold. Using the assumption of $(X, d, \preceq)$, we have $x_n \preceq u$. We shall show again that $fu = u$. Indeed, we have that

$$\begin{aligned}\frac{1}{s} d(u, fu) &\leq d(u, x_{n+1}) + d(fx_n, fu) \\ &\leq d(u, x_{n+1}) + \frac{1}{s^{1+\varepsilon}} M_I(x_n, u),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}M_I(x_n, u) = \max\left\{d(x_n, u), \frac{d(x_n, x_{n+1})\, d(u, fu)}{1 + d(x_n, u)}, \frac{d(x_n, x_{n+1})\, d(u, fu)}{1 + d(x_{n+1}, fu)}\right\} \to \max\{0, 0, 0\} = 0,\\ \text{as } n \to \infty.\end{aligned} \tag{2.10}$$

Therefore, from (2.9) and (2.10), we deduce that $d(u, fu) = 0$, so $u = fu$.

Now, let f be a rational Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$.

In this case, we speculate that

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \leq \frac{1}{s^{1+\varepsilon}} M_{II}(x_n, x_{n-1}), \tag{2.11}$$

where

$$\begin{aligned}M_{II}(x_n, x_{n-1}) &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1})\, d(x_n, x_n) + d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{1 + s\left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right]},\right.\\ &\qquad\left.\frac{d(x_n, x_{n+1})\, d(x_n, x_n) + d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{1 + s\left[d(x_n, x_n) + d(x_{n-1}, x_{n+1})\right]}\right\}\\ &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{1 + s\left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right]}, \frac{d(x_{n-1}, x_n)\, d(x_{n-1}, x_{n+1})}{1 + s d(x_{n-1}, x_{n+1})}\right\}\\ &\leq \max\left\{d(x_n, x_{n-1}), \frac{s d(x_{n-1}, x_n)\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]}{1 + s\left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right]}, \frac{d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1})}{s d(x_{n-1}, x_{n+1})}\right\}\\ &\leq \max\left\{d(x_n, x_{n-1}), \frac{s d(x_{n-1}, x_n)\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]}{s\left[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right]}, \frac{d(x_n, x_{n-1})}{s}\right\}\\ &= d(x_n, x_{n-1}).\end{aligned} \tag{2.12}$$

Hence, by (2.11) and (2.12), (2.7) holds. Again, by [12, Lemma 3.1] and [17, Lemma 2.3], $\{x_n\}$ is a b-Cauchy sequence. b-Completeness of X shows that $\{x_n\}$ b-converges to a point $u \in X$.

(i) If (1) is satisfied, then we have $u = fu$, that is, u is a fixed point of f.

(ii) Now, let (2) hold. As in the previous case we shall show again that $fu = u$. Actually, since (2) holds, we have $x_n \preceq u$. Therefore,

$$\begin{aligned}\frac{1}{s} d(u, fu) &\leq d(u, x_{n+1}) + d(fx_n, fu) \\ &\leq d(u, x_{n+1}) + \frac{1}{s^{1+\varepsilon}} M_{II}(x_n, u),\end{aligned} \tag{2.13}$$

where

$$M_{II}(x_n,u)=\max\left\{d(x_n,u),\frac{d(x_n,x_{n+1})\,d(x_n,fu)+d(u,fu)\,d(u,x_{n+1})}{1+s[d(x_n,x_{n+1})+d(u,fu)]},\right.$$
$$\left.\frac{d(x_n,x_{n+1})\,d(x_n,fu)+d(u,fu)\,d(u,x_{n+1})}{1+s[d(x_n,fu)+d(u,x_{n+1})]}\right\}.$$

Since $d(x_n,fu)\le s[d(x_n,u)+d(u,fu)]$, we further obtain that

$$\begin{aligned}M_{II}(x_n,u)&\le\max\{d(x_n,u),d(x_n,x_{n+1})\,s[d(x_n,u)+d(u,fu)]+d(u,fu)\,d(u,x_{n+1}),\\&\qquad d(x_n,x_{n+1})\,s[d(x_n,u)+d(u,fu)]+d(u,fu)\,d(u,x_{n+1})\}\\&\to\max\{0,0\cdot s[0+d(u,fu)]+d(u,fu)\cdot 0,0\cdot s[0+d(u,fu)]+d(u,fu)\cdot 0\}\\&=0,\quad\text{as } n\to\infty.\end{aligned}\tag{2.14}$$

Therefore, from (2.13) and (2.14) it follows that $d(u,fu)=0$, i.e., $u=fu$.

Finally, let f be a rational Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$.

