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Common fixed point theorems for non-compatible self-maps in b-metric spaces

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Abstract

By using R-weak commutativity of type (Ag) and non-compatible conditions of self-mapping pairs in b-metric space, without the conditions for the completeness of space and the continuity of mappings, we establish some new common fixed point theorems for two self-mappings. Our results differ from other already known results. An example is provided to support our new result.

Keywords: b-metric space, common fixed point theorem, R-weakly commuting mappings of type (Ag), non-compatible mapping pairs.
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1. Introduction and Preliminaries

Czerwik in [10] introduced the concept of $b-$metric spaces. Since then, several papers deal with fixed point theory for single-valued and multivalued operators in $b-$metric spaces (see also [2, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 19, 21, 24]). Pacurar [21] proved results on sequences of almost contractions and fixed points in $b-$metric spaces. Recently, Hussain and Shah [14] obtained results on KKM mappings in cone $b-$metric spaces. Khamsi ([16]) also showed that each cone metric space has a $b-$metric structure.

The aim of this paper is to present some common fixed point results for two mappings under generalized contractive condition in $b-$metric space, where the $b-$metric function is not necessarily continuous. Because many of the authors in their works have used the $b-$metric spaces in which the $b-$metric functions are continuous, the techniques used in this paper can be used for many of the results in the context of $b-$metric

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space. From this point of view the results obtained in this paper generalize and extend several earlier results obtained in a lot of papers concerning $b-$metric spaces.

Consistent with [10] and [24, p. 264], the following definition and results will be needed in the sequel.

Definition 1.1 ([10])**.** Let X be a (nonempty) set and $b \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a $b-$metric iff, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ iff $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a $b-$metric space.

It should be noted that the class of $b-$metric spaces is effectively larger than that of metric spaces since a $b-$metric is a metric when $b = 1$.

We present an example which shows that a $b-$metric on X need not be a metric on X. (see also [24, p. 264]):

Example 1.2. Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a $b-$metric with $b = 2^{p-1}$.

Obviously conditions (b1) and (b2) of Definition 1.1 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right),$$

and hence, $(a + c)^p \leq 2^{p-1}(a^p + c^p)$ holds.

Thus for each $x, y, z \in X$ we obtain

$$\begin{aligned}\rho(x, y) = (d(x, y))^p &\leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) = 2^{p-1}(\rho(x, z) + \rho(z, y)).\end{aligned}$$

So condition (b3) of Definition 1.1 holds and ρ is a $b-$metric.

It should be noted that in the preceding example, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, let $X = \mathbb{R}$ be the set of real numbers and $d(x, y) = |x - y|$ be the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on $\mathbb{R}$ with $b = 2$, but is not a metric on $\mathbb{R}$, because the triangle inequality does not hold.

Before stating and proving our results, we present some definitions and a proposition in $b-$metric space. We recall first the notions of convergence, closedness and completeness in a $b-$metric space.

Definition 1.3 ([7])**.** Let (X, d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

Proposition 1.4 (see remark 2.1 in [7])**.** *In a $b-$metric space (X, d) the following assertions hold:*

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is Cauchy,

(iii) in general, a $b-$metric is not continuous.

Definition 1.5 ([7])**.** The $b-$metric space (X, d) is complete if every Cauchy sequence in X converges.

It should be noted that, in general a $b-$metric function $d(x, y)$ for $b > 1$ is not jointly continuous in all two of its variables. Now we present an example of a $b-$metric which is not continuous.

Example 1.6 (see example 3 in [14])**.** Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}$ be defined by

$$D(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq \frac{5}{2}(D(m, n) + D(n, p)).$$

Thus, (X, D) is $b-$metric space with $b = \frac{5}{2}$. Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$D(2n, \infty) = \frac{1}{2n} \to 0 \text{ as } n \to \infty,$$

that is, $x_n \to \infty$, but $D(x_{2n}, 1) = 2 \neq D(\infty, 1)$ as $n \to \infty$.

Since in general a $b-$metric is not continuous, we need the following simple lemmas about the b-convergent sequences.

