



# Convergence of iterative schemes for solving fixed point problems for multi-valued nonself mappings and equilibrium problems

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## Abstract

In this paper, we use the viscosity approximation method to establish strong convergence theorems for a finite family of nonexpansive multi-valued nonself mappings and equilibrium problems in a Hilbert space under some suitable conditions. As applications, we provide an example and numerical results. ©2015 All rights reserved.

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## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $D$  be a nonempty subset of  $H$  and let  $F : D \times D \rightarrow \mathbb{R}$  be a bifunction, where  $\mathbb{R}$  is the set of real numbers. The equilibrium problem for  $F$  is to find  $u \in D$  such that

$$F(u, y) \geq 0 \quad \forall y \in D. \quad (1.1)$$

The set of solutions of (1.1) is denoted by  $EP(F)$ . Given a mapping  $S : D \rightarrow H$ , let  $F(x, y) = \langle Sx, y - x \rangle$  for all  $x, y \in D$ . Then  $z \in EP(F)$  if and only if  $F(z, y) = \langle Sz, y - z \rangle$  for all  $y \in D$ , i.e.,  $z$  is a solution

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of the variational inequality. The equilibrium problem (1.1) includes as special cases numerous problems in physics, optimization and economics. Methods for solving the equilibrium problem have been studied by many authors (see, for example, [4, 5, 6, 8, 13, 19]).

The set  $D$  is called *proximal* if for each  $x \in H$ , there exists an element  $y \in D$  such that  $\|x - y\| = d(x, D)$ , where  $d(x, D) = \inf\{\|x - z\| : z \in D\}$ . Let  $CB(D), K(D)$  and  $P(D)$  be the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of  $D$  respectively. The *Hausdorff metric* on  $CB(D)$  is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for  $A, B \in CB(D)$ . A single-valued mapping  $T : D \rightarrow D$  is called *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in D$ . A multi-valued mapping  $T : D \rightarrow CB(D)$  is said to be nonexpansive if  $H(Tx, Ty) \leq \|x - y\|$  for all  $x, y \in D$ . An element  $p \in D$  is called a *fixed point* of  $T : D \rightarrow D$  (resp.  $T : D \rightarrow CB(D)$ ) if  $p = Tp$  ( $p \in Tp$ , respectively). The set of fixed points of  $T$  is denoted by  $F(T)$ .

For single-valued nonexpansive mappings, in 2000, Moudafi [11] proved the following strong convergence theorem:

**Theorem M** ([11]). *Let  $D$  be a nonempty, closed and convex subset of a Hilbert space  $H$  and let  $T$  be a nonexpansive mapping of  $D$  into itself such that  $F(T)$  is nonempty. Let  $f$  be a contraction of  $D$  into itself and let  $\{x_n\}$  be a sequence defined as follows:  $x_1 \in D$  and*

$$x_{n+1} = \frac{1}{1 + \varepsilon_n}Tx_n + \frac{\varepsilon_n}{1 + \varepsilon_n}f(x_n)$$

for all  $n \in \mathbb{N}$ , where  $\{\varepsilon_n\} \subset (0, 1)$  satisfies

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \sum_{n=1}^{\infty} \varepsilon_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0.$$

Then  $\{x_n\}$  converges strongly to  $z \in F(T)$ , where  $z = P_{F(T)}f(z)$  and  $P_{F(T)}$  is the metric projection of  $H$  onto  $F(T)$ .

Such a method is called the viscosity approximation method. Recently, Takahashi-Takahashi [20] introduced an iterative scheme by the viscosity approximation method for finding a common element of the solutions set of (1.1) and the fixed points set of a nonexpansive mapping in a Hilbert space, and proved the following strong convergence theorem.

**Theorem TT** ([20]). *Let  $D$  be a nonempty, closed and convex subset of a Hilbert space  $H$ . Let  $F : D \times D \rightarrow \mathbb{R}$  be a bifunction satisfying the following assumptions:*

- (A1)  $F(x, x) = 0$  for all  $x \in D$ ;
- (A2)  $F$  is monotone, i.e.,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in D$ ;
- (A3) for each  $x, y, z \in D$ ,

$$\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

- (A4) for each  $x \in D$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous.

Let  $T : D \rightarrow H$  be a nonexpansive mapping such that  $F(T) \cap EP(F) \neq \emptyset$ ,  $f : H \rightarrow H$  be a contraction, and  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\liminf_{n \rightarrow \infty} r_n > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ .

Then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $z \in F(T) \cap EP(F)$ , where  $z = P_{F(T) \cap EP(F)}f(z)$ .

In recent years, fixed point theory for nonlinear multi-valued mappings in various spaces has been studied by many authors (see, for example, [3, 14, 16, 18] and the references therein).

