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Integrability and L^1-convergence of fuzzy trigonometric series with special fuzzy coefficients

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Abstract

In this paper, we generalize some classical results on the integrability of trigonometric series using the notion of integrability in fuzzy L^1-norm. Here, we introduce new classes of fuzzy coefficients and obtain the necessary and sufficient conditions for L^1-convergence of fuzzy trigonometric series. Also, an example is given for the existence of new classes of fuzzy coefficients.

Keywords: Fuzzy numbers, fuzzy trigonometric series, integrability, convergence in fuzzy L^1-norm,
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1. Introduction

A powerful tool in harmonic analysis is the concept of Fourier analysis. It is extremely useful in approximation theory, partial differential equations, probability theory etc. The study of integrability and L^1-convergence of Fourier series is based on the existence of sine and cosines. In the literature, so far available, various authors such as ([5], [6], [7], [15], [26], [27], [28]) have studied the L^1-convergence of trigonometric series under different classes of coefficient sequences.

Due to the rapid development of the fuzzy theory, the aim of this paper is to investigate new classes of sequences formed by the fuzzy real coefficients, which generalizes the classical results on Fourier Analysis. To accomplish this, we need to study the basic concepts of fuzzy theory given by various authors such as Zadeh [29], Kaleva [13], Puri and Ralescu [17], Goetchel and Voxman [8], Stojaković ([19], [20], [21]), Zhang and

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Guo [30], Ban [1], Puri and Ralescu [18], Stojaković ([8], [21]), Kim and Ghil [14], Stojaković and Stojaković ([22], [23]), Talo and Başar [25], Kadak and Başar ([9]–[12]) and so on. One of them set mapping operations to the case of interval valued fuzzy sets and then People move to the another direction to establish a new way of generalizing the interval valued fuzzy sets in terms of its level sets. The effectiveness of level sets come from not only their required memory capacity for fuzzy sets, but also their two valued nature. This nature contributes to an effective derivation of the fuzzy inference algorithm based on the families of the level sets. Beside this, the definition of fuzzy sets by level sets offers advantages over membership functions, when the fuzzy sets are in universe of discourse with many elements.

In this paper, we present a fuzzy notion of integrability and Lebesgue convergence of fuzzy trigonometric series. This paper is organized in six sections as follows: In section 2, we give some required definitions and consequences related with the fuzzy numbers, sequences and series of fuzzy numbers. In section 3, we define the integrals of fuzzy valued functions. In section 4, we introduce some new classes of sequences with special fuzzy coefficients. The section 5 and 6 accomplish the lemma and main results which are the generalization of some existing results.

2. Preliminaries, background and notation

In this section, we first recall some of the basic notions related to fuzzy numbers, sequence and series of fuzzy numbers.

Definition 2.1. A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:
(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.
(ii) u is fuzzy convex, i.e., $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.
(iii) u is upper semi-continuous.
(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{\{x \in \mathbb{R} : u(x) > 0}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it *as the space of fuzzy numbers* λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad 0 < \lambda \leq 1 \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad \lambda = 0. \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0, 1]$ which is defined by $[u]_\lambda = [u^-_\lambda, u^+_\lambda]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \begin{cases} 1 & , \quad x = r \ , \\ 0 & , \quad x \neq r. \end{cases}$$

Theorem 2.2. *[8] (Goetschel and voxman) For $u \in E^1$, denote $u^-(\lambda) = u^-_\lambda$ and $u^+(\lambda) = u^+_\lambda$. Then*
(i) $u^-(\lambda)$ is a bounded increasing function on $[0, 1]$.
(ii) $u^+(\lambda)$ is a bounded decreasing function on $[0, 1]$.
(iii) $u^-(1) \leq u^+(1)$.
(iv) $u^-(\lambda)$ and $u^+(\lambda)$ are left continuous on $(0, 1]$ and right continuous at 0.
(v) If $u^-(\lambda)$ and $u^+(\lambda)$ satisfy above $(i) - (iv)$, then there exist a unique $v \in E^1$ such that $v^-_\lambda = u^-(\lambda)$ and $v^+_\lambda = u^+(\lambda)$.

The above theorem implies that we can identify a fuzzy number u with the parameterized representation $\{(u^-_\lambda, u^+_\lambda) \mid 0 \leq \lambda \leq 1\}$. Suppose that $u, v \in E^1$ are fuzzy numbers represented by $\{(u^-_\lambda, u^+_\lambda) \mid 0 \leq \lambda \leq 1\}$ and $\{(v^-_\lambda, v^+_\lambda) \mid 0 \leq \lambda \leq 1\}$, respectively. If we define

$$(u \oplus v)(z) = \sup_{x+y=z} \min(u(x), v(y)), \tag{2.1}$$

$$(\alpha u)(z) = \begin{cases} u(z/\alpha), & \alpha \neq 0, \\ \tilde{0} & \alpha = 0, \quad where \quad \tilde{0} = \chi_{\{0\}}, \end{cases} \tag{2.2}$$

then

$$u \oplus v = \{(u_\lambda^- + v_\lambda^-, u_\lambda^+ + v_\lambda^+) | 0 \leq \lambda \leq 1\}$$

$$u \ominus v = \{\min(u_\lambda^- - v_\lambda^-, u_\lambda^+ - v_\lambda^+), \max(u_\lambda^- - v_\lambda^-, u_\lambda^+ - v_\lambda^+) \ \mid \ 0 \leq \lambda \leq 1\}$$

$$\alpha u = \begin{cases} \{(\alpha u_\lambda^-, \alpha u_\lambda^+) \ \mid \ 0 \leq \lambda \leq 1\}, & \alpha \geq 0, \\ \{(\alpha u_\lambda^+, \alpha u_\lambda^-) \ \mid \ 0 \leq \lambda \leq 1\}, & \alpha < 0. \end{cases}$$

We define a metric d on E^1 by

$$d(u, v) = \sup_{0 \leq \lambda \leq 1} d_H([u]_\lambda, [v]_\lambda) \tag{2.3}$$

where d_H is the hausdorff metric defined as

$$d_H([u]_\lambda, [v]_\lambda) = \max(|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|). \tag{2.4}$$

Also, $d(u, \tilde{0})$ will be denoted by $||u||$.

Definition 2.3. [24] The following basic statements hold:
(i) A sequence $\{u_k\}$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set E^1. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the general term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers.
(ii) A sequence $\{u_n\} \in w(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\epsilon > 0$ there exists an $n_0 = n(\epsilon) \in \mathbb{N}$ such that $D(u_n, u) < \epsilon$ for all $n \geq n_0$.
(iii) A sequence $\{u_n\} \in w(F)$ is called bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence $\{u_n\}$ is a bounded set. That is to say that a sequence $\{u_n\} \in w(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \preceq u_n \preceq M$ for all $n \in \mathbb{N}$. This means that $m^-(\lambda) \leq u_n^-(\lambda) \leq M^-(\lambda)$ and $m^+(\lambda) \leq u_n^+(\lambda) \leq M^+(\lambda)$ for all $\lambda \in [0, 1]$.

Remark 2.4. [24] According to Definition 2.3 the following remarks can be given;
(a) Obviously the sequence $\{u_n\} \in w(F)$ converges to a fuzzy number u if and only if $\{u_n^-(\lambda)\}$ and $\{u_n^+(\lambda)\}$ converge uniformly to $u^-(\lambda)$ and $u^+(\lambda)$ on $[0, 1]$, respectively.
(b) The boundedness of the sequence $\{u_n\} \in w(F)$ is equivalent to the fact that

$$\sup_{n \in N} D(u_n, \bar{0}) = \sup_{n \in N} \sup_{\lambda \in [0,1]} \max\{|u_n^-(\lambda)|, |u_n^+(\lambda)|\} < \infty.$$

If the sequence $\{u_k\} \in w(F)$ is bounded then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ uniformly bounded in $[0, 1]$.

