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Application of Schauder fixed point theorem to a coupled system of differential equations of fractional order

Mengru Hao, Chengbo Zhai

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R. China.

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Abstract

In this paper, by using Schauder fixed point theorem, we study the existence of at least one positive solution to a coupled system of fractional boundary value problems given by

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(t, y_1(t), y_2(t)) + e_1(t), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(t, y_1(t), y_2(t)) + e_2(t), \end{cases}$$

where $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, subject to the boundary conditions $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, for $0 \le i \le n-2$, and $[D_{0^+}^{\alpha} y_1(t)]_{t=1} = 0 = [D_{0^+}^{\alpha} y_2(t)]_{t=1}$, for $1 \le \alpha \le n-2$.

Keywords: Fractional differential equation, Schauder fixed point theorem, Positive solution.
2010 MSC: 47H10, 34A08, 34B18.

1. Introduction

In this paper, we are interested in the existence of positive solution of the following coupled system of fractional boundary value problems(FBVPs for short) given by:

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(t, y_1(t), y_2(t)) + e_1(t), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(t, y_1(t), y_2(t)) + e_2(t), \end{cases} \tag{1.1}$$

*Chengbo Zhai, Corresponding author
Email addresses: mengru314314@126.com (Mengru Hao), cbzhai@sxu.edu.cn (Chengbo Zhai)

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where $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, and $\lambda_1, \lambda_2 > 0$, with the boundary value condition:

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), 0 \le i \le n-2, \tag{1.2}$$

$$[D_{0^+}^{\alpha} y_1(t)]_{t=1} = 0 = [D_{0^+}^{\alpha} y_2(t)]_{t=1}, 1 \le \alpha \le n-2, \tag{1.3}$$

where $f, g \in C([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$, $a_1, a_2, e_1, e_2 \in C([0,1], [0,\infty))$.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. The fractional differential equations play an important role in various fields of science and engineering. With the help of fractional calculus, we can describe the natural phenomena and mathematical model more accurately. So, the fractional differential equations have received much attention and the theory and application have been greatly developed (see [1]-[25]). Recently, the existence of solutions of the initial and boundary value problems for nonlinear fractional equations are extensively studied (see [1],[2],[5]-[7],[9],[10],[18]-[22]), and some are coupled systems of nonlinear fractional differential equations (see [2],[5],[6],[9],[19],[20]).

In [9], by using Krasnoselskii fixed point theorem, C.S. Goodrich considered:

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \end{cases} \tag{1.4}$$

with the same boundary value problem like (1.2) and (1.3), and establish the existence of at least one positive solution.

In [18], M.ur Rehman, R.A. Khan investigated the existence of solutions for a class of nonlinear multi-point boundary value problems for fractional differential equations:

$$\begin{cases} D_t^{\alpha} y(t) = f(t, y(t), D_t^{\beta} y(t)), \quad t \in (0,1), \\ y(0) = 0, \quad D_t^{\beta} y(1) - \Sigma_{i=1}^{m-2} \zeta_i D_t^{\beta} y(\xi_i) = y_0, \end{cases} \tag{1.5}$$

where $1 < \alpha \le 2$, $0 < \beta < 1$, $0 < \xi_i < 1 (i = 1, 2, \ldots, m-2)$, $\zeta_i \ge 0$ with $\gamma = \sum\limits_{i=1}^{m-2} \zeta_i \xi_i^{\alpha-\beta-1} < 1$. Their analysis relied on the Schauder fixed point theorem.

In [19], S.R. Sun, Q.P. Li, Y.N Li considered an initial value problem for a coupled system of multi-term nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha} u(t) = f(t, v(t), D^{\beta_1} v(t), \ldots, D^{\beta_N} v(t)), \quad D^{\alpha-i} u(0) = 0, \quad i = 1, 2, \ldots, n_1, \\ D^{\sigma} v(t) = g(t, u(t), D^{\rho_1} u(t), \ldots, D^{\rho_N} u(t)), \quad D^{\sigma-j} v(0) = 0, \quad j = 1, 2, \ldots, n_2, \end{cases} \tag{1.6}$$

where $t \in (0,1]$, $\alpha > \beta_1 > \beta_2 > \cdots > \beta_N > 0$, $\sigma > \rho_1 > \rho_2 > \cdots > \rho_N > 0$, $n_1 = \alpha + 1$, $n_2 = \sigma + 1$ for $\alpha, \sigma \notin N$ and $n_1 = \alpha$, $n_2 = \sigma$ for $\alpha, \sigma \in N$, $\beta_q, \rho_q < 1$ for any $q \in \{1, 2, \ldots, N\}$. Also, by means of the Schauder fixed point theorem, an existence result for the solution are obtained.