One way of approximating the fixed points of nonlinear multi-valued mappings is to use the concept of the best approximation operator  $P_T$  defined by  $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$ . In 2003, Hussain-Khan [9] used the best approximation operator  $P_T$  to study the fixed points of a \*-nonexpansive multi-valued mapping and the strong convergence of its iterates to a fixed point defined on a closed and convex subset of a Hilbert space. By using the concept of best approximation operator, many authors have found fixed point results for multi-valued nonself mappings (see, for example, [10, 15, 21]).

In 2010, Zegeye-Shahzad [21] studied the convergence of the viscosity approximation process for nonexpansive nonself multi-valued mappings in Banach spaces.

**Theorem ZS** ([21]). *Let  $E$  be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm,  $D$  a nonempty closed convex subset of  $E$ , and  $T : D \rightarrow K(D)$  a multimap such that  $P_T$  is nonexpansive. For given  $x_0 \in D$ ,  $y_0 \in P_T x_0$ , let  $\{x_n\}$  be generated by the algorithm*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \\ y_n \in P_T(x_n) \text{ such that } \|y_{n-1} - y_n\| = d(y_{n-1}, P_T(x_n)), \quad n \geq 1 \end{cases}$$

(see, e.g., [18]), where  $f : D \rightarrow D$  is a contraction and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and
- (iii)  $\lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ .

If  $F(T) \neq \emptyset$ , then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

In 2011, Song-Cho [17] gave an example of a multi-valued mapping  $T$  which is not necessary nonexpansive but  $P_T$  is nonexpansive. It is an interesting problem to study the convergence of multi-valued mappings by using the best approximation operator.

Let  $H$  be a Hilbert space and  $D$  be a subset of  $H$ . A multi-valued mapping  $T : D \rightarrow CB(H)$  is said to satisfy the condition (A) if  $\|x - p\| = d(x, Tp)$  for all  $x \in H$  and  $p \in F(T)$ . We see that  $T$  satisfies the condition (A) if and only if  $Tp = \{p\}$  for all  $p \in F(T)$ . It is known that the best approximation operator  $P_T$  also satisfies the condition (A).

Motivated by Takahashi-Takahashi [20] and Zegeye-Shahzad [21], we introduce the viscosity approximation method for solving the equilibrium problem and the fixed points problem of a finite family of multi-valued nonself mappings in a Hilbert space. In the last section, we also give an example and numerical results for supporting our method.

## 2. Preliminaries and lemmas

Let  $D$  be a nonempty, closed and convex subset of a Hilbert space  $H$ . For every point  $x \in H$ , there exists a unique nearest point in  $D$ , denoted by  $P_D x$ , such that

$$\|x - P_D x\| \leq \|x - y\|, \quad \forall y \in D.$$

$P_D$  is called the *metric projection* of  $H$  onto  $D$ . It is known that  $P_D$  is a nonexpansive mapping of  $H$  onto  $D$ . We also recall the following facts regarding real Hilbert spaces.

**Lemma 2.1.** *Let  $D$  be a nonempty, closed and convex subset of a real Hilbert space  $H$  and let  $P_D$  be the metric projection of  $H$  onto  $D$ . Let  $x \in H$  and  $z \in D$ . Then  $z = P_D x$  if and only if*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

**Lemma 2.2.** *Let  $H$  be a real Hilbert space. Then the following relations hold:*

- (i)  $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \forall x, y \in H;$
- (ii)  $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$  and  $x, y \in H;$
- (iii)  $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H.$

By using Lemma 2.2 (ii), we can deduce the next result.

**Lemma 2.3.** *Let  $H$  be a real Hilbert space. Then for each  $m \in \mathbb{N}$*

$$\left\| \sum_{i=1}^m t_i x_i \right\|^2 = \sum_{i=1}^m t_i \|x_i\|^2 - \sum_{i=1, i \neq j}^m t_i t_j \|x_i - x_j\|^2,$$

where  $x_i \in H, t_i, t_j \in [0, 1]$  for all  $i, j = 1, 2, \dots, m,$  and  $\sum_{i=1}^m t_i = 1.$

**Lemma 2.4** ([4]). *Let  $D$  be a nonempty, closed and convex subset of a real Hilbert space  $H.$  Let  $F$  be a bifunction from  $D \times D$  to  $\mathbb{R}$  satisfying (A1)-(A4) and let  $r > 0$  and  $x \in H.$  Then there exists  $z \in D$  such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \text{for all } y \in D.$$

**Lemma 2.5** ([8]). *For  $r > 0$  and  $x \in H,$  define the mapping  $T_r : H \rightarrow D$  by*

$$T_r(x) = \left\{ z \in D : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in D \right\}.$$

Then the following hold:

- (i)  $T_r$  is single-valued;
- (ii)  $T_r$  is firmly nonexpansive, i.e., for any  $x, y \in H,$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (iii)  $F(T_r) = EP(F);$
- (iv)  $EP(F)$  is closed and convex.