Lemma 1.7 ([1])**.** *Let (X, d) be a $b-$metric space with $b \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y respectively, then we have*

$$\frac{1}{b^2}d(x, y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq b^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{b}d(x, z) \leq \liminf_{n\longrightarrow\infty} d(x_n, z) \leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq bd(x, z),$$

Proof. Using the triangle inequality in a $b-$metric space it is easy to see that

$$d(x, y) \leq bd(x, x_n) + b^2 d(x_n, y_n) + b^2 d(y_n, y),$$

and

$$d(x_n, y_n) \leq bd(x_n, x) + b^2 d(x, y) + b^2 d(y, y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result. Similarly, again using the triangle inequality we have:

$$d(x, z) \leq bd(x, x_n) + bd(x_n, z),$$

and

$$d(x_n, z) \leq bd(x_n, x) + bd(x, z).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the second desired result. □

In 2010, Vats *et al.* [26] introduced the concept of weakly compatible. Also, in 2010, Manro *et al.* [17] introduced the concepts of weakly commuting, R-weakly commuting mappings, and R-weakly commuting mappings of type (P), (A_f), and (A_g) in G-metric space.

We will introduce these concepts in b-metric space.

Definition 1.8. The self-mappings f and g of a $b-$metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition 1.9. A pair of self-mappings (f,g) of a $b-$metric space (X,d) are said to be

(a) R-weakly commuting mappings of type (A_f) if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx,gx), \quad for \quad all \quad x \quad in \quad X.$

(b) R-weakly commuting mappings of type (A_g) if there exists some positive real number R such that $d(gfx, ffx) \leq Rd(gx,fx), \quad for \quad all \quad x \quad in \quad X.$

Definition 1.10. The self-mapping f of a $b-$metric space (X,d) is said to be b-continuous at $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f(x_n)\}$ is b-convergent to $f(x)$.

Example 1.11. Let $d(x,y) = (x-y)^2$, $fx = 1$ and $gx = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & otherwise. \end{cases}$
Thus for each $x, y \in \mathbb{R}$ it is easy to see that the pair of self-mappings (f,g) of a $b-$metric space are R-weakly commuting mappings of type (A_f) and (A_g).

In this section, we recall some definitions of partial metric space and some of their properties. See [3, 13, 18, 20, 22, 25] for details.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2) $x = y$, but if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [12], [18].

Lemma 1.12. *Let (X,d) and (X,p) be a metric space and partial metric space respectively. Then*

(i) The function $\rho : X \times X \longrightarrow \mathbb{R}^+$ defined by $\rho(x,y) = d(x,y) + p(x,y)$, is a partial metric.

(ii) Let $\rho : X \times X \longrightarrow \mathbb{R}^+$ defined by $\rho(x,y) = d(x,y) + \max\{\omega(x), \omega(y)\}$, then ρ is a partial metric on X, where $\omega : X \longrightarrow \mathbb{R}^+$ is an arbitrary function.

(iii) Let $\rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\rho(x,y) = \max\{2^x, 2^y\}$, then ρ is a partial metric on $\mathbb{R}$.

(iv) Let $\rho : X \times X \longrightarrow \mathbb{R}^+$ defined by $\rho(x,y) = d(x,y) + a$, then ρ is a partial metric on X, where $a \geq 0$. Moreover, $\rho(x,x) = \rho(y,y)$ for all $x, y \in X$.

Each partial metric p on X generates a T_0 topology τ_p on X which has, as a base, the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Let (X,p) be a partial metric space. Then:

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Suppose that $\{x_n\}$ is a sequence in the partial metric space (X, p), then we define $L(x_n) = \{x | x_n \longrightarrow x\}$.

The following example shows that every convergent sequence $\{x_n\}$ in a partial metric space (X, p) may not be a Cauchy sequence. In particular, it shows that the limit is not unique.

Example 1.13. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Let

$$x_n = \begin{cases} 0 \quad , & n = 2k \\ \\ 1 \quad , & n = 2k + 1. \end{cases}$$

Then clearly it is convergent sequence and for every $x \geq 1$ we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$, hence $L(x_n) = [1, \infty)$. But $\lim_{n,m\to\infty} p(x_n, x_m)$ does not exist, that is it is not a Cauchy sequence.

The following Lemma shows that under certain conditions the limit is unique.

Lemma 1.14 ([23])**.** *Let $\{x_n\}$ be a convergent sequence in partial metric space (X, p), $x_n \longrightarrow x$ and $x_n \longrightarrow y$. If*

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then $x = y$.

Lemma 1.15 ([23, 15])**.** *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial metric space (X, p) such that*

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x),$$

and

$$\lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = p(y, y),$$

then $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$. In particular, $\lim_{n\to\infty} p(x_n, z) = p(x, z)$, for every $z \in X$.

Lemma 1.16. *If p is a partial metric on X, then the functions $p^s, p^m : X \times X \to \mathbb{R}^+$ given by*

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$p^m(x, y) = \max\{\ p(x, y) - p(x, x), p(x, y) - p(y, y)\ \}$$

for every $x, y \in X$, are equivalent metrics on X.