In [20], X.W. Su studied a boundary value problem of the coupled system:

$$\begin{cases} D^{\alpha} u(t) = f(t, v(t), D^{\mu} v(t)), \quad 0 < t < 1, \\ D^{\beta} v(t) = f(t, u(t), D^{\nu} u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.7}$$

where $1 < \alpha, \beta < 2$, $\mu, \nu > 0$, $\alpha - \nu \ge 1$, $\beta - \mu \ge 1$. Due to the Schauder fixed point theorem, an existence result for the solution is obtained.

In our paper, we also utilize Schauder fixed point theorem to obtain three results on the existence of positive solutions for the system of fractional boundary value problem (1.1)-(1.3).

With this context in mind, the outline of this paper is as follows. In Section 2 we shall recall certain results from the theory of the continuous fractional calculus. In Section 3 we shall provide some conditions under which FBVPs (1.1)-(1.3) will have at least one positive solution.

2. Preliminaries

We first wish to collect some basic lemmas that will be important to us in the sequel.

Definition 2.1. ([10]) Let $\nu > 0$ with $\nu \in R$. Suppose that $y : [a, \infty) \to R$. Then the ν-th Riemann-Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1} ds,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in R$, we define the ν-th Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}} ds,$$

where $n \in N$ is the unique positive integer satisfying $n-1 \le \nu < n$ and $t > a$.

Lemma 2.2. *([8]) Let $g \in C^n([0,1])$ be given. Then the unique solution to problem $-D_{0+}^{\nu} y(t) = g(t)$ together with the boundary conditions $y^{(i)}(0) = 0 = [D_{0+}^{\alpha} y(t)]_{t=1}$, where $1 \le \alpha \le n-2$ and $0 \le i \le n-2$, is*

$$y(t) = \int_0^1 G(t,s) g(s) ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, 0 \le s \le t \le 1, \\ \frac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

is the Green function for this problem.

Lemma 2.3. *([8]) Let G(t,s) be as given in the statement of Lemma 2.2. Then we find that:*

(i) $G(t,s)$ is a continuous function on the unit square $[0,1] \times [0,1]$;
(ii) $G(t,s) \ge 0$ for each $(t,s) \in [0,1] \times [0,1]$;
(iii) $\max_{t \in [0,1]} G(t,s) = G(1,s)$, for each $s \in [0,1]$.

Lemma 2.4. *([8]) Let G(t,s) be as given in the statement of Lemma 2.2. Then there exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t \in [\frac{1}{2},1]} G(t,s) \ge \gamma \max_{t \in [0,1]} G(t,s) = \gamma G(1,s). \tag{2.3}$$

3. Main results

In this paper, let E represent the Banach space of $C([0,1])$ when equipped with the usual supremum norm, $\|\cdot\|$. Then put $X := E \times E$, where X is equipped with the norm $\|(y_1, y_2)\| := \|y_1\| + \|y_2\|$ for $(y_1, y_2) \in X$. Observe that X is also a Banach space (see [8]). In addition, define the operators $T_1, T_2 : X \to E$ by

$$(T_1(y_1, y_2))(t) := \int_0^1 G_1(t,s)[\lambda_1 a_1(s) f(s, y_1(s), y_2(s)) + e_1(s)] ds$$

and

$$(T_2(y_1, y_2))(t) := \int_0^1 G_2(t,s)[\lambda_2 a_2(s) g(s, y_1(s), y_2(s)) + e_2(s)] ds,$$

where $G_1(t,s)$ is the Green function of Lemma 2.2 with ν replaced by ν_1 and, likewise, $G_2(t,s)$ is the Green function of Lemma 2.2 with ν replaced by ν_2. Now, we define an operator $S : X \to X$ by

$$\begin{aligned}(S(y_1,y_2))(t) : &= ((T_1(y_1,y_2))(t), (T_2(y_1,y_2))(t)) \\ &= (\textstyle\int_0^1 G_1(t,s)[\lambda_1 a_1(s) f(s,y_1(s),y_2(s)) + e_1(s)]ds, \\ &\quad \textstyle\int_0^1 G_2(t,s)[\lambda_2 a_2(s) g(s,y_1(s),y_2(s)) + e_2(s)]ds).\end{aligned} \tag{3.1}$$

We claim that whenever $(y_1,y_2) \in X$ is a fixed point of the operator defined in (1.1), it follows that $y_1(t)$ and $y_2(t)$ solve FBVPs (1.1)-(1.3).

We shall look for fixed points of the operator S, seeing as these fixed points coincide with solutions of FBVPs (1.1)-(1.3). For use in the sequel, let γ_1 and γ_2 the constants given by Lemma 2.4 associated, respectively, to the Green functions G_1 and G_2.