**Lemma 2.6** ([1]). *Let  $D$  be a nonempty and weakly compact subset of a Hilbert space  $H$  and  $T : D \rightarrow K(H)$  be a nonexpansive mapping. Then  $I - T$  is demiclosed.*

**Lemma 2.7** ([2]). *Let  $\{s_n\}$  be a sequence of nonnegative real numbers,  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  with  $\sum_{n=1}^\infty \alpha_n = \infty,$   $\{\beta_n\}$  be a sequence of nonnegative real numbers with  $\sum_{n=1}^\infty \beta_n < \infty,$  and  $\{\gamma_n\}$  be a sequence of real numbers with  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0.$  Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all  $n \in \mathbb{N}.$  Then  $\lim_{n \rightarrow \infty} s_n = 0.$

**Lemma 2.8** ([7]). *Let  $D$  be a closed and convex subset of a real Hilbert space  $H.$  Let  $T : D \rightarrow CB(D)$  be a nonexpansive multi-valued map with  $F(T) \neq \emptyset$  and  $Tp = \{p\}$  for each  $p \in F(T).$  Then  $F(T)$  is a closed and convex subset of  $D.$*

Using the above results, we study the convergence of the iteration (2.1) defined in the following. Let  $D$  be a nonempty, closed and convex subset of a Hilbert space  $H.$  Let  $T_i : D \rightarrow CB(H)$  be a multi-valued nonself mapping for all  $i \in \{1, 2, \dots, N\}, f : H \rightarrow H$  be a contraction and  $F : D \times D \rightarrow \mathbb{R}$  be a bifunction.

Let  $\{\alpha_{0,n}\}$  and  $\{\alpha_{i,n}\}$  be sequences in  $[0, 1]$  for all  $i \in \{1, 2, \dots, N\}$  with  $\sum_{k=0}^N \alpha_{k,n} = 1$ , and  $\{r_n\}$  be a sequence in  $(0, \infty)$ . For a given  $x_1 \in H$ , we find  $u_1 \in D$  such that

$$F(u_1, y) + \frac{1}{r_1} \langle y - u_1, u_1 - x_1 \rangle \geq 0, \quad \forall y \in D.$$

Let  $z_{i,1} \in T_i u_1$  for all  $i \in \{1, 2, \dots, N\}$  and compute  $x_2 \in H$  by

$$x_2 = \alpha_{0,1} f(x_1) + \sum_{i=1}^N \alpha_{i,1} z_{i,1}.$$

Find  $u_2 \in D$  such that

$$F(u_2, y) + \frac{1}{r_2} \langle y - u_2, u_2 - x_2 \rangle \geq 0, \quad \forall y \in D.$$

From Nadler’s Theorem (see [12]), there exist  $z_{i,2} \in T_i u_2$  for all  $i \in \{1, 2, \dots, N\}$  such that  $\|z_{i,2} - z_{i,1}\| \leq H(T_i u_2, T_i u_1)$  for all  $i \in \{1, 2, \dots, N\}$ .

Inductively, we construct the sequence  $\{x_n\} \subset H$  as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D, \\ x_{n+1} = \alpha_{0,n} f(x_n) + \sum_{i=1}^N \alpha_{i,n} z_{i,n}, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where  $z_{i,n} \in T_i u_n$  such that  $\|z_{i,n+1} - z_{i,n}\| \leq H(T_i u_{n+1}, T_i u_n)$  for all  $i \in \{1, 2, \dots, N\}$ .

### 3. Main results

In this section, we prove a strong convergence theorem for the iteration (2.1) to find a common element of the solutions set of an equilibrium problem and the common fixed points sets of a finite family of multi-valued nonself mappings.

**Theorem 3.1.** *Let  $D$  be a nonempty, and weakly compact subset of a Hilbert space  $H$ . Let  $F$  be a bifunction from  $D \times D$  to  $\mathbb{R}$  satisfying (A1)-(A4) and  $\{T_i\}_{i=1}^N$  a family of nonexpansive multi-valued mappings of  $D$  into  $K(H)$  such that  $\cap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$ . Let  $f$  be a contraction of  $H$  into itself. Let  $\{\alpha_{0,n}\}, \{\alpha_{i,n}\}$  be sequences in  $[0, 1]$  with  $\sum_{k=0}^N \alpha_{k,n} = 1$  and  $\{r_n\} \subset (0, \infty)$  be a sequence such that the following hold:*

- (i)  $\lim_{n \rightarrow \infty} \alpha_{0,n} = 0, \sum_{n=1}^{\infty} \alpha_{0,n} = \infty, \liminf_{n \rightarrow \infty} \alpha_{i,n} \alpha_{j,n} > 0$  for all  $i, j \in \{1, 2, \dots, N\}$  and  $\sum_{n=1}^{\infty} |\alpha_{k,n+1} - \alpha_{k,n}| < \infty$  for all  $k \in \{0, 1, 2, \dots, N\}$ ;
- (ii)  $\liminf_{n \rightarrow \infty} r_n > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ .

*If  $\{T_i\}_{i=1}^N$  satisfies the condition (A), then the sequences  $\{x_n\}$  and  $\{u_n\}$  generated by (2.1) converge strongly to  $z \in \cap_{i=1}^N F(T_i) \cap EP(F)$ , where  $z = P_{\cap_{i=1}^N F(T_i) \cap EP(F)} f(z)$ .*

*Proof.* Let  $Q = P_{\cap_{i=1}^N F(T_i) \cap EP(F)}$ . Since  $f$  is a contraction, there exists a constant  $\alpha \in [0, 1)$  such that  $\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha \|x - y\|$  for all  $x, y \in H$ . Hence  $Qf$  is a contraction of  $H$  into itself, so there exists a unique element  $z \in H$  such that  $z = Qf(z)$ . We divide the proof into five steps.

**Step 1.** We show that  $\{x_n\}$  is bounded.

Let  $p \in \cap_{i=1}^N F(T_i) \cap EP(F)$ . Then from  $u_n = T_{r_n} x_n$ , we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\| \tag{3.1}$$

for all  $n \geq 1$ . It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_{0,n} \|f(x_n) - p\| + \sum_{i=1}^N \alpha_{i,n} \|z_{i,n} - p\| \\ &= \alpha_{0,n} \|f(x_n) - p\| + \sum_{i=1}^N \alpha_{i,n} d(z_{i,n}, T_i p) \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_{0,n} \|f(x_n) - p\| + \sum_{i=1}^N \alpha_{i,n} H(T_i u_n, T_i p) \\
 &\leq \alpha_{0,n} (\|f(x_n) - f(p)\| + \|f(p) - p\|) + \sum_{i=1}^N \alpha_{i,n} \|u_n - p\| \\
 &\leq \left(1 - \alpha_{0,n}(1 - \alpha)\right) \|x_n - p\| + \alpha_{0,n} \|f(p) - p\| \\
 &\leq \max \left\{ \|x_n - p\|, \frac{1}{(1 - \alpha)} \|f(p) - p\| \right\}.
 \end{aligned}$$

By induction,

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{1}{(1 - \alpha)} \|f(p) - p\| \right\}, \quad \forall n \geq 1.$$

Hence  $\{x_n\}$  is bounded. The same holds for  $\{u_n\}$ ,  $\{f(x_n)\}$  and  $\{z_{i,n}\}$  for all  $i \in \{1, 2, \dots, N\}$ .

**Step 2.** We show that  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

From the definition of  $\{x_n\}$  there exist  $z_{i,n+1} \in T_i u_{n+1}$  and  $z_{i,n} \in T_i u_n$  for all  $i \in \{1, 2, \dots, N\}$  such that  $\|z_{i,n+1} - z_{i,n}\| \leq H(T_i u_{n+1}, T_i u_n)$ . Let  $K = \sup_{n \geq 1} \{\|f(x_n)\| + \sum_{i=1}^N \|z_{i,n}\|\}$ . Then, we have

$$\begin{aligned}
 \|x_{n+2} - x_{n+1}\| &= \|\alpha_{0,n+1} f(x_{n+1}) - \alpha_{0,n+1} f(x_n) + \alpha_{0,n+1} f(x_n) - \alpha_{0,n} f(x_n) \\
 &\quad + \sum_{i=1}^N \alpha_{i,n+1} z_{i,n+1} - \sum_{i=1}^N \alpha_{i,n+1} z_{i,n} + \sum_{i=1}^N \alpha_{i,n+1} z_{i,n} - \sum_{i=1}^N \alpha_{i,n} z_{i,n}\| \\
 &\leq \alpha_{0,n+1} \|f(x_{n+1}) - f(x_n)\| + |\alpha_{0,n+1} - \alpha_{0,n}| \|f(x_n)\| \\
 &\quad + \sum_{i=1}^N \alpha_{i,n+1} \|z_{i,n+1} - z_{i,n}\| + \sum_{i=1}^N |\alpha_{i,n+1} - \alpha_{i,n}| \|z_{i,n}\| \\
 &\leq \alpha_{0,n+1} \alpha \|x_{n+1} - x_n\| + \sum_{i=0}^N |\alpha_{i,n+1} - \alpha_{i,n}| K + \sum_{i=1}^N \alpha_{i,n+1} H(T_i u_{n+1}, T_i u_n) \\
 &\leq \alpha_{0,n+1} \alpha \|x_{n+1} - x_n\| + \sum_{i=0}^N |\alpha_{i,n+1} - \alpha_{i,n}| K + \sum_{i=1}^N \alpha_{i,n+1} \|u_{n+1} - u_n\|. \tag{3.2}
 \end{aligned}$$