Definition 2.5. [24] Let $\{u_k\} \in w(F)$. Then the expression $\sum_{\oplus} {}_k u_k$ is called series of fuzzy numbers with the level summation $\sum_{\oplus}$. Define a sequence $\{s_n\}$ via nth partial level sum of the series by

$$s_n = u_0 \oplus u_1 \oplus u_2 \oplus \cdots \oplus u_n,$$

for all $n \in \mathbb{N}$. If the sequence $\{s_n\}$ converges to a fuzzy number u then we say that the series $\sum_{\oplus} {}_k u_k$ of fuzzy numbers converges to u and write $\sum_{\oplus} {}_k u_k = u$ which implies that

$$\lim_{n \to \infty} \sum_{k=0}^{n} u_k^-(\lambda) = u^-(\lambda) \quad and \quad \lim_{n \to \infty} \sum_{k=0}^{n} u_k^+(\lambda) = u^+(\lambda),$$

where the summation is in the sense of classical summation and converges uniformly in $\lambda \in [0,1]$. Conversely if

$$\sum_k u_k^-(\lambda) = u^-(\lambda) \quad and \quad \sum_k u_k^+(\lambda) = u^+(\lambda),$$

converges uniformly in λ, then $u = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [0,1]\}$ defines a fuzzy number such that $u = \sum_{\oplus\ k} u_k$.

Theorem 2.6. *[2] The following statements for level addition $\oplus$ of fuzzy numbers and classical addition + of real scalars are valid:*
(i) $\overline{0}$ is neutral element with respect to $\oplus$, i.e. $u \oplus \overline{0} = \overline{0} \oplus u = u$ for all $u \in E^1$.
(ii) With respect to $\overline{0}$, none of $u \neq \bar{r}$, $r \in \mathbb{R}$ has opposite in E^1.
(iii) For any $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$ or $\alpha, \beta \leq 0$ and any $u \in E^1$, we have $(\alpha + \beta)u = \alpha u \oplus \beta u$.
(iv) For any $\alpha \in \mathbb{R}$ and any $u, v \in E^1$, we have $\alpha(u \oplus v) = \alpha u \oplus \alpha v$.
(v) For any $\alpha, \beta \in \mathbb{R}$ and any $u \in E^1$, we have $\alpha(\beta u) = (\alpha\beta)u$.

3. Integrals of fuzzy number valued functions

Let (Ω, Σ, μ) denote a complete σ-finite measure space. If $F : \Omega \to E^1$ is a fuzzy-number-valued function and B is a subset of $\mathbb{R}$, then $F^{-1}(B)$ denotes the fuzzy subset of Ω defined by $F^{-1}(B)(\omega) = \sup_{x \in B} F(\omega)(x)$ for every $\omega \in \Omega$. The fuzzy-number-valued function $F : \Omega \to E^1$ is called measurable if for every closed subset B of $\mathbb{R}$ the fuzzy set $F^{-1}(B)$ is measurable when considered as a function from Ω to $[0,1]$. This concept of measurability for fuzzy-set-valued functions was introduced by Butnariu [3] as a natural generalization of measurable multifunctions. Kaleva [13] defined $F : \Omega \to E^1$ to be strongly measurable if for each $\lambda \in [0,1]$ the set-valued function $F_\lambda : \Omega \to I(\mathbb{R})$ defined by $F_\lambda(\omega) = [F(\omega)]_\lambda$ is measurable, where $I(\mathbb{R})$ is the set of all closed bounded intervals on $\mathbb{R}$ endowed with the topology generated by the Hausdorff metric d_H defined as in (2.4).

Theorem 3.1. *([3], [4]) For $F : \Omega \to E^1$, $F(\omega) = \{(F_\lambda^-(\omega), F_\lambda^+(\omega)) \mid 0 \leq \lambda \leq 1\}$, the following conditions are equivalent.*
(i) F is measurable.
(ii) F is strongly measurable.
(iii) For each $\lambda \in [0,1]$, F_λ^- and F_λ^+ are measurable.

Let $F : \Omega \to E^1$, $F(\omega) = \{(F_\lambda^-(\omega), F_\lambda^+(\omega)) \mid 0 \leq \lambda \leq 1\}$, be measurable. If for each $\lambda \in [0,1]$, F_λ^- and F_λ^+ are integrable, it follows from Theorem 2.2 and the *Lebesgue-dominated convergence theorem* that the parameterized representation $\{\left(\int_A F_\lambda^- d\mu, \int_A F_\lambda^+ d\mu\right) \mid 0 \leq \lambda \leq 1\}$ is a fuzzy number for each $A \in \Sigma$. This enables us to define the integral of F without using the integral of set-valued function.

Definition 3.2. [14] A measurable function $F : \Omega \to E^1$, $F(\omega) = \{(F_\lambda^-(\omega), F_\lambda^+(\omega)) \mid 0 \leq \lambda \leq 1\}$ is called integrable if for each $\lambda \in [0,1]$, F_λ^- and F_λ^+ are integrable, or equivalently, if F_0^- and F_0^+ are integrable. In this case, the integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu = \left\{ \left(\int_A F_\lambda^- d\mu, \int_A F_\lambda^+ d\mu \right) \;\middle|\; 0 \leq \lambda \leq 1 \right\}. \tag{3.1}$$

Theorem 3.3. *[13] Let $F, G : \Omega \to E^1$ be integrable. Then*
(i) if a and b are real numbers, then $aF \oplus bG$ is integrable and for each $A \in \Sigma$, $\int_A (aF \oplus bG) d\mu = a \int_A F d\mu \oplus b \int_A G d\mu$.
(ii) if d is the metric on E^1 which defined as in (2.3), then $d(F, G)$ is a real-valued integrable function and for each $A \in \Sigma$,

$$d\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A d(F, G) d\mu. \tag{3.2}$$

Let $L(\Omega, \Sigma, \mu) = L$ denote the Banach space of (equivalence classes of) integrable function $f : \Omega \to R$ with the norm

$$||f|| = \int_\Omega |f(\omega)|d\mu = \left\{ \left(\int_\Omega |f_\lambda^-|d\mu, |f_\lambda^+|d\mu \right) \quad | \quad 0 \le \lambda \le 1 \right\}.$$

Remark 3.4. (i) F is integrable if and only if the real-valued function $\omega \longmapsto ||F(\omega)||$ is integrable.
(ii) $|| \int_A F d\mu|| \le \int_A ||F||d\mu$.

For the case of fuzzy-number-valued functions, we denote $L(E^1)$ the space of all integrable functions $F : \Omega \to E^1$, where two functions $F, G \in L(E^1)$ are considered to be identical if $F(\omega) = G(\omega)$ *a.e.*
For $F, G \in L(E^1)$, we define

$$\delta(F, G) = \int_\Omega d(F, G)d\mu. \tag{3.3}$$

It is easy to show that δ is metric on $L(E^1)$.
$L(E^1)$ is complete with respect to the metric.

$$D(F, G) = \sup_{0\le\lambda\le1} \int_\Omega \max(|F_\lambda^- - G_\lambda^-|, |F_\lambda^+ - G_\lambda^+|)d\mu. \tag{3.4}$$

4. Fuzzy trigonometric cosine and sine series for fuzzy valued periodic functions

In this section, we present the notion of periodic fuzzy valued functions and their harmonics with respect to the level sets.