Similar to previous two cases, we have

$$d(x_{n+1},x_n)=d(fx_n,fx_{n-1})\le\frac{1}{s^{1+\varepsilon}}M_{III}(x_n,x_{n-1}),\tag{2.15}$$

where

$$\begin{aligned}M_{III}(x_n,x_{n-1})&=\max\left\{d(x_n,x_{n-1}),\frac{d(x_n,x_{n+1})\,d(x_{n-1},x_n)}{1+s[d(x_n,x_{n-1})+d(x_n,x_n)+d(x_{n-1},x_{n+1})]},\right.\\&\qquad\left.\frac{d(x_n,x_n)\,d(x_n,x_{n-1})}{1+sd(x_n,x_{n+1})+s^3[d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)]}\right\}\\&=\max\left\{d(x_n,x_{n-1}),\frac{d(x_n,x_{n+1})\,d(x_{n-1},x_n)}{1+s[d(x_n,x_{n-1})+d(x_{n-1},x_{n+1})]},0\right\}\\&\le\max\left\{d(x_n,x_{n-1}),\frac{d(x_n,x_{n+1})\,d(x_{n-1},x_n)}{d(x_n,x_{n-1})}\right\}\\&=\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}.\end{aligned}$$

If $d(x_n,x_{n-1})\le d(x_n,x_{n+1})$, then by (2.15) it follows that $d(x_{n+1},x_n)\le\frac{1}{s^{1+\varepsilon}}d(x_n,x_{n+1})<d(x_n,x_{n+1})$. A contradiction. Hence, $M_{III}(x_n,x_{n-1})\le d(x_n,x_{n-1})$, that is, (2.7) holds. Therefore, by [12, Lemma 3.1] and [17, Lemma 2.3] , $\{x_n\}$ is a b-Cauchy sequence. Hence, from b-completeness of b-metric space (X,d), it follows that $x_n\to u\in X$.

Now, if (1) holds, then obviously $fu=u$, that is, u is a fixed point of f. In the case if (2) holds, we obtain that

$$\begin{aligned}\frac{1}{s}d(u,fu)&\le d(u,x_{n+1})+d(fx_n,fu)\\&\le d(u,x_{n+1})+\frac{1}{s^{1+\varepsilon}}M_{III}(x_n,u),\end{aligned}\tag{2.16}$$

where

$$M_{III}(x_n,u)=\max\left\{d(x_n,u),\frac{d(x_n,x_{n+1})d(u,fu)}{1+s[d(x_n,u)+d(x_n,fu)+d(u,x_{n+1})]},\right.$$
$$\left.\frac{d(x_n,fu)d(x_n,u)}{1+sd(x_n,x_{n+1})+s^3[d(u,x_{n+1})+d(u,fu)]}\right\}.$$

It is not hard to verify that

$$\begin{aligned}M_{III}(x_n,u)&\le\max\{d(x_n,u),d(x_n,x_{n+1})\,d(u,fu),s[d(x_n,u)+d(u,fu)]\,d(x_n,u)\}\\&\to\max\{0,0\cdot d(u,fu),s\cdot[0+d(u,fu)]\cdot 0\}=0,\quad\text{as } n\to\infty.\end{aligned}\tag{2.17}$$

From (2.16) and (2.17) it follows that $fu=u$, i.e., u is a fixed point of f. □

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X,d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that for every two comparable elements $x, y \in X$, it satisfies*

$$s^{\varepsilon} d\left(fx, fy\right) \leq \psi\left(M\left(x, y\right)\right), \tag{2.18}$$

where

$$M\left(x,y\right) = \max\left\{d\left(x,y\right), \frac{d\left(x,fx\right)d\left(y,fy\right)}{1+d\left(x,y\right)}, \frac{d\left(x,fx\right)d\left(y,fy\right)}{1+d\left(fx,fy\right)}\right\},$$

and $\psi : [0,\infty) \to [0,\infty)$ is a function with $\psi(t) < \frac{t}{s}$ for each $t > 0$ and $\psi(0) = 0$. If

(1) f is continuous, or,

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. First, for all $x, y \in X$, we have that $\psi\left(M\left(x,y\right)\right) \leq \frac{M(x,y)}{s}$. Now, from (2.18) it establishes that

$$d\left(fx, fy\right) \leq \frac{1}{s^{1+\varepsilon}} M\left(x,y\right),$$

for all comparable $x, y \in X$. Hence, Corollary 2.5 follows from Theorem 2.4. □

The following result is an ordered variant of the Geraghty theorem from [7, Theorem 3.8] for b-metric spaces.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X,d) is a b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that*

$$s^{\varepsilon} d\left(fx, fy\right) \leq \beta\left(d\left(x,y\right)\right) d\left(x,y\right) \tag{2.19}$$

for all comparable elements $x, y \in X$. If

(1) f is continuous, or,

(2) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Obviously (2.19) implies (2.1). □

Remark 2.7. Since the proofs of the main results of [21] are strongly dependent of the recent lemma of Aghajani et al. [3], it is too complex to deal with them. Theorem 2.4 covers all the results of [21] without utilizing the lemma mentioned above. Accordingly, our results are more meaningful and convenient in applications.