On the other hand, from  $u_n = T_{r_n} x_n$  and  $u_{n+1} = T_{r_{n+1}} x_{n+1}$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \tag{3.3}$$

for all  $y \in D$  and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \tag{3.4}$$

for all  $y \in D$ . Setting  $y = u_{n+1}$  in (3.3) and  $y = u_n$  in (3.4), we obtain

$$F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

It follows from (A2) that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Since  $\liminf_{n \rightarrow \infty} r_n > 0$ , there exists a real number  $a$  such that  $r_n > a > 0$  for all  $n \geq 1$ . Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{a} |r_{n+1} - r_n| M, \end{aligned} \tag{3.5}$$

where  $M = \sup\{\|u_n - x_n\| : n \geq 1\}$ . Combining (3.2) and (3.5), we obtain

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \alpha_{0,n+1} \alpha \|x_{n+1} - x_n\| + \sum_{i=0}^N |\alpha_{i,n+1} - \alpha_{i,n}| K + \sum_{i=1}^N \alpha_{i,n+1} \left( \|x_{n+1} - x_n\| + \frac{1}{a} |r_{n+1} - r_n| M \right) \\ &= (1 - \alpha_{0,n+1}(1 - \alpha)) \|x_{n+1} - x_n\| + \sum_{i=0}^N |\alpha_{i,n+1} - \alpha_{i,n}| K + \frac{1}{a} |r_{n+1} - r_n| M. \end{aligned}$$

By Lemma 2.7,  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Step 3.** We show that  $\lim_{n \rightarrow \infty} \|x_n - z_{i,n}\| = \lim_{n \rightarrow \infty} \|u_n - z_{i,n}\| = 0$  for all  $i \in \{1, 2, \dots, N\}$ .

From (3.5) and (ii), we have

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{3.6}$$

Let  $p \in \cap_{i=1}^N F(T_i) \cap EP(F)$ . From Lemma 2.3 and (3.1), we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n} \|z_{i,n} - p\|^2 - \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 \\ &= \alpha_{0,n} \|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n} d(z_{i,n}, T_i p)^2 - \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 \\ &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n} H(T_i u_n, T_i p)^2 - \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 \\ &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \|u_n - p\|^2 - \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 \\ &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 \end{aligned}$$

for all  $j, k \in \{1, 2, \dots, N\}$ . It follows that

$$\begin{aligned} \alpha_{j,n} \alpha_{k,n} \|z_{j,n} - z_{k,n}\|^2 &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_{0,n} \|f(x_n) - p\|^2 + \|x_{n+1} - x_n\| (\|x_n - p\| + \|x_{n+1} - p\|) \end{aligned}$$

for all  $j, k \in \{1, 2, \dots, N\}$ . From (i), we have that  $\|z_{j,n} - z_{k,n}\| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $j, k \in \{1, 2, \dots, N\}$ . This implies that

$$\|x_n - z_{i,n}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_{i,n}\| \rightarrow 0 \tag{3.7}$$

as  $n \rightarrow \infty$  for all  $i \in \{1, 2, \dots, N\}$ . For  $p \in \bigcap_{i=1}^N F(T_i) \cap EP(F)$ , we see that

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p \rangle \\ &= \langle u_n - p, x_n - p \rangle \\ &= \frac{1}{2} \left( \|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 \right), \end{aligned}$$

which yields

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Therefore, using the convexity of  $\|\cdot\|^2$ , we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_{0,n}f(x_n) + \sum_{i=1}^N \alpha_{i,n}z_{i,n} - p\|^2 \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n}\|z_{i,n} - p\|^2 \\ &= \alpha_{0,n}\|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n}d(z_{i,n}, T_i p)^2 \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + \sum_{i=1}^N \alpha_{i,n}H(T_i u_n, T_i p)^2 \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + (1 - \alpha_{0,n})\|u_n - p\|^2 \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + (1 - \alpha_{0,n})(\|x_n - p\|^2 - \|x_n - u_n\|^2) \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1 - \alpha_{0,n})\|x_n - u_n\|^2, \end{aligned}$$

whence

$$\begin{aligned} (1 - \alpha_{0,n})\|x_n - u_n\|^2 &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_{0,n}\|f(x_n) - p\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|). \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \alpha_{0,n} = 0$  and  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , we have

$$\|x_n - u_n\| \rightarrow 0 \tag{3.8}$$

as  $n \rightarrow \infty$ . It follows from (3.7) and (3.8) that, for each  $i = 1, 2, \dots, N$ ,

$$\|z_{i,n} - u_n\| \leq \|z_{i,n} - x_n\| + \|x_n - u_n\| \rightarrow 0 \tag{3.9}$$

as  $n \rightarrow \infty$ .

**Step 4.** We show that  $\limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle \leq 0$ , where  $z = P_{\bigcap_{i=1}^N F(T_i) \cap EP(F)} f(z)$ .

Since  $\{x_n\}$  is bounded, we can choose a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that

$$\lim_{i \rightarrow \infty} \langle f(z) - z, x_{n_i} - z \rangle = \limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle.$$

Since  $\{u_n\}$  is bounded, we infer that  $u_{n_i} \rightharpoonup q \in D$  and also  $x_{n_i} \rightarrow q$ . We will now show that  $q \in EP(F)$ .

