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A viscosity method for solving convex feasibility problems

Yunpeng Zhang[a,*], Yanling Li[b]

[a]College of Electric Power, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China.
[b]School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power University, Zhengzhou 450011, China.

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Abstract

In this paper, generalized equilibrium problems and strict pseudocontractions are investigated based on a viscosity algorithm. Strong convergence theorems are established in the framework of real Hilbert spaces.

Keywords: Equilibrium problem, variational inequality, nonexpansive mapping, fixed point, viscosity algorithm.
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1. Introduction

Fixed point and equilibrium problems have been extensively studied based on iterative algorithms because of its applications in nonlinear analysis, optimization, economics, game theory, mechanics, medicine and so forth, see [1, 2, 8, 9, 10, 11, 12, 15, 19] and the references therein. Viscosity algorithms are first introduced by Moudafi [18] in Hilbert spaces to study fixed points of nonexpansive mappings. The fixed point of nonexpansive mappings is revealed that it is also a unique solution of some variational inequality. The viscosity algorithms recently were extensively studied by many authors in different spaces, for more detail; see [5]-[7], [13, 14, 20, 21, 24, 23, 27] and the references therein.

In this paper, we consider the problem of approximating a common element of fixed point sets of strict pseudocontractions and solution sets of generalized equilibrium problems. Theorems of strong convergence are established in real Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity algorithm is proposed and analyzed. Theorems of strong convergence are established, too. Some corollaries are also provided.

*Corresponding author
Email addresses: zhangypliyl@yeah.net (Yunpeng Zhang), hsliyl@sina.com (Yanling Li)

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2. Preliminaries

From now on, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ and that C is a nonempty closed convex subset of H. P_C denotes the metric projection from H onto C.

Let S be a mapping on C. $F(S)$ stands for the fixed point set of S. Recall that S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be κ-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Brower and Petryshyn [4]. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we also call it an α-strongly monotone mapping. A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also call it an α-inverse-strongly monotone mapping. It is clear that A is inverse-strongly monotone if and only if A^{-1} is strongly monotone.

Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

It is known that $x \in C$ is a solution to (2.1) if and only if x is a fixed point of the mapping $P_C(I - rA)$, where $r > 0$ is a constant and I is the identity mapping. Recently, projection methods have been intensively investigated for solving solutions of variational inequality (2.1) by many authors in the framework of Hilbert spaces.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $A : C \to H$ an inverse-strongly monotone mapping. In this paper, we consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$, *i.e.*,

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C\}.$$

To study generalized equilibrium problem (2.2), we may assume that F satisfies the following conditions:

A1. $F(x, x) = 0$ for all $x \in C$;

A2. F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

A3. for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

A4. for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

If $F \equiv 0$, then generalized equilibrium problem (2.2) is reduced to classical variational inequality (2.1).

If $A \equiv 0$, then generalized equilibrium problem (2.2) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{2.3}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, *i.e.*,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Recently, problems (2.1), (2.2) and (2.3) were studied based on Halpern-like methods by many authors; see [17], [22], [26], [28]–[31] and the references therein. The advantage of Halpern-like methods is that compact assumptions are relaxed due to contractive conditions. Motivated by the research going on this direction, we study the problem of solving common solutions of generalized equilibrium problem (2.2) and fixed points of a strict pseudocontraction. Possible computation errors are taken into account. Strong convergence theorems are established in the framework of real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 ([3]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 2.2 ([4]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $S : C \to C$ be a strict pseudocontraction. Then $I - S$ is demi-closed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 2.3 ([4]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $S : C \to C$ be a κ-strict pseudocontraction. Define a mapping T by $T = \lambda I + (1 - \lambda)S$, where λ is a constant in $(0, 1)$. If $\lambda \in [\kappa, 1)$ then T is nonexpansive and $F(T) = F(S)$.*

Lemma 2.4 ([25]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 ([16]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + e_n,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$, $\{e_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\sum_{n=1}^{\infty} |e_n| < \infty$;

(iii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

The following lemma was proved in [26]. For the sake of completeness, we still give the proof.