Next, we suppose that $f : [0,1] \times R^+ \to R$ is a Caratheodory function, that is,

(i) for almost all $t \in [0,1]$, $f(t,\cdot) : R^+ \to R$ is continuous;
(ii) for every $l \in R^+$, $f(\cdot,l) : [0,1] \to R$ is measurable.

We give a notation: if for almost all $t \in [0,1]$, $b \geq 0$, $b \in L^1(0,1)$, we denote $b \succ 0$.

For the sake of convenience, we set

$$r^* = \max\left\{\sup_{t\in[0,1]}\int_0^1 \frac{G_1(t,s)}{t^{\nu_1-1}}e_1(s)ds, \quad \sup_{t\in[0,1]}\int_0^1 \frac{G_2(t,s)}{t^{\nu_2-1}}e_2(s)ds\right\},$$

$$r_* = \min\left\{\inf_{t\in[0,1]}\int_0^1 \frac{G_1(t,s)}{t^{\nu_1-1}}e_1(s)ds, \quad \inf_{t\in[0,1]}\int_0^1 \frac{G_2(t,s)}{t^{\nu_2-1}}e_2(s)ds\right\}.$$

Now, we define a set P by

$$P := \{(y_1,y_2) \in X : y_1(t), y_2(t) \geq 0, \ t \in [0,1]\}.$$

Now we consider the problem (1.1)-(1.3), where f is a Caratheodory function.

Case 1. $r_* > 0$**.**

Theorem 3.1. *Suppose that there exist $b \succ 0$ and $\lambda > 0$ such that*

$$0 \leq f(t,y_1,y_2), \quad g(t,y_1,y_2) \leq \frac{b(t)}{y_1^\lambda}, \quad \forall (y_1,y_2) \in P \text{ and } y_1 \neq 0, t \in [0,1],$$

where

$$\lambda_1 \int_0^1 \frac{G_1(1,s)a_1(s)b(s)}{s^{\lambda(\overline{\nu}-1)}}ds, \quad \lambda_2 \int_0^1 \frac{G_2(1,s)a_2(s)b(s)}{s^{\lambda(\overline{\nu}-1)}}ds < +\infty, \tag{3.2}$$

where $\overline{\nu} = \max\{\nu_1,\nu_2\}$. If $r_* > 0$, then FBVPs (1.1)-(1.3) have at least one positive solution.

Proof. Let $\Omega = \{(y_1,y_2) \in P : t^{\overline{\nu}-1}r \leq y_1(t), y_2(t) \leq t^{\nu^*-1}R, \forall t \in [0,1]\}$, where $\nu^* = \min\{\nu_1,\nu_2\}$ and $R > r > 0$ are undetermined positive constants. Then Ω is a bounded convex closed set.

Furthermore, a relatively straightforward application of the Arzela-Ascoil theorem, which we omit, reveals that S is a completely continuous operator. Next, we show that $S(\Omega) \subset \Omega$. In fact, we fix $r := r_*$ and from assumption, we have $r > 0$. For all $t \in [0,1]$ and $(y_1,y_2) \in \Omega$, we get

$$T_1(y_1,y_2)(t) \geq \int_0^1 G_1(t,s)e_1(s)ds \geq t^{\nu_1-1}r_* = t^{\nu_1-1}r \geq t^{\overline{\nu}-1}r.$$

On the other hand, we set

$$\beta^* = \max\left\{\lambda_1\frac{1}{t^{\nu_1-1}}\int_0^1 \frac{G_1(1,s)a_1(s)b(s)}{s^{\lambda(\overline{\nu}-1)}}ds, \quad \lambda_2\frac{1}{t^{\nu_2-1}}\int_0^1 \frac{G_2(1,s)a_2(s)b(s)}{s^{\lambda(\overline{\nu}-1)}}ds\right\}.$$

Hence,

$$\begin{aligned} T_1(y_1,y_2)(t) &\le \lambda_1\int_0^1 G_1(1,s)a_1(s)f(s,y_1(s),y_2(s))ds+\int_0^1 G_1(t,s)e_1(s)ds \\ &\le \lambda_1\int_0^1 \frac{G_1(1,s)a_1(s)b(s)}{y_1^\lambda(s)}ds+r^*t^{\nu_1-1} \\ &\le t^{\nu_1-1}(\frac{\beta^*}{r^\lambda}+r^*)\le t^{\nu^*-1}(\frac{\beta^*}{r^\lambda}+r^*). \end{aligned}$$

Set $R=(\frac{\beta^*}{r^\lambda}+r^*)$, then we have $T_1(y_1,y_2)(t)\ge t^{\overline{\nu}-1}r$ and $T_1(y_1,y_2)(t)\le t^{\nu^*-1}R$. Similarly, we get $T_2(y_1,y_2)(t)\ge t^{\overline{\nu}-1}r$ and $T_2(y_1,y_2)(t)\le t^{\nu^*-1}R$.