From  $u_n = T_{r_n}x_n$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D.$$

Also, from (A2),

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}).$$

Since  $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rightarrow 0$  and  $u_{n_i} \rightarrow q$ , from (A4) we have

$$0 \geq F(y, q)$$

for all  $y \in D$ . For  $t$  with  $0 < t \leq 1$  and  $y \in D$ , let  $y_t = ty + (1 - t)q$ . Since  $y \in D$  and  $q \in D$ ,  $y_t \in D$ , hence  $F(y_t, q) \leq 0$ . Consequently, from (A1) and (A4) we get

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, q) \leq tF(y_t, y)$$

and thus  $0 \leq F(y_t, y)$ . It follows that  $0 \leq F(q, y)$  for all  $y \in D$  by (A3), and hence  $q \in EP(F)$ . Since  $\lim_{n \rightarrow \infty} \|z_{i,n} - u_n\| = 0$  and  $u_{n_i} \rightarrow q$ , using Lemma 2.6, we obtain that  $q \in F(T_i)$  for all  $i \in \{1, 2, \dots, N\}$ . Therefore  $q \in \bigcap_{i=1}^N F(T_i) \cap EP(F)$ . By Lemma 2.1, we have

$$\limsup_{n \rightarrow \infty} \langle f(z) - z, x_n - z \rangle = \lim_{i \rightarrow \infty} \langle f(z) - z, x_{n_i} - z \rangle = \langle f(z) - z, q - z \rangle \leq 0. \tag{3.10}$$

**Step 5.** We show that  $x_n \rightarrow z$  as  $n \rightarrow \infty$ .

From Lemma 2.2 (iii) we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \sum_{i=1}^N \alpha_{i,n}^2 \|z_{i,n} - z\|^2 + 2\alpha_{0,n} \langle f(x_n) - z, x_{n+1} - z \rangle \\ &= \sum_{i=1}^N \alpha_{i,n}^2 d(z_{i,n}, T_i z)^2 + 2\alpha_{0,n} \langle f(x_n) - z, x_{n+1} - z \rangle \\ &\leq \sum_{i=1}^N \alpha_{i,n}^2 H(T_i u_n, T_i z)^2 + 2\alpha_{0,n} \langle f(x_n) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_{0,n})^2 \|u_n - z\|^2 + 2\alpha_{0,n} \langle f(x_n) - f(z), x_{n+1} - z \rangle + 2\alpha_{0,n} \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_{0,n})^2 \|x_n - z\|^2 + 2\alpha_{0,n} \alpha \|x_n - z\| \|x_{n+1} - z\| + 2\alpha_{0,n} \langle f(z) - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_{0,n})^2 \|x_n - z\|^2 + \alpha_{0,n} \alpha \left\{ \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \right\} + 2\alpha_{0,n} \langle f(z) - z, x_{n+1} - z \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \frac{(1 - \alpha_{0,n})^2 + \alpha_{0,n} \alpha}{1 - \alpha_{0,n} \alpha} \|x_n - z\|^2 + \frac{2\alpha_{0,n}}{1 - \alpha_{0,n} \alpha} \langle f(z) - z, x_{n+1} - z \rangle \\ &= \frac{1 - 2\alpha_{0,n} + \alpha_{0,n} \alpha}{1 - \alpha_{0,n} \alpha} \|x_n - z\|^2 + \frac{\alpha_{0,n}^2}{1 - \alpha_{0,n} \alpha} \|x_n - z\|^2 + \frac{2\alpha_{0,n}}{1 - \alpha_{0,n} \alpha} \langle f(z) - z, x_{n+1} - z \rangle \\ &= \left( 1 - \frac{2(1 - \alpha)\alpha_{0,n}}{1 - \alpha_{0,n} \alpha} \right) \|x_n - z\|^2 \\ &\quad + \frac{2(1 - \alpha)\alpha_{0,n}}{1 - \alpha_{0,n} \alpha} \left\{ \frac{\alpha_{0,n}}{2(1 - \alpha)} \|x_n - z\|^2 + \frac{1}{1 - \alpha} \langle f(z) - z, x_{n+1} - z \rangle \right\}. \end{aligned}$$

Put

$$\gamma_n = \frac{\alpha_{0,n}}{2(1 - \alpha)} \|x_n - z\|^2 + \frac{1}{1 - \alpha} \langle f(z) - z, x_{n+1} - z \rangle.$$

It follows from (i) and (3.10) that  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ , so  $\lim_{n \rightarrow \infty} \|x_n - z\|^2 = 0$  by Lemma 2.7. This implies that  $\{x_n\}$  converges strongly to  $z \in \bigcap_{i=1}^N F(T_i) \cap EP(F)$ . It is easily seen that  $\{u_n\}$  also converges strongly to  $z$ . We thus complete the proof.  $\square$

If, for each  $i = 1, 2, \dots, N$ ,  $T_i p = \{p\}$  for all  $p \in F(T_i)$ , then  $\{T_i\}_{i=1}^N$  satisfies the condition (A). We then obtain the following result.