Lemma 2.6. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying A1-A4. T_r is defined as in* Lemma 2.1. *Then*

$$\|T_s x - T_t x\| \leq \frac{|s-t|}{s} |T_s x - x|.$$

Proof. Put $u = T_s x$ and $v = T_t x$. It follows that $F(u,v) + \frac{1}{s}\langle v-u, u-x\rangle \geq 0$ and $F(v,u) + \frac{1}{t}\langle u-v, v-x\rangle \geq 0$. Hence, we have

$$\frac{1}{s}\langle v-u, u-x\rangle + \frac{1}{t}\langle u-v, v-x\rangle \geq 0.$$

This implies that $\langle u-v, u-x-\frac{t}{s}(u-x)\rangle \geq 0$. It follows that

$$\|u-v\|^2 \leq \frac{s-t}{s}\langle u-v, u-x\rangle.$$

This proves this lemma. □

3. Main results

Theorem 3.1. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies A1-A4. Let $S : C \to C$ be a κ-strict pseudocontraction and let $f : C \to C$ be a μ-contraction. Assume that $F(S) \cap EP(F, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y-y_n\rangle + \frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\big(\lambda_n y_n + (1-\lambda_n)Sy_n\big) + \delta_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

a. $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

b. $\lim_{n\to\infty} \alpha_n =$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

c. $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

d. $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

e. $0 < \kappa < \lambda_n \leq \lambda < 1$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where λ, r, r' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP(F,A)} f(\bar{x})$.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Let $p \in F(S) \cap EP(F, A)$ be fixed arbitrarily. For any $x, y \in C$, we see that

$$\begin{aligned}\|(I-r_nA)x-(I-r_nA)y\|^2 &= \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax-Ay\|^2.\end{aligned}$$

Using restriction e, we see that $\|(I-r_nA)x-(I-r_nA)y\| \le \|x-y\|$. This proves that $I-r_nA$ is nonexpansive. Put $S_n = \lambda_n I + (1-\lambda_n)S$. It follows from Lemma 2.3 that S_n is nonexpansive and $F(S_n) = F(S)$. Hence, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\| + \beta_n\|x_n-p\| + \gamma_n\|S_ny_n-p\| + \delta_n\|e_n-p\|\\ &\le \alpha_n\mu\|x_n-p\| + \alpha_n\|f(p)-p\| + \beta_n\|x_n-p\| + \gamma_n\|y_n-p\| + \delta_n\|e_n-p\|\\ &\le (1-\alpha_n(1-\mu))\|x_n-p\| + \alpha_n\|f(p)-p\| + \delta_n\|e_n-p\|\\ &\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\mu}\right\} + \delta_n\|e_n-p\|\\ &\le \max\left\{\|x_{n-1}-p\|, \frac{\|f(p)-p\|}{1-\mu}\right\} + \delta_{n-1}\|e_{n-1}-p\| + \delta_n\|e_n-p\|\\ &\le \cdots\\ &\le \max\left\{\|x_1-p\|, \frac{\|f(p)-p\|}{1-\mu}\right\} + M\sum_{i=1}^{\infty}\delta_i,\end{aligned}$$

where $M = \sup_{n\ge1}\{\|e_n-p\|\}$. This shows that $\{x_n\}$ is bounded, so is $\{y_n\}$. Putting $z_n = (I-r_nA)x_n$, we see that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|(I-r_{n+1}A)x_{n+1} - (I-r_{n+1}A)x_n\| + \|(I-r_{n+1}A)x_n - (I-r_nA)x_n\|\\ &\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|Ax_n\|.\end{aligned}$$