Example 2.8. Let $X = \{0, 1, 3\}$ and define the partial order on X by

$$\preceq := \left\{(0,0), (1,1), (3,3), (1,3)\right\}.$$

Consider the function $f : X \to X$ given by $f0 = 1, f1 = f3 = 3$, which is increasing with respect to $\preceq$. Take $x_0 = 0$, it ensures us that $fx_0 = 1$, so $x_0 \preceq fx_0$. Define first the b-metric d on X by $d(x,y) = (x-y)^2$, then

(X, d) is a b-complete b-metric space with $s = \frac{9}{5}$. Let $\beta \in \mathcal{F}_s$ be given by $\beta(t) = \frac{5}{9}e^{-\frac{t}{9}}$ for $t \in (0, \infty)$ and $\beta(0) \in [0, \frac{5}{9})$. Since

$$s^{\varepsilon} d(f0, f0) = s^{\varepsilon} d(f1, f1) = s^{\varepsilon} d(f3, f3) = s^{\varepsilon} d(f1, f3) = 0 \leq \beta(d(x, y)) \, d(x, y),$$

for all comparable $x, y \in X$, then we claim that f satisfies all the assumptions of Corollary 2.6, and thus it has (a unique) fixed point (which is $u = 3$).

In the following, we shall consider the periodic property for the rational Geraghty contractive mappings.

Let X be a nonempty set and denote $F(f) = \{x \in X : fx = x\}$ as the fixed point set of a mapping $f : X \to X$. It is clear that if f is a map which has a fixed point x, then x is also a fixed point of f^n for every $n \in \mathbb{N}$, but the converse does not hold. If $F(f) = F(f^n)$ for every $n \in \mathbb{N}$, then f is said to have property P. For more details, we refer the reader to [2], [7] and [11] and their references therein.

Definition 2.9 ([20]). Let $(X, \preceq)$ be a partially ordered set and let f be a self-map on X. The map f is said to be weakly isotone increasing if for all $x \in X$, one has $fx \preceq ffx$, i.e., if the sequence $\{f^n x\}_{n=0,1,2,\ldots}$ is nondecreasing for every $x \in X$.

Theorem 2.10. *Let X and f be same as in Theorem 2.4. Then f has the property P if f is a weakly isotone increasing.*

Proof. From Theorem 2.4, it follows that $F(f) \neq \emptyset$. Let $x \in F(f^n)$ for some $n > 1$. We shall show that $x = fx$. Assume that $x \neq fx$, i.e., $d(x, fx) > 0$. For given x we have that $f^{k-1}x \preceq f^k x$ for $k = 2, 3, 4, \ldots$. as f is a weakly isotone increasing. If $f^{n-1}x = f^n x$, then it establishes that $x = fx$. A contradiction. Therefore, let $f^{n-1}x \prec f^n x$. In order to end the proof, we firstly consider the case $M_I(x, y)$. Then by (2.8), we have

$$d(x, fx) = d\left(f^n x, f^{n+1} x\right) = d\left(ff^{n-1}x, ff^n x\right) \leq \frac{1}{s^{1+\varepsilon}} M_I\left(f^{n-1}x, f^n x\right), \tag{2.20}$$

where

$$\begin{aligned} M_I\left(f^{n-1}x, f^n x\right) &= \max\left\{d\left(f^{n-1}x, f^n x\right), \frac{d\left(f^{n-1}x, f^n x\right) d\left(f^n x, f^{n+1}x\right)}{1 + d\left(f^{n-1}x, f^n x\right)}, \frac{d\left(f^{n-1}x, f^n x\right) d\left(f^n x, f^{n+1}x\right)}{1 + d\left(f^n x, f^{n+1}x\right)}\right\} \\ &\leq \max\left\{d\left(f^{n-1}x, f^n x\right), \frac{d\left(f^{n-1}x, f^n x\right) d\left(f^n x, f^{n+1}x\right)}{d\left(f^{n-1}x, f^n x\right)}, \frac{d\left(f^{n-1}x, f^n x\right) d\left(f^n x, f^{n+1}x\right)}{d\left(f^n x, f^{n+1}x\right)}\right\} \\ &= \max\left\{d\left(f^{n-1}x, f^n x\right), d\left(f^n x, f^{n+1}x\right)\right\}. \end{aligned}$$

If $M_I\left(f^{n-1}x, f^n x\right) = d\left(f^n x, f^{n+1}x\right)$, then from (2.20), it follows that

$$d(x, fx) = d\left(f^n x, f^{n+1}x\right) \leq \frac{1}{s^{1+\varepsilon}} d\left(f^n x, f^{n+1}x\right).$$

A contradiction. Hence, we conclude that

$$d(x, fx) = d\left(f^n x, f^{n+1}x\right) \leq \frac{1}{s^{1+\varepsilon}} d\left(f^{n-1}x, f^n x\right).$$

Repeating the above process, we get

$$d(x, fx) = d\left(f^n x, f^{n+1}x\right) \leq \frac{1}{s^{1+\varepsilon}} d\left(f^{n-1}x, f^n x\right) \leq \ldots \leq \left(\frac{1}{s^{1+\varepsilon}}\right)^n d(x, fx) < d(x, fx).$$

This is again a contradiction. Thus, $x = fx$.

Similarly, we have the proof for the cases $M_{II}(x, y)$ and $M_{III}(x, y)$. $\square$

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