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Cone-adapted continuous shearlet transform and reconstruction formula

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Abstract

The shearlet system generated by unitary representation of the shearlet group becomes unattractive due to biasedness towards one axis. Therefore, in this paper we study the cone-adapted shearlet system to cover whole $\mathbb{R}^2$ and for giving equal treatment of all directions. Since the horizontal and vertical cones are treated similarly by just interchanging w_1 and w_2,$w = (w_1, w_2) \in \mathbb{R}^2$, we study only horizontal cone and derived some basic results concerning to continuous shearlet transform.

Keywords: Shearlets, continuous shearlet transform, cone-adapted shearlet system, Parseval formula.
2010 MSC: 42C40, 68U10, 47G20.

1. Introduction

The wavelet gave the understanding of many problems in various sciences, engineering and other disciplines.The n-dimensional continuous wavelet transform is able to describe the local regularity of functions and distribution and detect the location of singularity points through it decay at fine scale, it does not provide additional information about the geometry of the set of singularities. Several constructions have been introduced, starting with the wedgelets [4] and ridgelets [1]. Among the most successful constructions proposed in the literature, the curvelets [2] and shearlets [6] achieve this additional flexibility by defining a collection of analyzing functions ranging not only over various scales and locations, like traditional wavelets, but also over various orientations and with highly anisotropic supports. Shearlets were developed by Labate *et al.* [7] in 2005 as the first directional representation system which allows a unified treatment of

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the continuum and digital world similar to wavelets. The shearlets provide an alternative approach to the curvelets, and exhibit some very distinctive features. Similarly to the curvelets, the shearlets are a multiscale directional system and unlike the curvelets the shearlets form an affine system. That is, they are generated by dilating and translating one single generating function, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix. The wavelet transform associated with above more general dilation groups is called shearlet transform. similarly to the theory of affine systems, the continuous shearlets are associated with the whole range of scaling, shear, and translation indices $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, whereas the discrete shearlet systems are associated with a sequence in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ of discrete scaling, shear and translation indices.

Shearlet systems obtained by two procedures: One system being generated by unitary representation of the shearlet group and equipped with a particularly 'nice' mathematical structure, but due to biasedness towards one axis it becomes unattractive for applications point of view, the other system being generated by cone-adapted shearlets, by ensuring an equal treatment of all directions. The main advantage of this system is that it provide a unified treatment of the continuum and digital world. Therefore, in this paper we consider only cone-adapted shearlet system introduced by Sören Häuser and Gabriele Steidl [8] and obtained some basic results concerning continuous shearlet transform similar to wavelet transform.

(Cone-adapted) continuous shearlet systems. We use the parabolic scaling matrices A_a or $\tilde{A}_a, a > 0$, and shear matrices $S_s, s \in \mathbb{R}$, defined by
$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ or $\tilde{A}_a = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix}$ and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, respectively. We partition the frequency plane into the following four cones:

$$c^k = \left\{ \begin{array}{c} (w_1, w_2) \in \mathbb{R}^2 : |w_1| \geq \frac{1}{2}, |w_2| < |w_1|, k = h, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_2| \geq \frac{1}{2}, |w_2| > |w_1|, k = v, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_2| \geq \frac{1}{2}, |w_2| = |w_1|, k = \times, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_1| < 1, |w_2| < 1, k = 0. \end{array} \right\}$$

Here we denote $c^h, c^v, c^\times$ and c^0 by the horizontal, vertical, intersection of both, cones and low frequency part respectively. Also we have $\mathbb{R}^2 = c^h \cup c^v \cup c^\times \cup c^0$ with an overlapping domain $c^\square = (-1, 1)^2 \diagdown (\frac{-1}{2}, \frac{1}{2})^2$. The domain $c^\square$ is represented by translations of some scaling function. Anisotropy now comes into play when encoding the high frequency content of a signal, which corresponds to the cones $c^k, k \in (h, v, \times)$.