Definition 4.1. [12] (Periodicity) A fuzzy-valued function f^t is called periodic if there exists a constant $P > 0$ for which $f^t(x + P) = f^t(x)$ for any $x, t \in [a, b]$. Thus it can easily seen that the conditions $f_\lambda^-(t + P) = f_\lambda^-(t)$ and $f_\lambda^+(t + P) = f_\lambda^+(t)$ hold for all $t \in [a, b]$ and $\lambda \in [0, 1]$. Such a constant $P > 0$ is called a period of the function f^t.

Let

$$\frac{1}{2}a_0 \oplus \sum_{\oplus\ k=1}^{\infty} a_k \cos kx, \tag{4.1}$$

and

$$\sum_{\oplus\ k=1}^{\infty} b_k \sin kx, \tag{4.2}$$

be fuzzy trigonometric cosine series and sine series as defined in [12] and $s_n(x)$ and $\tilde{s}_n^t(x)$ be the nth partial level sum of the series (4.1) and (4.2) given by $s_n(x) = \frac{a_0}{2} \oplus \sum_{\oplus\ k=1}^{n} a_k \cos kx$ and $\tilde{s}_n^t(x) = \sum_{\oplus\ k=1}^{n} b_k \sin kx$, where $\{a_k\} = \left\{ \left((a_k)_\lambda^-, (a_k)_\lambda^+\right) \quad | \quad 0 \le \lambda \le 1 \right\}$ and $\{b_k\} = \left\{ \left((b_k)_\lambda^-, (b_k)_\lambda^+\right) \quad | \quad 0 \le \lambda \le 1 \right\}$ for $k = 1, 2, ...n$ are the fuzzy coefficients.

Let $\{a_k\} = \left\{ \left((a_k)_\lambda^-, (a_k)_\lambda^+\right) \quad | \quad 0 \le \lambda \le 1 \right\}$ be a fuzzy sequence. Then we write

$$\begin{aligned} \Delta a_k &= a_k \ominus a_{k+1} \\ &= \left(\min\{(a_k)_\lambda^- - (a_{k+1})_\lambda^-, (a_k)_\lambda^+ - (a_{k+1})_\lambda^+\}, \max\{(a_k)_\lambda^- - (a_{k+1})_\lambda^-, (a_k)_\lambda^+ - (a_{k+1})_\lambda^+\}\right) \\ &= \left((\Delta a_k)_\lambda^-, (\Delta a_k)_\lambda^+\right) \\ \Delta^2 a_k &= \Delta(\Delta a_k) = \left(\Delta(\Delta a_k)_\lambda^-, \Delta(\Delta a_k)_\lambda^+\right). \end{aligned}$$

Definition 4.2. A sequence $\{a_k\} \in w(F)$ is said to be decreasing sequence if $a_{k+1} \prec a_k$ *i.e.* $(a_k)_\lambda^- < (a_{k+1})_\lambda^-$ and $(a_k)_\lambda^+ > (a_{k+1})_\lambda^+$.

Remark 4.3. If $\{a_k\}$ is a decreasing fuzzy sequence, then it can be easily seen that $(\Delta a_k)^-_\lambda = (a_k)^-_\lambda - (a_{k+1})^-_\lambda = \Delta(a_k)^-_\lambda$ and $(\Delta a_k)^+_\lambda = (a_k)^+_\lambda - (a_{k+1})^+_\lambda = \Delta(a_k)^+_\lambda$.

Now, we introduce some new classes of fuzzy coefficients as follows:

Definition 4.4. A sequence $\{a_k\} \in w(F)$ is said to be fuzzy null sequence if $\lim\limits_{k\to\infty} a_k = [0]_\lambda$ with respect to the level sets. *i.e.*

$$\left\{\left(\lim_{k\to\infty}(a_k)^-_\lambda = 0^-_\lambda, \lim_{k\to\infty}(a_k)^+_\lambda = 0^+_\lambda\right) \;\middle|\; 0 \le \lambda \le 1\right\}. \tag{4.3}$$

Definition 4.5. A decreasing sequence $\{a_k\} \in w(F)$ is said to belong to class $\mathbf{BV(F)}$ with respect to the level set if (4.3) is satisfied and the series $\sum\limits_{\oplus\; k=0}^{\infty} |\Delta a_k|$ is convergent.

Definition 4.6. A decreasing sequence $\{a_k\} \in w(F)$ is said to belong to class $\mathbf{K_p(F)}$ (for $\lambda \in [0,1]$) if (4.3) is satisfied and the series

$$\sum_{\oplus\; m=1}^{\infty} 2^{m/q}\left(\sum_{\oplus\; k\in I_m} |\Delta a_k|^p\right)^{1/p}, \qquad for \quad some \quad p > 1 \tag{4.4}$$

is convergent. (4.4) can be written as

$$\left\{\left(\sum_{m=1}^{\infty} 2^{m/q}\left(\sum_{k\in I_m}\left|(\Delta a_k)^-_\lambda\right|^p\right)^{1/p} < \infty, \sum_{m=1}^{\infty} 2^{m/q}\left(\sum_{k\in I_m}\left|(\Delta a_k)^+_\lambda\right|^p\right)^{1/p} < \infty\right) \;\middle|\; 0 \le \lambda \le 1\right\}. \tag{4.5}$$

Here $p > 1$ be any real number and q be the conjugate exponent to p *i.e.* $\frac{1}{p} + \frac{1}{q} = 1$ and I_m be the dyadic interval $[2^{m-1}, 2^m)$ and $m \ge 1$.

By Cauchy condensation test, it can be easily seen that (4.5) is equivalent to

$$\left\{\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{2n-1}\left|(\Delta a_k)^-_\lambda\right|^p\right)^{1/p} < \infty, \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{2n-1}\left|(\Delta a_k)^+_\lambda\right|^p\right)^{1/p} < \infty\right) \;\middle|\; 0 \le \lambda \le 1\right\}. \tag{4.6}$$

Further, by applying Holder's inequality to (4.6), we get

$$\mathbf{K_1(F)} = \left\{\left(\sum_{k=1}^{\infty}\left|(\Delta a_k)^-_\lambda\right|, \sum_{k=1}^{\infty}\left|(\Delta a_k)^+_\lambda\right|\right) \;\middle|\; 0 \le \lambda \le 1\right\} \preceq \mathbf{K_p(F)}$$

$$\implies \quad \sum_{\oplus\; k=1}^{\infty} |\Delta a_k| = \mathbf{K_1(F)} \preceq \mathbf{K_p(F)}, \qquad p > 1. \tag{4.7}$$

Hence, the class $\mathbf{K_p(F)}$ contains the class of bounded variation.

Now, we present an example for the existence of above classes as follows:

Example 4.7. Let $\{a_k\} = \{(a_k)^-_\lambda, (a_k)^+_\lambda\} = \left\{\left(\frac{1}{k+1} + \lambda\left(\frac{1}{k} - \frac{1}{k+1}\right), \frac{1}{k-1} - \lambda\left(\frac{1}{k-1} - \frac{1}{k}\right)\right) \;\middle|\; 0 \le \lambda \le 1\right\}$, be fuzzy sequence. Since $(a_k)^-_\lambda$ and $(a_k)^+_\lambda$ tends to zero as $k \to \infty$ therefore it is fuzzy null sequence. As (4.5) is equivalent to (4.6), it is sufficient to show that the series $\sum\limits_{\oplus\; n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{\oplus\; k=n}^{2n-1}|\Delta a_k|^p\right)^{1/p}$ is convergent.