This implies from Lemma 2.6 that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|T_{r_{n+1}}z_{n+1} - T_{r_n}z_n\|\\ &\le \|T_{r_{n+1}}z_{n+1} - T_{r_{n+1}}z_n\| + \|T_{r_{n+1}}z_n - T_{r_n}z_n\|\\ &\le \|z_{n+1}-z_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}z_n - z_n\|\\ &\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|Ax_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}z_n - z_n\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|S_{n+1}y_{n+1} - S_ny_n\| &\le \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_ny_n\|\\ &\le \|y_{n+1}-y_n\| + \|S_{n+1}y_n - S_ny_n\|\\ &\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|Ax_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\|T_{r_{n+1}}z_n - z_n\|\\ &\quad + |\lambda_{n+1}-\lambda_n|\|Sy_n - y_n\|.\end{aligned} \tag{3.1}$$

Let $\zeta_n = \frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$. It follows that

$$\begin{aligned}\zeta_{n+1}-\zeta_n &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}S_{n+1}y_{n+1} + \delta_{n+1}e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_nf(x_n) + \gamma_nS_ny_n + \delta_ne_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}(f(x_{n+1}) - S_{n+1}y_n) + (1-\beta_{n+1})S_{n+1}y_{n+1} + \delta_{n+1}(e_{n+1} - S_{n+1}y_n)}{1-\beta_{n+1}}\\ &\quad - \frac{\alpha_n(f(x_n) - S_ny_n) + (1-\beta_n)S_ny_n + \delta_n(e_n - S_ny_n)}{1-\beta_n}\\ &= \frac{\alpha_{n+1}(f(x_{n+1}) - S_{n+1}y_{n+1})}{1-\beta_{n+1}} + \frac{\delta_{n+1}(e_{n+1} - S_{n+1}y_{n+1})}{1-\beta_{n+1}}\end{aligned}$$

$$- \frac{\alpha_n(f(x_n) - S_n y_n)}{1-\beta_n} - \frac{\delta_n(e_n - S_n y_n)}{1-\beta_n} + S_{n+1}y_{n+1} - S_n y_n.$$

This implies from (3.1) that

$$\begin{aligned}
&\|\zeta_{n+1} - \zeta_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}\|f(x_{n+1}) - S_{n+1}y_{n+1}\|}{1-\beta_{n+1}} + \frac{\delta_{n+1}\|e_{n+1} - S_{n+1}y_{n+1}\|}{1-\beta_{n+1}} \\
&\quad + \frac{\alpha_n\|f(x_n) - S_n y_n\|}{1-\beta_n} + \frac{\delta_n\|e_n - S_n y_n\|}{1-\beta_n}\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|T_{r_{n+1}}z_n - z_n\| + |\lambda_{n+1} - \lambda_n|\|Sy_n - y_n\|.
\end{aligned}$$

It follows from restrictions b-e that

$$\limsup_{n\to\infty}(\|\zeta_{n+1} - \zeta_n\| - \|x_{n+1} - x_n\|) \le 0.$$

With the aid of Lemma 2.4, we see that $\lim_{n\to\infty}\|\zeta_n - x_n\| = 0$, which in turn implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.2}$$

Notice that

$$\begin{aligned}
\|y_n - p\|^2 &\le \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + {r_n}^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.3}$$

Since $\|\cdot\|^2$ is convex, we see from (3.3) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n y_n - p\|^2 + \delta_n\|e_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + \delta_n\|e_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - r_n(2\alpha - r_n)\gamma_n\|Ax_n - Ap\|^2 + \delta_n\|e_n - p\|^2.
\end{aligned}$$

This yields that

$$\begin{aligned}
&r_n(2\alpha - r_n)\gamma_n\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n\|e_n - p\|^2 \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - p\|^2 + \delta_n\|e_n - p\|^2.
\end{aligned}$$