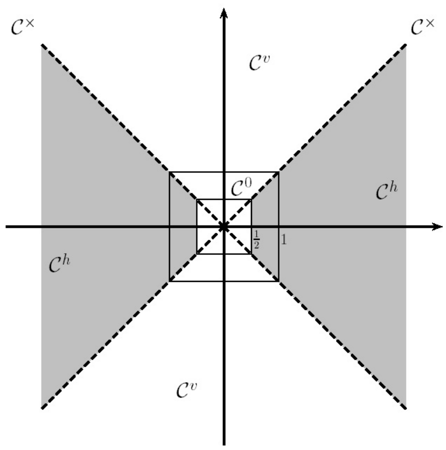

Figure 1

The shearlets $\varphi_{a,s,t}, \tilde{\varphi}_{a,s,t} \in L^2(\mathbb{R}^2)$ for functions $\varphi, \tilde{\varphi} \in L^2(\mathbb{R}^2)$ is defined as

$$\varphi_{a,s,t}(x) = \left\{ a^{-\frac{3}{4}} \varphi(A_a^{-1} S_s^{-1}(x-t)) : a \in (0,1], s \in [-(1+a^{\frac{1}{2}}), 1+a^{\frac{1}{2}}], t \in \mathbb{R}^2 \right\}$$

and

$$\tilde{\varphi}_{a,s,t}(x) = \left\{ a^{-\frac{3}{4}} \tilde{\varphi}(\tilde{A}_a^{-1} S_s^{-1}(x-t)) : a \in (0,1], s \in [-(1+a^{\frac{1}{2}}), 1+a^{\frac{1}{2}}], t \in \mathbb{R}^2 \right\}.$$

The shearlets φ and $\tilde{\varphi}$ defined above are suited for the horizontal and vertical cones. For each set $c^k, k \in (h, v, \times)$, we define a characteristic function $\chi_{c^k}(w)$ as

$$\chi_{c^k}(w) = \left\{ \begin{array}{l} 1 : w \in c^k, \\ 0 : otherwise, \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} \hat{\varphi}^h(w_1, w_2) = \hat{\varphi}(w_1, w_2) = \hat{\varphi}_1(w_1)\hat{\varphi}_2(\frac{w_2}{w_1})\chi_{c^h}, \\ \hat{\varphi}^v(w_1, w_2) = \hat{\varphi}(w_2, w_1) = \hat{\varphi}_1(w_2)\hat{\varphi}_2(\frac{w_1}{w_2})\chi_{c^v}, \\ \hat{\varphi}^\times(w_1, w_2)\hat{\varphi}(w_1, w_2)\chi_{c^\times}. \end{array} \right\}$$

The shearlets $\hat{\varphi}^h, \hat{\varphi}^v$ (and $\hat{\varphi}^\times$) are called shearlets on the cone [5]. These functions cover three of the four parts of $\mathbb{R}^2$, the remaining part c^0 is corresponding to scaling function $\phi \in L^2(\mathbb{R}^2)$. Let

$$L^2(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{R}^2; \int_{\mathbb{R}^2} |f(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

The space $L^2(\mathbb{R}^2)$ is the Hilbert space of all square (Lebesgue) integrable functions endowed with the inner product $< f, g >= \int_{\mathbb{R}^2} f\overline{g}$. The Fourier transform is the unitary operator that maps $f \in L^2(\mathbb{R}^2)$ into the function $\hat{f}$ defined by

$$\hat{f}(w) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i <x,w>} \,\mathrm{d}x,$$

when $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and by the appropriate limit for the general $f \in L^2(\mathbb{R}^2)$. The function $\hat{f}$ is also square integrable. Indeed, Fourier transform maps $L^2(\mathbb{R}^2)$ one-to-one onto itself and the inverse Fourier transform is defined by

$$\check{f}(x) = \int_{\mathbb{R}^2} f(w) e^{2\pi i <x,w>} \,\mathrm{d}w.$$

Setting

$$S_{cone} = \left\{ (a, s, t) : a \in (0,1], s \in [-(1+a^{\frac{1}{2}}), 1+a^{\frac{1}{2}}], t \in \mathbb{R}^2 \right\},$$

the associated (Cone-adapted) continuous shearlet transform $SH_{\phi,\varphi,\tilde{\varphi}}(f) : \mathbb{R}^2 \times S^2_{cone} \to \mathcal{C}^3$ of some function $f \in L^2(\mathbb{R}^2)$ is given by

$$SH_{\phi,\varphi,\tilde{\varphi}} f(t', (a,s,t), (\tilde{a}, \tilde{s}, \tilde{t})) = (< f, \phi_{t'} >, < f, \varphi_{a,s,t} >, < f, \tilde{\varphi}_{\tilde{a},\tilde{s},\tilde{t}} >).$$