First, we check the behavior of the series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{k=n}^{2n-1}\left|(\Delta a_k)^-_\lambda\right|^p\right)^{1/p}$ for $0 \le \lambda \le 1$.

we have

$$\begin{aligned}(\Delta a_k)^-_\lambda &= \min\{(a_k)^-_\lambda - (a_{k+1})^-_\lambda, (a_k)^+_\lambda - (a_{k+1})^+_\lambda\}\\ &= \frac{1}{(k+1)(k+2)} + \frac{2\lambda}{k(k+1)(k+2)}, \qquad 0 \le \lambda \le 1.\end{aligned}$$

Consider $J_1 = \sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{k=n}^{2n-1}\left|(\Delta a_k)^-_\lambda\right|^p\right)^{1/p}$.

By Minkowski Inequality,

$$\begin{aligned}J_1 &\le \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\left[\left\{\sum_{k=n}^{2n-1}\left|\frac{1}{(k+1)(k+2)}\right|^p\right\}^{1/p} + \left\{\sum_{k=n}^{2n-1}\left|\frac{2\lambda}{(k-1)k(k+1)}\right|^p\right\}^{1/p}\right]\\ &\le \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\left[\begin{array}{c}\left\{\frac{1}{(n+1)^p(n+2)^p} + \frac{1}{(n+2)^p(n+3)^p} + ...\right\}^{1/p} + \\ 2\lambda\left\{\frac{1}{(n-1)^p n^p(n+1)^p} + \frac{1}{n^p(n+1)^p(n+2)^p} + ...\right\}^{1/p}\end{array}\right]\\ &\le \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\left[\left\{\frac{n}{(n+1)^p(n+2)^p}\right\}^{1/p} + 2\lambda\left\{\frac{n}{n^p(n+1)^p(n+2)^p}\right\}^{1/p}\right]\\ &\le \sum_{n=1}^{\infty}\left[\frac{1}{(n+1)(n+2)} + \frac{2\lambda}{(n-1)n(n+1)}\right] = \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} + \sum_{n=1}^{\infty}\frac{2\lambda}{(n-1)n(n+1)} < \infty\end{aligned}$$

similarly, $J_2 = \sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{k=n}^{2n-1}\left|(\Delta a_k)^-_\lambda\right|^p\right)^{1/p} < \infty$. Therefore, the series $\sum\limits_{\oplus\, n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{\oplus\, k=n}^{2n-1}|\Delta a_k|^p\right)^{1/p}$ is convergent.

Hence, the fuzzy sequence $\{a_k\}$ belongs to class $\mathbf{K_p(F)}$.

5. Lemma

Lemma 5.1. *Let $\{a_k\}$ be a sequence of fuzzy numbers. Then for any $1 < p \le 2$ and $n \ge 1$,*

$$\frac{1}{n}\int_0^\pi\left|\sum_{\oplus\, k=n}^{2n-1} a_k D_k(x)\right| dx \preceq M_p\left(\frac{1}{n}\sum_{\oplus\, k=n}^{2n-1}|a_k|^p\right)^{\frac{1}{p}} \tag{5.1}$$

where M_p is an absolute constant.

Proof. We write

$$\begin{aligned}\frac{1}{n}\int_0^\pi\left|\sum_{\oplus\, k=n}^{2n-1} a_k D_k(x)\right| dx &= \left\{\left(\frac{1}{n}\int_0^\pi\left|\sum_{k=n}^{2n-1}(a_k)^-_\lambda D_k(x)\right| dx, \frac{1}{n}\int_0^\pi\left|\sum_{k=n}^{2n-1}(a_k)^+_\lambda D_k(x)\right| dx\right) \;\middle|\; 0 \le \lambda \le 1\right\}\\ &= (J_1, J_2).\end{aligned}$$

Consider

$$J_1 = \frac{1}{n}\int_0^\pi\left|\sum_{k=n}^{2n-1}(a_k)^-_\lambda D_k(x)\right| dx$$

$$= \frac{1}{n}\int_0^{\pi/n}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- D_k(x)\right|dx + \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- D_k(x)\right|dx$$

$$= J_{11} + J_{12}.$$

Now,

$$J_{11} = \frac{1}{n}\int_0^{\pi/n}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- D_k(x)\right|dx \le \frac{1}{n}\int_0^{\pi/n}\sum_{k=n}^{2n-1}|(a_k)_\lambda^-||D_k(x)|dx.$$

Here, $D_k(x)$ is the Dirichlet kernel and $D_k(x) \le k+1$ for all $x \in (0,\pi)$. Thus, we get

$$\begin{aligned} J_{11} &\le \frac{1}{n}\int_0^{\pi/n}\sum_{k=n}^{2n-1}|(a_k)_\lambda^-|(k+1)dx \le \frac{1}{n}\sum_{k=n}^{2n-1}|(a_k)_\lambda^-|(k+1)\int_0^{\pi/n}dx \\ &\le \frac{\pi}{n^2}(2n-1+1)\sum_{k=n}^{2n-1}|(a_k)_\lambda^-| = \frac{2\pi}{n}\sum_{k=n} 2n-1|(a_k)_\lambda^-|. \end{aligned}$$

By using Holder's inequality, we have

$$J_{11} \le 2\pi\left(\frac{1}{n}\sum_{k=n}^{2n-1}|(a_k)_\lambda^-|^p\right)^{1/p}. \tag{5.2}$$

Next, we consider

$$J_{12} = \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- D_k(x)\right|dx = \frac{1}{n}\int_{\pi/n}^{\pi}\frac{1}{2\sin(\frac{x}{2})}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- \sin\left(k+\frac{1}{2}\right)x\right|dx.$$

Again by applying Holder's inequality for $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, we get

$$J_{12} \le \frac{1}{2n}\left(\int_{\pi/n}^{\pi}\frac{dx}{\left(\sin\left(\frac{x}{2}\right)\right)^p}\right)^{1/p}\left(\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q}.$$

Further, It can be easily noted that for $x \in \left(\frac{\pi}{n},\pi\right)$, $\displaystyle\int_{\pi/n}^{\pi}\frac{dx}{(\sin(x/2))^p} \le \pi^p\int_{\pi/n}^{\pi}\frac{dx}{x^p} \le \frac{\pi}{p-1}n^{p-1}.$

Therefore,

$$J_{12} \le \frac{1}{2n}\left(\frac{\pi}{p-1}\right)^{1/p}(n^{p-1})^{1/p}\left(\int_0^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q}.$$

Now, by the use of Hausdorff Young's inequality (for $1 < p \le 2$), we get

$$\begin{aligned} &\left(\frac{1}{\pi}\int_0^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- \sin\left(k+\frac{1}{2}\right)x\right|^q dx\right)^{1/q} \\ &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=n}^{2n-1}(a_k)_\lambda^- e^{i(k+\frac{1}{2})x}\right|^q dx\right)^{1/q} \end{aligned}$$

$$\leq \left(\sum_{k=n}^{2n-1} |(a_k)_\lambda^-|^p \right)^{1/p}.$$

Thus, we get

$$J_{12} \leq \frac{\pi}{2}(p-1)^{-1/p} n^{-1/p} \left(\sum_{k=n}^{2n-1} |(a_k)_\lambda^-|^p \right)^{1/p}. \tag{5.3}$$

Combining (5.2) and (5.3), we have

$$J_1 \leq \pi \left(2 + \frac{1}{2}(p-1)^{-1/p} \right) \left(\frac{1}{n} \sum_{k=n}^{2n-1} |(a_k)_\lambda^-|^p \right)^{1/p}.$$