In view of restrictions b-e, we obtain from (3.2) that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\le \langle (x_n - r_n Ax_n) - (p - r_n Ap), y_n - p\rangle \\
&= \frac{1}{2}\big(\|(x_n - r_n Ax_n) - (p - r_n Ap)\|^2 + \|y_n - p\|^2 \\
&\quad - \|(x_n - r_n Ax_n) - (p - r_n Ap) - (y_n - p)\|^2 \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n - r_n(Ax_n - Ap)\|^2\big) \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 - r_n^2\|Ax_n - Ap\|^2 \\
&\quad + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|\big) \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|\big).
\end{aligned}$$

It follows that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|.$$

This further implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n y_n - p\|^2 + \delta_n\|e_n - p\|^2\\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + \delta_n\|e_n - p\|^2\\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2 + \delta_n\|e_n - p\|^2\\ &\quad + 2r_n\gamma_n\|x_n - y_n\|\|Ax_n - Ap\|,\end{aligned}$$

which yields that

$$\begin{aligned}\gamma_n\|x_n - y_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2r_n\gamma_n\|x_n - y_n\|\|Ax_n - Ap\| + \delta_n\|e_n - p\|^2\\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|\\ &\quad + 2r_n\gamma_n\|x_n - y_n\|\|Ax_n - Ap\| + \delta_n\|e_n - p\|^2.\end{aligned}$$

In view of restrictions b-e, we obtain from (3.2) and (3.4) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.5}$$

Notice that

$$\gamma_n(S_n y_n - x_n) = (x_{n+1} - x_n) + \alpha_n(x_n - f(x_n)) + \delta_n(x_n - e_n).$$

By use of restrictions b-d, we obtain from (3.2) that

$$\lim_{n\to\infty}\|x_n - S_n y_n\| = 0. \tag{3.6}$$

Since $\|S_n x_n - x_n\| \le \|S_n x_n - S_n y_n\| + \|S_n y_n - x_n\|$, and S_n is nonexpansive, we see from (3.5) and (3.6) that

$$\lim_{n\to\infty}\|x_n - S_n x_n\| = 0. \tag{3.7}$$

Notice that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - \big(\lambda_n x_n + (1-\lambda_n)Sx_n\big)\|\\ &\quad + \|\big(\lambda_n x_n + (1-\lambda_n)Sx_n\big) - x_n\|\\ &\le \lambda_n\|Sx_n - x_n\| + \|S_n x_n - x_n\|.\end{aligned}$$

It follows from (3.7) and restriction e that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.8}$$

Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \le 0$, where $\bar{x} = P_{F(S)\cap EP(F,A)}f(\bar{x})$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\rangle.$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly some point x. We may assume, without loss of generality, that $\{x_{n_i}\}$ converges weakly to x. Now, we are in a position to show $x \in F(S)\cap EP(F,A)$. By use of Lemma 2.2, we see that $x \in F(S)$.

Next, we show $x \in EP(F,A)$. From (3.5), we see that $\{y_{n_i}\}$ converges weakly to x. It follows that

$$F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

By use of condition A2, we see that

$$\langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq F(y, y_n), \quad \forall y \in C.$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, y - y_{n_i}\rangle + \langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, y_{n_i}), \quad \forall y \in C. \tag{3.9}$$

For t with $0 < t \leq 1$ and $y \in C$, let $u_t = ty + (1-t)x$. Since $y \in C$ and $x \in C$, we have $u_t \in C$. In view of (3.9), we find that

$$\begin{aligned}\langle u_t - y_{n_i}, Au_t\rangle &\geq \langle u_t - y_{n_i}, Au_t\rangle - \langle Ax_{n_i}, u_t - y_{n_i}\rangle - \langle u_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\ &\quad + F(u_t, y_{n_i}) \\ &= \langle u_t - y_{n_i}, Au_t - Ay_{n_i}\rangle + \langle u_t - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle \\ &\quad - \langle u_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(u_t, y_{n_i}).\end{aligned}$$