Since the low frequency part already has been studied extensively and the horizontal and vertical cones are treated similarly by just interchanging w_1 and w_2, therefore, from now we will consider only horizontal cone c^h and define

$$L^2(c^h) = \left\{ f \in L^2(\mathbb{R}^2) : supp\hat{f} \subseteq c^h \right\}.$$

Remark 1.1. The continuous shearlet transform projects the function f onto the functions $\varphi_{a,s,t}$, at scale a, orientation s and location t.

Definition 1.2. A function $\varphi \in L^2(\mathbb{R}^2)$ is called a continuous shearlet if and only if it satisfies the admissibility condition

$$C_\varphi = \int_{\mathbb{R}^2} \frac{\hat{\varphi}(w_1, w_2)}{|w_1|^2} \,\mathrm{d}w_1 \,\mathrm{d}w_2 < \infty. \tag{1.1}$$

Here φ is called admissible shearlet.

2. Some basic results related to (cone-adapted) continuous shearlet transform

Like the continuous wavelet transform, the continuous shearlet transform has weaker conditions, especially the orthogonality not necessary for an invertible continuous shearlet transform. So, it is convenient to focus our attention on Parseval frame shearlets rather than on orthonormal shearlets.

Theorem 2.1 (**Parseval's formula for shearlet transform**). *If φ be an admissible shearlet and the continuous shearlet transform of $\varphi \in L^2(c^h)$ defined by $SH_\varphi(f)(a,s,t) =< f, \varphi_{a,s,t} >_{L^2(c^h)}$, then for any $f, f^* \in L^2(c^h)$ we have the formula*

$$\int_{(a,s,t)\in SH} SH_\varphi(f)(a,s,t)\overline{SH_\varphi(f^*)(a,s,t)}\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3} = C_\varphi < f, f^* >_{L^2(c^h)} . \tag{2.1}$$

Proof. We have

$$\begin{aligned}
&\int_{(a,s,t)\in SH} < f, \varphi_{a,s,t} > \overline{< f^*, \varphi_{a,s,t} >}\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+}\left[a^{\frac{3}{4}}\int_{\mathbb{R}^2}\hat{f}(w)\overline{\hat{\varphi}_1(aw_1)\hat{\varphi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))}e^{2\pi i<w,t>}\,\mathrm{d}w\right] \\
&\quad\times\left[a^{\frac{3}{4}}\int_{\mathbb{R}^2}\overline{\hat{f^*}(w)}\hat{\varphi}_1(aw_1)\hat{\varphi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))e^{-2\pi i<w,t>}\,\mathrm{d}w\right]\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3} \\
&= int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} a^{-\frac{3}{2}}\left[\int_{\mathbb{R}^2}\overline{\overline{\hat{f}(w)}\hat{\varphi}_1(aw_1)\hat{\varphi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))e^{2\pi i<w,t>}}\,\mathrm{d}w\right] \\
&\quad\times\left[\int_{\mathbb{R}^2}\overline{\hat{f^*}(w)}\hat{\varphi}_1(aw_1)\hat{\varphi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))e^{-2\pi i<w,t>}dw\right]\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} a^{-\frac{3}{2}}\hat{f}(w)\overline{\hat{f^*}(w)}|\hat{\varphi}_1(aw_1)\hat{\varphi}_2(a^{-\frac{1}{2}}(\frac{w_2}{w_1}+s))|^2\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}w,
\end{aligned}$$

putting

$$aw_1 = \zeta_1, \left(a^{-\frac{1}{2}}\left(\frac{w_2}{w_1}+s\right)\right) = \frac{\zeta_2}{\zeta_1} \Rightarrow \mathrm{d}s = a^{\frac{1}{2}}\frac{\mathrm{d}\zeta_2}{\zeta_1}, \mathrm{d}a = a\frac{\mathrm{d}\zeta_1}{\zeta_1},$$

now we have

$$\int_{c^h}\hat{f}(w)\overline{\hat{f^*}(w)}\,\mathrm{d}w\int_{\mathbb{R}^2}\frac{\hat{\varphi}(\zeta_1,\zeta_2)}{|\zeta_1|^2}\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2 = C_\varphi < f, f^* >_{L^2(c^h)} .$$