Similarly, $J_2 \leq \pi \left(2 + \frac{1}{2}(p-1)^{-1/p}\right) \left(\frac{1}{n} \sum_{k=n}^{2n-1} |(a_k)_\lambda^+|^p \right)^{1/p}$ Hence,

$$\begin{aligned}(J_1, J_2) &\preceq \left\{ \left(\begin{array}{c} \pi \left(2 + \frac{1}{2}(p-1)^{-1/p} \right) \left(\frac{1}{n} \sum_{k=n}^{2n-1} |(a_k)_\lambda^-|^p \right)^{1/p}, \\ \pi \left(2 + \frac{1}{2}(p-1)^{-1/p} \right) \left(\frac{1}{n} \sum_{k=n}^{2n-1} |(a_k)_\lambda^+|^p \right)^{1/p} \end{array} \right) \;\middle|\; 0 \leq \lambda \leq 1 \right\} \\ &= \pi \left(2 + \frac{1}{2}(p-1)^{-1/p} \right) \left(\frac{1}{n} \sum_{\oplus\, k=n}^{2n-1} |a_k|^p \right)^{1/p}.\end{aligned}$$

Thus the conclusion of lemma holds. □

6. Main Results

In this section, we show that more general results hold in fuzzy Fourier analysis.

The first main result of this section read as follows:

Theorem 6.1. *If $\{a_k\}$ is a sequence of fuzzy coefficients belonging to class $\mathbf{K_p}(\mathbf{F})$, then*
(i) $f^t \in L^1(0, \pi)$.
(ii) series (4.1) is the Fourier series of f^t.
(iii) the series (4.1) converges to fuzzy valued function f^t in $L^1(0, \pi)$-norm if and only if
$\left\{ \lim_{n\to\infty} a_n \ln n = [0]_\lambda \;\middle|\; 0 \leq \lambda \leq 1 \right\}$.

Proof. (i) Let

$$\begin{aligned} s_n^t(x) &= \frac{1}{2}a_0 \oplus \sum_{\oplus\, k=1}^{n} a_k \cos kx \\ &= \left\{ \left(\frac{1}{2}(a_0)_\lambda^- + \sum_{k=1}^{n} (a_k)_\lambda^- \cos kx, \frac{1}{2}(a_0)_\lambda^+ + \sum_{k=1}^{n} (a_k)_\lambda^+ \cos kx \right) \;\middle|\; 0 \leq \lambda \leq 1 \right\}. \end{aligned}$$

Using summation by parts, we get

$$\begin{aligned} & s_n^t(x) \\ &= \left\{ \left(\sum_{k=0}^{n-1} D_k(x) \Delta(a_k)_\lambda^- + (a_n)_\lambda^- D_n(x), \sum_{k=0}^{n-1} D_k(x) \Delta(a_k)_\lambda^+ + (a_n)_\lambda^+ D_n(x) \right) \;\middle|\; 0 \leq \lambda \leq 1 \right\}. \end{aligned}$$

As $\{a_k\}$ belongs to class $\mathbf{K_p(F)}$ so $\{a_k\}$ is a decreasing sequence, then we have

$$s_n^t(x) = \left\{ \left(\sum_{k=0}^{n-1} D_k(x)(\Delta a_k)_\lambda^- + (a_n)_\lambda^- D_n(x), \sum_{k=0}^{n-1} D_k(x)(\Delta a_k)_\lambda^+ + (a_n)_\lambda^+ D_n(x) \right) \;\middle|\; 0 \le \lambda \le 1 \right\}. \tag{6.1}$$

By (4.3) and (4.7), the series $\left\{ \left(\sum_{k=0}^{\infty} D_k(x)(\Delta a_k)_\lambda^-, \sum_{k=0}^{\infty} D_k(x)(\Delta a_k)_\lambda^+ \right) \;\middle|\; 0 \le \lambda \le 1 \right\}$ converges absolutely and

$$\left\{ \left(\lim_{n\to\infty} (a_n)_\lambda^- D_n(x), \lim_{n\to\infty} (a_n)_\lambda^+ D_n(x) \right) \;\middle|\; 0 \le \lambda \le 1 \right\} = \left\{ \left(0_\lambda^-, 0_\lambda^+\right) \;\middle|\; 0 \le \lambda \le 1 \right\}$$

at every point $x \ne 0$ (mod2π). Thus the series (4.1) converges except possibly at $x = 0$ (mod2π), and

$$\begin{aligned} &\left\{ \left(\frac{1}{2}(a_0)_\lambda^- + \sum_{k=1}^{\infty} (a_k)_\lambda^- \cos kx, \frac{1}{2}(a_0)_\lambda^+ + \sum_{k=1}^{\infty} (a_k)_\lambda^+ \cos kx \right) \;\middle|\; 0 \le \lambda \le 1 \right\} \\ =& \left\{ \begin{pmatrix} \sum_{k=0}^{\infty} D_k(x)(\Delta a_k)_\lambda^-, \\ \sum_{k=0}^{\infty} D_k(x)(\Delta a_k)_\lambda^+ \end{pmatrix} \;\middle|\; 0 \le \lambda \le 1 \right\} &(6.2) \\ =& \left\{ \left(f_\lambda^-(x), f_\lambda^+(x) \right) \;\middle|\; 0 \le \lambda \le 1 \right\} \\ =& f^t. &(6.3) \end{aligned}$$

Therefore f^t exists in $(0, \pi)$. We group the terms in (6.2) as follows

$$\begin{aligned} &\left\{ \left(f_\lambda^-(x), f_\lambda^+(x) \right) \;\middle|\; 0 \le \lambda \le 1 \right\} \\ =& \left\{ \begin{pmatrix} D_0(x)(\Delta a_0)_\lambda^- + \sum_{m=1}^{\infty} \sum_{k \in I_m} D_k(x)(\Delta a_k)_\lambda^-, \\ D_0(x)(\Delta a_0)_\lambda^+ + \sum_{m=1}^{\infty} \sum_{k \in I_m} D_k(x)(\Delta a_k)_\lambda^+ \end{pmatrix} \;\middle|\; 0 \le \lambda \le 1 \right\} \\ f^t(x) =& D_0(x)\Delta a_0 \oplus \sum_{\oplus\ m=1}^{\infty} \sum_{\oplus\ k \in I_m} D_k(x)\Delta a_k. \end{aligned}$$

By using Lemma 5.1, we get for $1 \le p \le 2$

$$\int_0^\pi |f^t(x)| dx \preceq \frac{\pi}{2} |\Delta a_0| \oplus M_p \sum_{\oplus\ m=1}^{\infty} 2^{m/q} \left(\sum_{\oplus\ k \in I_m} |\Delta a_k|^p \right)^{1/p}. \tag{6.4}$$

The conclusion of (i) holds, by using given hypothesis.
(ii) Now, for proving this part, we consider $f^t(x) = \lim_{n\to\infty} s_n^t(x)$. [by (i) as $f^t \in L^1(0, \pi)$]

Let $l \ge 0$ be fixed integer, therefore

$$\begin{aligned} &\frac{2}{\pi} \int_0^\pi f^t(x) \cos lx dx \\ =& \lim_{n\to\infty} \frac{2}{\pi} \int_0^\pi s_n^t(x) \cos lx dx \end{aligned}$$

$$= \lim_{n\to\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left\{\frac{a_0}{2}\oplus\sum_{\oplus\ k=1}^{n} a_k\cos kx\right\}\cos lxdx\right]$$

$$= \left\{\left(\begin{array}{l}\lim\limits_{n\to\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left\{\frac{(a_0)_\lambda^-}{2}+\sum_{k=1}^{n}(a_k)_\lambda^-\cos kx\right\}\cos lxdx\right],\\ \lim\limits_{n\to\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left\{\frac{(a_0)_\lambda^+}{2}+\sum_{k=1}^{n}(a_k)_\lambda^+\cos kx\right\}\cos lxdx\right]\end{array}\right)\ \middle|\ 0\le\lambda\le 1\right\}$$

$$= \left\{\left(\lim_{n\to\infty}\left[\frac{2}{\pi}\times\frac{\pi}{2}(a_l)_\lambda^-\right],\lim_{n\to\infty}\left[\frac{2}{\pi}\times\frac{\pi}{2}(a_l)_\lambda^+\right]\right)\ \middle|\ 0\le\lambda\le 1\right\}$$

$$= a_l.$$

This proves that (4.1) is the Fourier series of f^t.