**Corollary 3.2.** *Let  $D$  be a nonempty and weakly compact subset of a Hilbert space  $H$ . Let  $F$  be a bifunction from  $D \times D$  to  $\mathbb{R}$  satisfying (A1)-(A4) and  $\{T_i\}_{i=1}^N$  be nonexpansive multi-valued mappings of  $D$  into  $K(H)$  such that  $\bigcap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$ . Let  $f$  be a contraction of  $H$  into itself, and let  $\{\alpha_{0,n}\}$ ,  $\{\alpha_{i,n}\}$  ( $i = 1, 2, \dots, N$ ) and  $\{r_n\}$  be as in Theorem 3.1. If, for each  $i = 1, 2, \dots, N$ ,  $T_i p = \{p\}$  for all  $p \in F(T_i)$ , then the sequences  $\{x_n\}$  and  $\{u_n\}$  generated by (2.1) converge strongly to  $z \in \bigcap_{i=1}^N F(T_i) \cap EP(F)$ , where  $z = P_{\bigcap_{i=1}^N F(T_i) \cap EP(F)} f(z)$ .*

Since  $P_{T_i}$  ( $i = 1, 2, \dots, N$ ) satisfies the condition (A), we also obtain

**Corollary 3.3.** *Let  $D$  be a nonempty and weakly compact subset of a Hilbert space  $H$ . Let  $F$  be a bifunction from  $D \times D$  to  $\mathbb{R}$  satisfying (A1)-(A4) and  $\{T_i\}_{i=1}^N$  be a family of multi-valued mappings of  $D$  into  $P(H)$  such that  $\bigcap_{i=1}^N F(T_i) \cap EP(F) \neq \emptyset$  and  $F(T_i)$  is closed and convex for all  $i \in \{1, 2, \dots, N\}$ . Let  $f$  be a contraction of  $H$  into itself, and let  $\{\alpha_{0,n}\}$ ,  $\{\alpha_{i,n}\}$  ( $i = 1, 2, \dots, N$ ), and  $\{r_n\}$  be as in Theorem 3.1. Let the sequences  $\{x_n\}$  and  $\{u_n\}$  be generated as follows:*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D, \\ x_{n+1} = \alpha_{0,n} f(x_n) + \sum_{i=1}^N \alpha_{i,n} z_{i,n}, \end{cases} \tag{3.11}$$

where  $z_{i,n} \in P_{T_i} u_n$  such that  $\|z_{i,n+1} - z_{i,n}\| \leq H(P_{T_i} u_{n+1}, P_{T_i} u_n)$ .

If  $P_{T_i}$  is nonexpansive and  $I - T_i$  is demiclosed at 0 for all  $i \in \{1, 2, \dots, N\}$ , then the sequences  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $z \in \bigcap_{i=1}^N F(T_i) \cap EP(F)$ , where  $z = P_{\bigcap_{i=1}^N F(T_i) \cap EP(F)} f(z)$ .

#### 4. Example and numerical results

In this section, we give an example and numerical results supporting our main theorem.

**Example 4.1.** Let  $H = \mathbb{R}$  and  $D = [0, 1]$ . Let  $F(x, y) = -9x^2 + xy + 8y^2$ ,  $f(x) = \frac{x}{2}$ ,  $T_1 x = [0, \frac{x}{2}]$ ,  $T_2 x = [0, \sin x]$  and let  $\alpha_{0,n} = \frac{1}{80n}$ ,  $\alpha_{1,n} = \alpha_{2,n} = \frac{80n-1}{160n}$  and  $r_n = \frac{n}{n+1}$ .

It is easy to check that  $F$  satisfies all the conditions in Theorem 3.1. For each  $r > 0$  and  $x \in [0, 1]$ , Lemma 2.4 ensures that there exists  $z \in [0, 1]$  such that, for any  $y \in [0, 1]$ ,

$$\begin{aligned} F(x, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 &\Leftrightarrow -9z^2 + zy + 8y^2 + \frac{1}{r} (y - z)(z - x) \geq 0 \\ &\Leftrightarrow 8ry^2 + (zr + z + x)y + (xz - 9rz^2 - z^2) \geq 0. \end{aligned}$$

Put  $G(y) = 8ry^2 + (zr + z + x)y + (xz - 9rz^2 - z^2)$ . Then  $G$  is a quadratic function of  $y$  with coefficients  $a = 8r$ ,  $b = zr + z + x$  and  $c = xz - 9rz^2 - z^2$ . We next compute the discriminant  $\Delta$  of  $G$  as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= ((1+r)z - x)^2 - 32r(xz - 9rz^2 - z^2) \\ &= x^2 - 2x(1+r)z + (1+r)^2 z^2 - 32rxz + 288r^2 z^2 + 32rz^2 \\ &= x^2 - 34rxz - 2xz + 289r^2 z^2 + 34rz^2 + z^2 \\ &= x^2 - 2x(17rz + z) + (17rz + z)^2 \\ &= (x - (17rz + z))^2. \end{aligned}$$