Consequently, $S(\Omega)\subset\Omega$. In summary, each of the conditions of Schauder fixed point theorem is satisfied. The proof is complete. □

Case 2. $r_*=0$.

Theorem 3.2. *Suppose that there exist $b\succ 0$, $\hat{b}\succ 0$ and $1>\lambda>0$, such that*

$$\frac{\hat{b}(t)}{y_1^\lambda}\le f(t,y_1,y_2),\quad g(t,y_1,y_2)\le\frac{b(t)}{y_1^\lambda},\quad \forall (y_1,y_2)\in P \text{ and } y_1\ne 0, t\in[0,1] \tag{3.3}$$

and (3.2) *is satisfied. If $r_*=0$, then FBVPs* (1.1)-(1.3) *have at least one positive solution.*

Proof. Like Theorem 3.1, we just need to search the fixed $0<r<R$, such that $S(\Omega)\subset\Omega$. Similarly, we have

$$T_1(y_1,y_2)(t)\le t^{\nu^*-1}(\frac{\beta^*}{r^\lambda}+r^*).$$

On the other hand, set

$$\hat{\beta}_*=\min\left\{\lambda_1\gamma_1\frac{1}{t^{\nu_1-1}}\int_{\frac12}^1\frac{G_1(1,s)a_1(s)\hat{b}(s)}{s^{\lambda(\nu^*-1)}}ds,\quad \lambda_2\gamma_2\frac{1}{t^{\nu_2-1}}\int_{\frac12}^1\frac{G_2(1,s)a_2(s)\hat{b}(s)}{s^{\lambda(\nu^*-1)}}ds\right\},$$

where γ_1,γ_2 are defined by Lemma 2.4. So,

$$\begin{aligned} T_1(y_1,y_2)(t) &\ge \lambda_1\int_0^1 G_1(t,s)a_1(s)f(s,y_1(s),y_2(s))ds \\ &\ge \lambda_1\int_{\frac12}^1 G_1(t,s)a_1(s)f(s,y_1(s),y_2(s))ds \\ &\ge \lambda_1\gamma_1\int_{\frac12}^1 G_1(1,s)a_1(s)\frac{\hat{b}(s)}{y_1^\lambda(s)}ds \\ &\ge t^{\nu_1-1}\frac{\hat{\beta}_*}{R^\lambda}\ge t^{\overline{\nu}-1}\frac{\hat{\beta}_*}{R^\lambda}. \end{aligned}$$

Hence, we only need to find r and R satisfying $\frac{\beta^*}{r^\lambda}+r^*\le R$ and $\frac{\hat{\beta}_*}{R^\lambda}\ge r$. Like Theorem 3.1, we can obtain the conclusion. □

Case 3. $r^*<0$.

Theorem 3.3. *Suppose that there exist $b\succ 0$, $\hat{b}\succ 0$ and $1>\lambda>0$, such that* (3.2) *and* (3.3) *are satisfied. If $r^*<0$ with*

$$r_*\ge[\frac{\hat{\beta}_*}{(\beta^*)^\lambda}\lambda^2]^{\frac{1}{1-\lambda^2}}(1-\frac{1}{\lambda^2}), \tag{3.4}$$

then FBVPs (1.1)-(1.3) *have at least one positive solution.*

Proof. Like Theorem 3.2, we only need to find $0<r<R$ satisfying

$$\frac{\beta^*}{r^\lambda}\le R,\quad \frac{\hat{\beta}_*}{R^\lambda}+r_*\ge r. \tag{3.5}$$

Let $R = \frac{\beta^*}{r^\lambda}$, if $\frac{\hat{\beta}_*}{(\beta^*)^\lambda} r^{\lambda^2} + r_* \geq r$, then $\frac{\hat{\beta}_*}{R^\lambda} + r_* \geq r$ is satisfied. Or it is equal to

$$r_* \geq f(r) := r - \frac{\hat{\beta}_*}{(\beta^*)^\lambda} r^{\lambda^2}.$$

It is easy to prove that f has the minimum value at r_0, where $r_0 = [\frac{\lambda^2 \hat{\beta}_*}{(\beta^*)^\lambda}]^{\frac{1}{1-\lambda^2}}$.
Now, set $r = r_0$, thus when $r_* \geq f(r_0)$, (3.5) is satisfied, that is to say, (3.4) is satisfied. The proof is complete. □

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