(iii) Next, we prove that $\lim\limits_{n\to\infty}\int_0^{\pi}|s_n^t(x)\ominus f^t(x)|=0$ if and only if $\left\{\lim\limits_{n\to\infty}a_n\ln n=[0]_\lambda\ \middle|\ 0\le\lambda\le 1\right\}$.

Consider,

$$\lim_{n\to\infty}\int_0^{\pi}\left|s_n^t(x)\ominus f^t(x)\right|dx$$

$$= \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\left(\int_0^{\pi}\left|(s_n)_\lambda^-(x)-f_\lambda^-\right|dx,\int_0^{\pi}\left|(s_n)_\lambda^+(x)-f_\lambda^+(x)\right|dx\right)$$

$$= \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\left(\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}(a_k)_\lambda^-\cos kx\right|dx,\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}(a_k)_\lambda^+\cos kx\right|dx\right)$$

$$= (J_1,J_2),$$

where

$$J_1$$

$$= \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}(a_k)_\lambda^-\cos kx\right|dx=\lim_{n\to\infty}\sup_{\lambda\in[0,1]}\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta(a_k)_\lambda^-D_k(x)-(a_{n+1})_\lambda^-D_n(x)\right|dx$$

$$= \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\int_0^{\pi}\left|\sum_{k=n+1}^{\infty}(\Delta a_k)_\lambda^-D_k(x)-(a_{n+1})_\lambda^-D_n(x)\right|dx$$

$$\le \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\int_0^{\pi}\left[\sum_{k=n+1}^{\infty}\left|(\Delta a_k)_\lambda^-D_k(x)\right|+\left|(a_{n+1})_\lambda^-D_n(x)\right|\right]dx.$$

Similarly,

$$J_2\le\lim_{n\to\infty}\sup_{\lambda\in[0,1]}\int_0^{\pi}\left[\sum_{k=n+1}^{\infty}\left|(\Delta a_k)_\lambda^+D_k(x)\right|+\left|(a_{n+1})_\lambda^+D_n(x)\right|\right]dx$$

$$\lim_{n\to\infty}\int_0^\pi \left|s_n^t(x)\ominus f^t(x)\right|dx \preceq \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\begin{pmatrix}\int_0^\pi\sum_{k=n+1}^\infty \left|(\Delta a_k)_\lambda^- D_k(x)\right| + \int_0^\pi \left|(a_{n+1})_\lambda^- D_n(x)\right|dx,\\ \int_0^\pi\sum_{k=n+1}^\infty \left|(\Delta a_k)_\lambda^+ D_k(x)\right| + \int_0^\pi \left|(a_{n+1})_\lambda^+ D_n(x)\right|dx\end{pmatrix}.$$

Since (see e.g. [31], Vol I, p. 67) $\int_0^\pi |D_n(x)|dx \asymp \ln n$ (iii) follows. Here the symbol $a_n \asymp b_n$ means that there are positive constants K_1 and K_2 such that $K_1 \le a_n/b_n \le K_2$ for all n large enough. □

Remark 6.2. If $(a_k)_\lambda^- = (a_k)_\lambda^+$ in above theorem then the sequence $\{a_k\}$ of fuzzy numbers becomes the sequence of real numbers, then the following corollary is an application of Theorem 6.1.

Corollary 6.3. *[16] If $\{a_k\}$ is a null sequence and the series $\sum_{m=1}^\infty 2^{m/q}\left(\sum_{k\in I_m}|\Delta a_k|^p\right)^{1/p}$ is convergent for some $p>1$, then (i) $f\in L^1(0,\pi)$. (ii) $\frac{1}{2}a_0+\sum_{k=1}^\infty a_k\cos kx$ is the Fourier series of f. (iii) $\lim_{n\to\infty}\int_0^\pi |s_n(x)-f(x)|dx = 0$ if and only if $\lim_{n\to\infty} a_n\ln n = 0$.*

Here, $s_n(x) = \frac{a_0}{2}+\sum_{k=1}^n a_k\cos kx$ and $\lim_{n\to\infty} s_n(x) = f(x)$.

The second main result of this section is as follows:

Theorem 6.4. *Let $\{b_k\}$ be a sequence of fuzzy coefficients belonging to class $\mathbf{K_p(F)}$, then*
(i) $g^t\in L^1(0,\pi)$ if and only if $\underset{\oplus\ k=1}{\sum^\infty}\frac{|b_k|}{k}<\infty$.
(ii) if $g^t\in L^1(0,\pi)$, then (4.2) is the Fourier series of g^t.
(iii) the series (4.2) converges to the fuzzy valued function g^t in $L^1(0,\pi)$-norm if and only if $\left\{\lim_{n\to\infty} b_n\ln n = [0]_\lambda \ \middle|\ 0\le\lambda\le 1\right\}$.

Proof. (i) First we show that the limit function g^t exists in $(0,\pi)$. For this we consider
$\widetilde{s}_n^t(x) = \underset{\oplus\ k=1}{\sum^n} b_k\sin kx = \left\{\left(\sum_{k=1}^n (b_k)_\lambda^-\sin kx, \sum_{k=1}^n (b_k)_\lambda^+\sin kx\right)\ \middle|\ 0\le\lambda\le 1\right\}$.
Performing summation by parts, we get

$$\widetilde{s}_n^t(x) = \left\{\left(\sum_{k=1}^n \Delta(b_k)_\lambda^-\widetilde{D}_k(x)+(b_n)_\lambda^-\widetilde{D}_n(x), \sum_{k=1}^n \Delta(b_k)_\lambda^+\widetilde{D}_k(x)+(b_n)_\lambda^+\widetilde{D}_n(x)\right)\ \middle|\ 0\le\lambda\le 1\right\}.$$