We know that  $G(y) \geq 0$  for all  $y \in [0, 1]$  if it has at most one solution in  $[0, 1]$ . So  $\Delta \leq 0$  and hence  $x = 17rz + z$ . Now we have  $z = T_r x = \frac{x}{(17r+1)}$ . Algorithm (2.1) becomes

$$x_{n+1} = \frac{x_n}{160n} + \frac{80n - 1}{160n}(z_{1,n} + z_{2,n}), \quad \forall n \geq 1,$$

where  $z_{1,n} \in \left[0, \frac{x_n}{2(17(\frac{n}{n+1})+1)}\right]$ ,  $z_{2,n} \in \left[0, \sin\left(\frac{x_n}{17(\frac{n}{n+1})+1}\right)\right]$  are such that

$$|z_{1,n+1} - z_{1,n}| \leq H\left(\left[0, \frac{x_{n+1}}{2(17(\frac{n+1}{n+2})+1)}\right], \left[0, \frac{x_n}{2(17(\frac{n}{n+1})+1)}\right]\right) = \left|\frac{x_{n+1}}{2(17(\frac{n+1}{n+2})+1)} - \frac{x_n}{2(17(\frac{n}{n+1})+1)}\right|$$

and

$$\begin{aligned} |z_{2,n+1} - z_{2,n}| &\leq H\left(\left[0, \sin\left(\frac{x_{n+1}}{17(\frac{n+1}{n+2})+1}\right)\right], \left[0, \sin\left(\frac{x_n}{17(\frac{n}{n+1})+1}\right)\right]\right) \\ &= \left|\sin\left(\frac{x_{n+1}}{17(\frac{n+1}{n+2})+1}\right) - \sin\left(\frac{x_n}{17(\frac{n}{n+1})+1}\right)\right| \end{aligned}$$

for all  $n \geq 1$ . Choose  $x_1 = 1$  and take randomly  $z_{1,n}, z_{2,n}$  satisfying the above conditions. We then have

$n$	$z_{1,n}$	$z_{2,n}$	$x_n$	$ x_{n+1} - x_n $
1	2.68763620E-02	3.18914138E-02	1.00000000E+00	9.67674761E-01
2	1.03550828E-03	1.27284146E-03	3.23252386E-02	3.11085285E-02
3	1.90384562E-05	4.33038000E-06	1.21671005E-03	1.20316104E-03
4	1.08825740E-07	5.58886787E-08	1.35490137E-05	1.34499262E-05
5	8.68977905E-10	2.18228584E-10	9.90875836E-08	9.84418966E-08
6	2.07077239E-11	1.68185074E-11	6.45686971E-10	6.26380855E-10
7	1.33453390E-13	1.77800965E-13	1.93061163E-11	1.91356494E-11
8	3.32911503E-15	1.17262475E-16	1.70466913E-13	1.68627738E-13
9	2.34667065E-17	1.85492397E-17	1.83917524E-15	1.81704404E-15
10	1.94968136E-19	2.60343261E-19	2.21311977E-17	2.18912261E-17
⋮	⋮	⋮	⋮	⋮
50	7.95313036E-94	5.60964135E-94	3.24048135E-92	6.27814377E-93

**Table 1** Numerical results of Example 4.1 being randomized the first time.

$n$	$z_{1,n}$	$z_{2,n}$	$x_n$	$ x_{n+1} - x_n $
1	3.87646646E-02	2.11808262E-02	1.00000000E+00	9.67089584E-01
2	9.08427596E-04	5.17399890E-04	3.29104157E-02	3.21319091E-02
3	9.82766561E-07	1.29163897E-06	7.78506635E-04	7.76156570E-04
4	6.53081245E-08	1.19028080E-08	2.35006562E-06	2.30861909E-06
5	1.33075088E-09	3.64430816E-10	4.14465346E-08	4.05575361E-08
6	2.68341046E-11	1.50264282E-11	8.88998510E-10	8.67311870E-10
7	2.73653976E-13	4.49152288E-13	2.16866396E-11	2.13088585E-11
8	1.15014750E-14	6.10767047E-16	3.77781126E-13	3.71471069E-13
9	8.23396605E-17	8.77593543E-17	6.31005777E-15	6.22117079E-15
10	9.57108690E-19	5.14323658E-19	8.88869817E-17	8.81015976E-17
⋮	⋮	⋮	⋮	⋮
50	8.74865895E-97	6.97685899E-97	2.03284925E-94	2.03284925E-94

**Table 2** Numerical results of Example 4.1 being randomized the second time.

From Table 1 and Table 2, we see that 0 is the solution of the equilibrium problem and it is the common fixed point of  $T_1$  and  $T_2$  in Example 4.1.

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