Lemma 1.17 ([18], [20])**.** *Let (X, p) be a partial metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 1.18. The self-mappings f and g of a partial metric space (X, p) are said to be compatible if $\lim_{n\to\infty} p(fgx_n, gfx_n) = p(u, u)$ for some $u \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition 1.19. A pair of self-mappings (f, g) of a partial metric space (X, p) are said to be

(a) R-weakly commuting mappings of type (A_g) if there exists some positive real number R such that $p(gfx, ffx) \leq Rp(gx, fx), \quad for \quad all \quad x \quad in \quad X.$

(b) weakly commuting mappings of type (A_g) if $p(gfx, ffx) \leq p(gx, fx), \quad for \quad all \quad x \quad in \quad X.$

2. Main results

The following is the main result of this section.

Theorem 2.1. *Let (X, d) be a b-metric space and (f, g) be a pair of non-compatible selfmappings with $\overline{fX} \subseteq gX$ (here $\overline{fX}$ denotes the closure of fX). Assume the following conditions are satisfied*

$$d(fx, fy) \leq \frac{k}{b^2}\ max\{d(gx, gy), d(fx, gx), d(fy, gy)\} \tag{2.1}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) are a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point (say z) and both f and g are not b-continuous at z.

Proof. Since f and g are non-compatible mappings, there exists a sequence $\{x_n\} \subset X$, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \quad z \in X,$$

but either $\lim_{n\to\infty} d(fgx_n, gfx_n)$ or $\lim_{n\to\infty} d(gfx_n, fgx_n)$ does not exist or exists and is different from 0. Since $z \in \overline{fX} \subset gX$, there must exist a $u \in X$ satisfying $z = gu$. We can assert that $fu = gu$. From condition (2.1) and Lemma 1.7, we get

$$\begin{aligned}
\frac{1}{b} d(fu, gu) &\leq \limsup_{n\longrightarrow\infty} d(fu, fx_n) \\
&\leq \limsup_{n\longrightarrow\infty} \frac{k}{b^2} \max\{d(gu, gx_n), d(fu, gx_n), d(fx_n, gu)\} \\
&\leq \frac{k}{b} \max\{d(gu, gu), d(fu, gu), d(gu, gu)\} \\
&= \frac{k}{b} d(fu, gu).
\end{aligned}$$

That is, $d(fu, gu) \leq kd(fu, gu)$, hence we get $fu = gu$. Since (f, g) are a pair of R-weakly commuting mappings of type (A_g), we have $d(gfu, ffu) \leq Rd(gu, fu) = 0$. It means $ffu = gfu$. Next, we prove $ffu = fu$. From condition (2.1), $fu = gu$ and $ffu = gfu$, we have

$$\begin{aligned}
d(fu, ffu) &\leq \frac{k}{b^2} \max\{d(gu, gfu), d(fu, gfu), d(gu, ffu)\} \\
&= \frac{k}{b^2} d(fu, ffu) \\
&\leq kd(fu, ffu).
\end{aligned}$$

Hence, we have $fu = ffu$, which implies that $fu = ffu = gfu$, and so $z = fu$ is a common fixed point of f and g. Next we prove that the common fixed point z is unique. Actually, suppose w is also a common fixed point of f and g, then using the condition (2.1), we have

$$\begin{aligned}
d(z, w) &= d(fz, fw) \\
&\leq \frac{k}{b^2} \max\{d(gz, gw), d(fz, gw), d(fw, gz)\} \\
&= \frac{k}{b^2} d(z, w) \\
&\leq kd(z, w),
\end{aligned}$$

which implies that $z = w$, so uniqueness is proved. Now, we prove that f and g are not b-continuous at z. In fact, if f is b-continuous at z, we consider the sequence $\{x_n\}$; then we have $\lim_{n\to\infty} ffx_n = fz = z$, $\lim_{n\to\infty} fgx_n = fz = z$. Since f and g are R-weakly commuting mappings of type Lemma 1.7 we have

$$\begin{aligned}\frac{1}{b^2}d(\lim_{n\to\infty} gfx_n, z) \leq & \limsup_{n\longrightarrow\infty} d(gfx_n, ffx_n)\\ \leq & \limsup_{n\longrightarrow\infty} Rd(gx_n, fx_n)\\ \leq & Rb^2 d(z, z) = 0,\end{aligned}$$