As $\{b_k\}$ belongs to class $\mathbf{K_p(F)}$, then

$$\widetilde{s}_n^t(x) = \left\{\left(\sum_{k=1}^n (\Delta b_k)_\lambda^-\widetilde{D}_k(x)+(b_n)_\lambda^-\widetilde{D}_n(x), \sum_{k=1}^n (\Delta b_k)_\lambda^+\widetilde{D}_k(x)+(b_n)_\lambda^+\widetilde{D}_n(x)\right)\ \middle|\ 0\le\lambda\le 1\right\}$$

where $\widetilde{D}_k(x)$ is the conjugate Dirichlet Kernel.
As $\widetilde{D}_k(x)$ is bounded in $(0,\pi)$ therefore, by given hypothesis $g^t(x)=\lim_{n\to\infty}\widetilde{s}_n^t(x)$ exists in $(0,\pi)$.
Now, consider

$$\int_0^\pi |g^t(x)|dx = \left\{\left(\int_0^\pi |g_\lambda^-(x)|dx, \int_0^\pi |g_\lambda^+(x)|dx\right)\ \middle|\ 0\le\lambda\le 1\right\}$$

$$= \left\{\left(\int_0^\pi |\sum_{k=1}^\infty (b_k)_\lambda^- \sin kx|dx, \int_0^\pi |\sum_{k=1}^\infty (b_k)_\lambda^+ \sin kx|dx\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$\preceq \left\{\left(\int_0^\pi \sum_{k=1}^\infty |(b_k)_\lambda^-||\sin kx|dx, \int_0^\pi \sum_{k=1}^\infty |(b_k)_\lambda^+||\sin kx|dx\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$\preceq \left\{\left(\sum_{k=1}^\infty |(b_k)_\lambda^-| \left[-\frac{\cos kx}{k}\right]_0^\pi, \sum_{k=1}^\infty |(b_k)_\lambda^+| \left[-\frac{\cos kx}{k}\right]_0^\pi\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$= \left\{\left(\sum_{k=1}^\infty |(b_k)_\lambda^-| \left[\frac{-\cos k\pi + 1}{k}\right], \sum_{k=1}^\infty |(b_k)_\lambda^+| \left[\frac{-\cos k\pi + 1}{k}\right]\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$= \left\{\left(\sum_{k=1}^\infty |(b_k)_\lambda^-| \left[\frac{-(-1)^k + 1}{k}\right], \sum_{k=1}^\infty |(b_k)_\lambda^-| \left[\frac{-(-1)^k + 1}{k}\right]\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$= \left\{\left(2\sum_{k=1}^\infty \frac{|(b_k)_\lambda^-|}{k}, 2\sum_{k=1}^\infty \frac{|(b_k)_\lambda^+|}{k}\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$= 2 \sum_{\oplus\, k=1}^\infty \frac{|b_k|}{k}.$$

Thus, $g^t(x) \in L^1(0,\pi)$ if and only if $\sum_{\oplus\, k=1}^\infty \frac{|b_k|}{k}$ is convergent.

(ii) Let $l \ge 0$ be fixed integer. Consider,

$$\frac{2}{\pi}\int_0^\pi g^t(x) \sin lx dx$$

$$= \lim_{n\to\infty} \left[\frac{2}{\pi}\int_0^\pi \widetilde{s}_n^t(x) \sin lx dx\right]$$

$$= \lim_{n\to\infty} \left[\frac{2}{\pi}\int_0^\pi \left\{\sum_{\oplus\, k=1}^n b_k \sin kx\right\} \sin lx dx\right]$$

$$= \left\{\left(\begin{array}{c} \lim\limits_{n\to\infty} \left[\frac{2}{\pi}\int_0^\pi \left\{\sum_{k=1}^n (b_k)_\lambda^- \sin kx\right\} \sin lx dx\right], \\ \lim\limits_{n\to\infty} \left[\frac{2}{\pi}\int_0^\pi \left\{\sum_{k=1}^n (b_k)_\lambda^- \sin kx\right\} \sin lx dx\right] \end{array}\right) \;\Bigg|\; 0 \le \lambda \le 1\right\}$$

$$= \left\{\left(\lim_{n\to\infty} \frac{2}{\pi} \times \frac{\pi}{2}(b_l)_\lambda^-, \lim_{n\to\infty} \frac{2}{\pi} \times \frac{\pi}{2}(b_l)_\lambda^+\right) \;\Big|\; 0 \le \lambda \le 1\right\}$$

$$= b_l.$$

(iii) Let $\sigma_n^t(x)$ denote the first arithmetic mean of series (4.2)

$$\begin{aligned}
&\sigma_n^t(x) \\
=\; & \sum_{\oplus\ k=1}^{n} \left(1 - \frac{k}{n+1}\right) b_k \sin kx \\
=\; & \left(\sigma_\lambda^-(x), \sigma_\lambda^+(x)\right) \\
=\; & \left\{ \left(\sum_{k=n}^{n} \left(1 - \frac{k}{n+1}\right) (b_k)_\lambda^- \sin kx, \sum_{k=n}^{n} \left(1 - \frac{k}{n+1}\right) (b_k)_\lambda^+ \sin kx \right) \;\middle|\; 0 \le \lambda \le 1 \right\}.
\end{aligned}$$

Since $g^t \in L^1(0,\pi)$ by (i), we get
$\lim\limits_{n\to\infty} \int\limits_0^\pi |\sigma_n^t(x) \ominus g^t(x)| dx = [0]_\lambda$ for $\lambda \in [0,1]$

$$i.e \sup_{\lambda\in[0,1]} \left(\lim_{n\to\infty} \int_0^\pi |(\sigma_n)_\lambda^-(x) - g_\lambda^-(x)| dx, \lim_{n\to\infty} \int_0^\pi |(\sigma_n)_\lambda^+(x) - g_\lambda^+(x)| dx \right) = 0$$

since

$$\left| \int_0^\pi |\widetilde{s}_n^t(x) \ominus f^t(x)| dx \ominus \int_0^\pi |\widetilde{s}_n^t(x) \ominus \sigma_n^t(x)| dx \right| \le \int_0^\pi |\sigma_n^t(x) \ominus f^t(x)| dx.$$

It is enough to prove that
$\lim\limits_{n\to\infty} \int\limits_0^\pi \left|\widetilde{s}_n^t(x) \ominus \sigma_n^t(x)\right| = [0]_\lambda$ if and only if $\lim\limits_{n\to\infty} b_n \ln n = [0]_\lambda$. Consider,

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^\pi \left|\widetilde{s}_n^t(x) \ominus \sigma_n^t(x)\right| dx \\
=\; & \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \left(\int_0^\pi \left|(\widetilde{s}_n)_\lambda^-(x) - (\sigma_n)_\lambda^-(x)\right| dx, \int_0^\pi \left|(\widetilde{s}_n)_\lambda^+(x) - (\sigma_n)_\lambda^+(x)\right| dx \right) \\
=\; & \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \left(\int_0^\pi \left| \frac{1}{n+1} \sum_{k=1}^{n} (b_k)_\lambda^- k \sin kx \right| dx, \int_0^\pi \left| \frac{1}{n+1} \sum_{k=1}^{n} (b_k)_\lambda^+ k \sin kx \right| dx \right) \\
=\; & (J_1, J_2).
\end{aligned}$$

First, we take

$$\begin{aligned}
&J_1 \\
=\; & \lim_{n\to\infty} \sup_{\lambda\in[0,1]} \int_0^\pi \left| \frac{1}{n+1} \sum_{k=1}^{n} k (b_k)_\lambda^- \sin kx \right| dx \\
=\; & \lim_{n\to\infty}
\end{aligned}$$

$$\sup_{\lambda\in[0,1]}\left[\int_0^\pi\left|\frac{1}{n+1}\sum_{k=1}^{k_0-1}[(b_k)_\lambda^- - (b_{k_0})_\lambda^-]k\sin kx - \frac{1}{n+1}\left\{\sum_{k=k_0}^{n-1}D_k'(x)\Delta(b_k)_\lambda^- + (b_n)_\lambda^- D_n'(x)\right\}\right|dx\right]$$

$$= \lim_{n\to\infty}$$

$$\sup_{\lambda\in[0,1]}\left[\int_0^\pi\left|\frac{1}{n+1}\sum_{k=1}^{k_0-1}[(b_k)_\lambda^- - (b_{k_0})_\lambda^-]k\sin kx - \frac{1}{n+1}\left\{\sum_{k=k_0}^{n-1}D_k'(x)(\Delta b_k)_\lambda^- + (b_n)_\lambda^- D_n'(x)\right\}\right|dx\right].$$