□

Theorem 2.2. *If $f \in L^2(c^h)$, then f can be reconstructed by the formula*

$$f(x) = C_\varphi^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}(x)\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3}. \tag{2.2}$$

Proof. For any $\varphi \in L^2(c^h)$, in view of Theorem 2.1, we have

$$\begin{aligned}
< f, f^* > &= C_\varphi^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} SH_\varphi(f)(a,s,t)\overline{SH_\varphi(f^*)(a,s,t)}\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3} \\
&= C_\varphi^{-1}\int_{(a,s,t)\in SH} SH_\varphi(f)(a,s,t) < \varphi_{a,s,t}, f^* > \frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3} \\
&=< C_\varphi^{-1}\int_{(a,s,t)\in SH} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}\frac{\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t}{a^3}, f^* >,
\end{aligned}$$

it gives (2.2). □

Theorem 2.3. *If $f \in (L^1 \cap L^2)(\mathbb{R}^2)$ and $\hat{f} \in L^1(\mathbb{R}^2)$ then the reconstruction formula (2.2) is valid point wise in the sense*

$$f(x) = C_\varphi^{-1} \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}} SH_\varphi(f)(a,s,t) \varphi_{a,s,t}(x) \, \mathrm{d}s \, \mathrm{d}t \right) \frac{\mathrm{d}a}{a^3}, x \in \mathbb{R}^2,$$

where for each x, both the inner integrals and outer integral are absolutely convergent, but possibly not the triple integral.

Proof. In view of Fubini's theorem and the formula (2.2) we have

$$f(x) = C_\varphi^{-1} \int_{\mathbb{R}^+} \left(a^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \left| \hat{\varphi}_1 (aw_1) \hat{\varphi}_2 \left(a^{-\frac{1}{2}} \left(\frac{w_2}{w_1} + s \right) \right) \right|^2 \hat{f}(w) e^{2\pi i <x,w>} \, \mathrm{d}s \, \mathrm{d}w \right) \mathrm{d}a,$$

where the triple integral is absolutely convergent since $\hat{f}$ is bounded. Thus to complete the proof we to show that, for all $a > 0, s \in \mathbb{R}, x \in \mathbb{R}^2$,

$$\begin{aligned} &a^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left| \hat{\varphi}_1(aw_1) \hat{\varphi}_2 \left(a^{-\frac{1}{2}} \left(\frac{w_2}{w_1} + s \right) \right) \right|^2 \hat{f}(w) e^{2\pi i <x,w>} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}w \\ &\quad = a^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^+} \varphi_{a,s,t}(x) SH_\varphi(f)(a,s,t) \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t, \end{aligned} \tag{2.3}$$

with absolutely convergent integral on the right hand side. This absolute convergence follows because $SH_\varphi(f)(a,s,.)$ is L^2 as a convolution product of L^1 and L^2 functions, while $\varphi_{a,s,t}$ is also a L^2 function in t. We can write (2.3) as

$$a^{-\frac{3}{2}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \hat{\varphi}_{a,s}(w_1,w_2) \hat{h}_{a,s}(w_1,w_2) \hat{f}(w) e^{2\pi i <x,w>} \, \mathrm{d}w \, \mathrm{d}s \, \mathrm{d}a = (\varphi_{a,s} * (h_{a,s} * f))(x), \tag{2.4}$$

where $h_{a,s}(x) = \overline{\varphi_{a,s}(-x)}$. Both sides of (2.4) are equal to

$$a^{-\frac{3}{2}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \hat{\varphi}_{a,s}(w_1,w_2) (h_{a,s} \hat{*} f)(w) e^{2\pi i <x,w>} \, \mathrm{d}w \, \mathrm{d}s \, \mathrm{d}a.$$

From the left hand side of (2.4) it is clear from the fact that $h_{a,s} \in L^2(\mathbb{R}^2)$ and $f \in L^1(\mathbb{R}^2)$, while this follows for the right hand side of (2.4) by application of the Parseval formula for the Fourier transform. □

Corollary 2.4. *Let $f \in L^2(\mathbb{R}^2)$. Then*
(i). The mapping $(a,s,t) \to \varphi_{a,s,t} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \to c^h$ is continuous.
(ii). $SH_\varphi(f)(a,s,t)$ is continuous on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$.
(iii). $|SH_\varphi(f)(a,s,t)| \leq \|f\|_2 \|\varphi\|_2$.
(iv). $\lim_{t \to \infty} SH_\varphi(f)(a,s,t) = 0$, uniformly for a, s in bounded subset of $\mathbb{R}^+ \times \mathbb{R}$.