it follows that $\lim_{n\to\infty} gfx_n = z$. Hence, by Lemma 1.7 we can get

$$\limsup_{n\longrightarrow\infty} d(fgx_n, gfx_n) \leq b^2 d(z, z) = 0$$

therefore,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

This contradicts with f and g being non-compatible, so f is not b-continuous at z. If g is b-continuous at z, then we have

$$\lim_{n\to\infty} gfx_n = gz = z, \quad \lim_{n\to\infty} ggx_n = gz = z.$$

Since f and g are R-weakly commuting mappings of type (A_g), we get

$$d(gfx_n, ffx_n) \leq Rd(gx_n, fx_n),$$

so by Lemma 1.7 we have

$$\begin{aligned}\frac{1}{b^2}d(z, \lim_{n\to\infty} ffx_n) \leq & \limsup_{n\longrightarrow\infty} d(gfx_n, ffx_n)\\ \leq & \limsup_{n\longrightarrow\infty} Rd(gx_n, fx_n)\\ \leq & Rb^2 d(z, z) = 0,\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} ffx_n = z = fz.$$

This contradicts with f being not b-continuous at z, which implies that g is not b-continuous at z. This completes the proof. □

Corollary 2.2. *Let (X, d) be a metric space and (f, g) be a pair of non-compatible selfmappings with $\overline{fX} \subseteq gX$ (here $\overline{fX}$ denotes the closure of fX). Assume the following conditions are satisfied*

$$d(fx, fy) \leq k\ max\{d(gx, gy), d(fx, gx), d(fy, gy)\} \tag{2.2}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) are a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point (say z) and both f and g are not continuous at z.

Proof. It is enough to set $b = 1$ in Theorem 2.1. □

Corollary 2.3. *Let (X, p) be a partial metric space and (f, g) be a pair of non-compatible selfmappings with $\overline{fX} \subseteq gX$ (here $\overline{fX}$ denotes the closure of fX). Assume the following conditions are satisfied*

$$p(fx, fy) \leq k\ max\{p(gx, gy), p(fx, gx), p(fy, gy)\} \tag{2.3}$$

for all $x, y \in X$ and $0 < k < 1$. If $p(gx, gx) = p(fy, fy)$ for all $x, y \in X$ and (f, g) are a pair of weakly commuting mappings of type (A_g), then f and g have a unique common fixed point (say z) and both f and g are not continuous at z.

Proof. From condition (2.3) we have

$$2p(fx,fy) \le k\ max\{2p(gx,gy), 2p(fx,gx), 2p(fy,gy)\},$$

hence

$$\begin{aligned} &2p(fx,fy) - p(fx,fx) - p(fy,fy) + p(fx,fx) + p(fy,fy) \\ &\quad \le k\ max \left\{ \begin{array}{c} 2p(gx,gy) - p(gx,gx) - p(gy,gy) + p(gx,gx) + p(gy,gy), \\ 2p(fx,gx) - p(fx,fx) - p(gx,gx) + p(fx,fx) + p(gx,gx), \\ 2p(fy,gy) - p(fy,fy) - p(gy,gy) + p(fy,fy) + p(gy,gy) \end{array} \right\}. \end{aligned}$$

Therefore,

$$p^s(fx,fy) + p(fx,fx) + p(fy,fy) \le k\ max \left\{ \begin{array}{c} p^s(gx,gy) + p(gx,gx) + p(gy,gy), \\ p^s(fx,fy) + p(fx,fx) + p(gx,gx), \\ p^s(fy,gy) + p(fy,fy) + p(gy,gy) \end{array} \right\}.$$

Let

$$\max \left\{ \begin{array}{c} p^s(gx,gy) + p(gx,gx) + p(gy,gy), \\ p^s(fx,fy) + p(fx,fx) + p(gx,gx), \\ p^s(fy,gy) + p(fy,fy) + p(gy,gy) \end{array} \right\} = p^s(gx,gy) + p(gx,gx) + p(gy,gy).$$

In this case we have

$$p^s(fx,fy) + p(fx,fx) + p(fy,fy) \le kp^s(gx,gy) + kp(gx,gx) + kp(gy,gy).$$

Since, $p(fx,fx) = p(gy,gy)$ and $p(fy,fy) = p(gx,gx)$ it follows that

$$p^s(fx,fy) \le kp^s(gx,gy) + p(gx,gx)(k-1) + p(gy,gy)(k-1) \le kp^s(gx,gy).$$

Since,

$$\begin{aligned} kp(gx,gx) + kp(gy,gy) - p(fx,fx) - p(fy,fy) \\ &= kp(gx,gx) + kp(gy,gy) - p(gy,gy) - p(gx,gx) \\ &= p(gx,gx)(k-1) + p(gy,gy)(k-1) \le 0. \end{aligned}$$