Similarly,

$$J_2 = \lim_{n\to\infty}\sup_{\lambda\in[0,1]}\left[\int_0^\pi\left|\frac{1}{n+1}\sum_{k=1}^{k_0-1}[(b_k)_\lambda^+ - (b_{k_0})_\lambda^+]k\sin kx - \frac{1}{n+1}\left\{\sum_{k=k_0}^{n-1}D_k'(x)(\Delta b_k)_\lambda^+ + (b_n)_\lambda^+ D_n'(x)\right\}\right|dx\right]$$

where $k_0 = 2^{m_0-1}$ is a fixed integer and "prime" means differentiation with respect to x. Let

$$\begin{aligned}
\sum &= \left|\int_0^\pi |\widetilde{s}_n^t(x)\ominus\sigma_n^t(x)|dx \ominus \frac{b_n}{n+1}\int_0^\pi |D_n'(x)|dx\right| \\
&= \sup_{\lambda\in[0,1]}\left(\begin{array}{l}\left|\int_0^\pi |(\widetilde{s}_n)_\lambda^-(x) - (\sigma)_\lambda^-(x)|dx - \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx\right|, \\ \left|\int_0^\pi |(\widetilde{s}_n)_\lambda^+(x) - (\sigma)_\lambda^+(x)|dx - \frac{(b_n)_\lambda^+}{n+1}\int_0^\pi |D_n'(x)|dx\right|\end{array}\right) \\
&= \left(\sum_1, \sum_2\right).
\end{aligned}$$

Consider,

$$\begin{aligned}
\sum_1 &= \sup_{\lambda\in[0,1]}\left|\int_0^\pi |(\widetilde{s}_n)_\lambda^-(x) - (\sigma)_\lambda^-(x)|dx - \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx\right| \\
&\le \sup_{\lambda\in[0,1]}\left|\begin{array}{r}\frac{1}{n+1}\int_0^\pi\sum_{k=1}^{k_0-1}\left|\left[(b_k)_\lambda^- - (b_{k_0})_\lambda^-\right]k\sin kx\right|dx + \frac{1}{n+1}\int_0^\pi\sum_{k=k_0}^{n-1}\left|\Delta(b_k)_\lambda^- D_k'(x)\right|dx + \\ \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx - \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx\end{array}\right| \\
&\le \sup_{\lambda\in[0,1]}\left|\begin{array}{r}\frac{1}{n+1}\int_0^\pi\sum_{k=1}^{k_0-1}\left|\left[(b_k)_\lambda^- - (b_{k_0})_\lambda^-\right]k\sin kx\right|dx + \frac{1}{n+1}\int_0^\pi\sum_{k=k_0}^{n-1}\left|(\Delta b_k)_\lambda^- D_k'(x)\right|dx + \\ \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx - \frac{(b_n)_\lambda^-}{n+1}\int_0^\pi |D_n'(x)|dx\end{array}\right|
\end{aligned}$$

$$\leq \sup_{\lambda\in[0,1]} \left[\frac{\pi}{n+1} \sum_{k=1}^{k_0-1} k|(b_k)^-_\lambda - (b_{k_0})^-_\lambda| + \frac{1}{n+1} \int_0^\pi \left| \sum_{k=k_0}^{n-1} D'_k(x)(\Delta b_k)^-_\lambda \right| dx \right].$$

Applying Bernstein's inequality ([31], Vol.2, p.11) and using Lemma 5.1, we get

$$\sum_1 \leq$$

$$\sup_{\lambda\in[0,1]} \left[\frac{\pi}{n+1} \sum_{k=1}^{k_0-1} k|(b_k)^- - (b_{k_0})^-| + 2 \int_0^\pi \left| \sum_{k=k_0}^{n-1} D_k(x)(\Delta b_k)^-_\lambda \right| dx \right]$$

$$\leq \sup_{\lambda\in[0,1]} \left[\frac{\pi}{n+1} \sum_{k=1}^{k_0-1} k|(b_k)^-_\lambda - (b_{k_0})^-_\lambda| + 2M_p \sum_{m=m_0}^{\infty} 2^{m/q} \left(\sum_{k\in I_m} |(\Delta b_k)^-_\lambda|^p \right)^{1/p} \right].$$

Similarly,

$$\sum_2 \leq \sup_{\lambda\in[0,1]} \left[\frac{\pi}{n+1} \sum_{k=1}^{k_0-1} k|(b_k)^+_\lambda - (b_{k_0})^+_\lambda| + 2M_p \sum_{m=m_0}^{\infty} 2^{m/q} \left(\sum_{k\in I_m} |(\Delta b_k)^+_\lambda|^p \right)^{1/p} \right].$$

Therefore,

$$\sum \preceq \frac{\pi}{n+1} \sum_{\oplus\ k=1}^{k_0-1} k|b_k \ominus b_{k_0}| \oplus 2M_p \sum_{\oplus\ m=m_0}^{\infty} 2^{m/q} \left(\sum_{\oplus\ k\in I_m} |\Delta b_k|^p \right)^{1/p} = \sum{}' + \sum{}''.$$

Given any $\epsilon > 0$ by class $\mathbf{K_p(F)}$ we choose m_0 so large that $\sum'' \prec \epsilon$. Then setting $k_0 = 2^{m_0-1}$, we take n so large that $\sum' \prec \epsilon$. To sum up we have $\sum \prec 2\epsilon$, if n is sufficiently large. This means that

$$\lim_{n\to\infty} \left[\int_0^\pi |s^t_n(x) \ominus \sigma^t_n(x)| dx \ominus \frac{b_n}{n+1} \int_0^\pi |D'_n(x)| dx \right] = [0]_\lambda,$$

if and only if $\lim_{n\to\infty} b_n \ln n = [0]_\lambda$ $\quad\because\quad$ $\frac{1}{n}\int_0^\pi |D'_n(x)|dx \approx \ln n$

This proves the main result. □

Remark 6.5. If $(b_k)^-_\lambda = (b_k)^+_\lambda$ in above theorem then the following corollary is an application of above theorem.

Corollary 6.6. *[16] If $\{b_k\}$ is a null sequence and the series $\sum_{m=1}^{\infty} 2^{m/q} \left(\sum_{k\in I_m} |\Delta b_k|^p \right)^{1/p}$ is convergent for some $p > 1$, then (i) $g \in L^1(0,\pi)$ if and only if $\sum_{k=1}^{\infty} \frac{|b_k|}{k} < \infty$. (ii) If $g \in L^1(0,\pi)$ then $\sum_{k=1}^{\infty} b_k \sin kx$ is the Fourier series of g. (iii) $\lim_{n\to\infty} \int_0^\pi |\tilde{s}_n(x) - g(x)| dx = 0$ if and only if $\lim_{n\to\infty} b_n \ln n = 0$.*

Here $\tilde{s}_n(x) = \sum_{k=1}^{n} b_k \sin kx$ and $\lim_{n\to\infty} \tilde{s}_n(x) = g(x)$.

7. Conclusion

Some useful results have been obtained by using level sets for defining series of fuzzy valued functions like Fourier series. The applications of the obtained results include the generalization of integrability and L^1-convergence of fuzzy trigonometric series of fuzzy valued functions. One of the purpose of this work is to extend the classical Fourier analysis to the fuzzy Fourier analysis. Futurework will be dedicated to obtain necessary and sufficient conditions for fuzzy integrability and L^1-convergence of fuzzy trigonometric series under more generalized conditions on fuzzy coefficients.

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