Since A is monotone, we see that $\langle u_t - y_{n_i}, Au_t - Ay_{n_i}\rangle \geq 0$. By use of condition A4, we arrive at

$$\langle u_t - x, Au_t\rangle \geq F(u_t, x). \tag{3.10}$$

Using conditions A1 and A4, we find from (3.10) that

$$\begin{aligned}0 = F(u_t, u_t) &\leq tF(u_t, y) + (1-t)F(u_t, x) \\ &\leq tF(u_t, y) + (1-t)\langle u_t - x, Au_t\rangle \\ &= tF(u_t, u) + (1-t)t\langle y - x, Au_t\rangle.\end{aligned}$$

Hence, we have $F(u_t, y) + (1-t)\langle y - x, Au_t\rangle \geq 0$. Letting $t \to 0$, we find

$$F(x, y) + \langle y - x, Ax\rangle \geq 0,$$

which implies that $x \in EP(F, A)$. It follows that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0.$$

Note that

$$\begin{aligned}&\|x_{n+1} - \bar{x}\|^2 \\ &\leq \alpha_n\langle f(x_n) - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\quad + \gamma_n\|S_n y_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \delta_n\|e_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\leq \alpha_n\langle f(x_n) - f(\bar{x}), x_{n+1} - \bar{x}\rangle + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\quad + \gamma_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \delta_n\|e_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\leq \frac{\alpha_n\mu + \beta_n + \gamma_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad + \frac{\delta_n}{2}(\|e_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2).\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \leq \big(1 - \alpha_n(1-\mu)\big)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\ + \delta_n\|e_n - \bar{x}\|.\end{aligned}$$

By use of Lemma 2.5, we find that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof. □

If S is nonexpansive, we draw from Theorem 3.1 the following result.

Corollary 3.2. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies A1-A4. Let $S : C \to C$ be a nonexpansive mapping and let $f : C \to C$ be a μ-contraction. Assume that $F(S) \cap EP(F, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n + \delta_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

a. $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

b. $\lim_{n\to\infty} \alpha_n =$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

c. $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

d. $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

e. $0 < r \leq r_n \leq r' < 2\alpha$,

where r, r' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP(F,A)} f(\bar{x})$.

Further, if S is the identity on C, then we have the following result on generalized equilibrium problem (2.2).

Corollary 3.3. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies A1-A4. Let $f : C \to C$ be a μ-contraction. Assume that $EP(F, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n + \delta_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

a. $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

b. $\lim_{n\to\infty} \alpha_n =$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

c. $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

d. $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

e. $0 < r \leq r_n \leq r' < 2\alpha$,

where r, r' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{EP(F,A)} f(\bar{x})$.

Next, we give a result on equilibrium problem (2.3).

Corollary 3.4. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies A1-A4. Let $S : C \to C$ be a κ-strict pseudocontraction and let $f : C \to C$ be a μ-contraction. Assume that $F(S) \cap EP(F) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\big(\lambda_n y_n + (1-\lambda_n)Sy_n\big) + \delta_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

a. $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

b. $\lim_{n\to\infty} \alpha_n =$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

c $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

d. $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

e. $0 < \kappa < \lambda_n \leq \lambda < 1$ *and* $0 < r \leq r_n$,

where λ, r, r' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP(F)} f(\bar{x})$.

Finally, we give a result on common solutions of solution sets of variational inequality (2.1) and fixed point set of a strict pseudocontraction.

Corollary 3.5. *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Let $S : C \to C$ be a κ-strict pseudocontraction and let $f : C \to C$ be a μ-contraction. Assume that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - r_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\big(\lambda_n y_n + (1-\lambda_n)Sy_n\big) + \delta_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

a. $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

b. $\lim_{n\to\infty} \alpha_n =$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

c. $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

d. $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

e. $0 < \kappa < \lambda_n \leq \lambda < 1$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where λ, r, r' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap VI(F,A)} f(\bar{x})$.

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