Hence we have

$$p^s(fx,fy) \le k\ max\{p^s(gx,gy), p^s(fx,gx), p^s(fy,gy)\}.$$

Moreover, since (f,g) are a pair of weakly commuting mappings of type (A_g) in partial metric space (X,p), we have $p(gfx,ffx) \le p(gx,fx)$. Hence $2p(gfx,ffx) \le 2p(gx,fx)$, therefore

$$p^s(gfx,ffx) + p(gfx,gfx) + p(ffx,ffx) \le p^s(gx,fx) + p(gx,gx) + p(fx,fx).$$

Since, $p(gfx,gfx) = p(gx,gx)$ and $p(ffx,ffx) = p(fx,fx)$ it follows that

$$p^s(gfx,ffx) \le p^s(gx,fx).$$

That is (f,g) are a pair of R-weakly commuting mappings of type (A_g) in metric space (X,p^s) for $R=1$. Therefore, all conditions of Corollary 2.2 are satisfied, hence f and g have a unique common fixed point (say z) and both f and g are not continuous at z. □

Next, we give an example to support Theorem 2.1.

Example 2.4. Let $X = [2, 20]$ and let d be metric on $X \times X \longrightarrow (0, +\infty)$ defined as $d(x, y) = (x - y)^2$. We define mappings f and g on X by

$$fx = \begin{cases} 2, & x = 2 \ or \ x \in (5, 20] \\ 6, & x \in (2, 5], \end{cases} \quad and \ gx = \begin{cases} 2, & x = 2 \\ 18, & x \in (2, 5] \\ \frac{x+1}{3}, & x \in (5, 20]. \end{cases}$$

Clearly, from the above functions we know that $\overline{f(X)} \subseteq g(X)$, and the pair (f, g) are noncompatible self-maps. To see that f and g are non-compatible, consider a sequence $\{x_n = 5 + \frac{1}{n}\}$. We have $fx_n \longrightarrow 2, gx_n \longrightarrow 2, fgx_n \longrightarrow 6 \ \ and \ \ gfx_n \longrightarrow 2$. Thus

$$\lim_{n \to \infty} d(gfx_n, fgx_n) = 16 \neq 0.$$

On the other hand, there exists $R = 1$ such that

$$d(gfx, ffx) = \begin{cases} (2-2)^2, & x = 2 \\ (\frac{7}{3} - 2)^2, & x \in (2, 5] \\ (2-2)^2 = 0, & x \in (5, 20] \end{cases},$$

and

$$d(fx, gx) = \begin{cases} (2-2)^2 = 0, & x = 2 \\ (18-6)^2, & x \in (2, 5] \\ (\frac{x+1}{3} - 2)^2, & x \in (5, 20] \end{cases},$$

for all $x \in X$, hence it is easy to see that in every case we have

$$d(gfx, ffx) \leq d(gx, fx).$$

That is, the pair (f, g) are R-weakly commuting mappings of type (A_g). Now we prove that the mappings f and g satisfy the condition (2.1) of Theorem 2.1 with $k = \frac{1}{2}$. For this, we consider the following cases:

Case (1) If $x, y \in \{2\} \cup (5, 20]$, then we have

$$\begin{aligned} d(fx, fy) &= d(2, 2) = 0 \\ &\leq k \ max\{d(gx, gy), d(fx, gx), d(fy, gy)\}, \end{aligned}$$

and hence (2.1) is obviously satisfied.

Case (2) If $x, y \in (2, 5]$, then we have

$$\begin{aligned} d(fx, fy) &= d(6, 6) = 0 \\ &\leq k \ max\{d(gx, gy), d(fx, gx), d(fy, gy)\} \end{aligned}$$

for all x, y in X, and hence (2.1) is obviously satisfied.

Case (3) If $x \in \{2\} \cup (5, 20]$ and $y \in (2, 5]$, then we have $d(fx, fy) = d(2, 6) = 16$ and

$$d(gx, gy) = \begin{cases} (2-18)^2, & x = 2 \\ (\frac{x+1}{3} - 18)^2, & x \in (5, 20] \end{cases}.$$

Thus we obtain $[d(fx, fy) \leq k \ max\{d(gx, gy), d(fx, gx), d(fy, gy)\}]$ for all x, y in X. Thus all the conditions of Theorem 2.1 are satisfied and 2 is a unique point in X such that $f2 = g2 = 2$.

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