Proof. Parts (i),(ii) and (iii) follows similar to wavelet transform.
(iv). Let f be a continuous function with compact support i.e., $f(x) = 0$ if $|x| \geq r, x \in \mathbb{R}^2, x = (x_1, x_2), r = \max(r_1, r_2)$. Then

$$\begin{aligned} |SH_\varphi(f)(a,s,t)| &\leq \|f\|_2 \left(\int_{|x|<r} a^{-\frac{3}{2}} |\varphi(A_a^{-1} S_s^{-1}(x-t))|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &= \|f\|_2 \left(a^{\frac{3}{4}} \int_{|y + A_a^{-1} S_s^{-1} t| \leq A_a^{-1} S_s^{-1} r} |\varphi(y)|^2 \, \mathrm{d}y \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \|f\|_2 \left(a^{\frac{3}{4}} \int_{|y| \geq A_a^{-1} S_s^{-1}(|t|-r)} |\varphi(y)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}}.$$

Let $f_o \in L^2(\mathbb{R}^2)$ and $V(a,s)$ be some neighborhood of $(a,s) \in \mathbb{R}^+ \times \mathbb{R}$. Take f continuous with compact support such that

$$\|f - f_o\|_2 \leq \frac{\varepsilon}{2\|\varphi\|_2}.$$

Then

$$\begin{aligned} |SH_\varphi(f)(a,s,t)| &\leq |SH_\varphi(f)(a,s,t) - SH_\varphi(f_o)(a,s,t)| + |SH_\varphi(f)(a,s,t)| \\ &\leq \|f - f_o\|_2 \|\varphi\|_2 + |SH_\varphi(f)(a,s,t)| \\ &\leq \frac{\varepsilon}{2} + |SH_\varphi(f09a,s,t)|. \end{aligned}$$

There exist $k > 0$ such that, if $(a,s) \in V$ and $|t| > k, t \in \bigcup(t_o), \bigcup(t_o)$ is a small neighborhood of t_o, then

$$|SH_\varphi(f)(a,s,t)| < \frac{\varepsilon}{2}.$$

Hence the proof is completed. □

Theorem 2.5. *Let $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3 > 0$. Then*

$$\left\| C_\varphi^{-1} \int_{-\gamma_1}^{\gamma_1} \int_{-\gamma_2}^{\gamma_2} \int_{-\gamma 3}^{\gamma_3} \int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t) < \varphi_{a,s,t}, f^* > a^{-3} \,\mathrm{d}a \,\mathrm{d}s \,\mathrm{d}t \right\|_2 = 4\gamma_1\gamma_2\gamma_3 a^3 \left[\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right] \|f\|_2 \|f^*\|_2.$$

Proof. We have

$$\begin{aligned} &\left\| C_\varphi^{-1} \int_{-\gamma_1}^{\gamma_1} \int_{-\gamma_2}^{\gamma_2} \int_{-\gamma_3}^{\gamma_3} \int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t) < \varphi_{a,s,t}, f^* > a^{-3} \,\mathrm{d}a \,\mathrm{d}s \,\mathrm{d}t \right\|_2 \\ &\quad \leq C_\varphi^{-1} \int_{-\gamma_1}^{\gamma_1} \int_{-\gamma_2}^{\gamma_2} \int_{-\gamma_3}^{\gamma_3} \int_{-\alpha}^{\beta} \|f\|_2 \|f^*\|_2 \|\varphi\|_2^2 a^{-3} \,\mathrm{d}a \,\mathrm{d}s \,\mathrm{d}t \\ &\quad = \|f\|_2 \|f^*\|_2 C_\varphi^{-1} \|\varphi\|_2^2 \int_{-\gamma_1}^{\gamma_1} \int_{-\gamma_2}^{\gamma_2} \int_{-\gamma_3}^{\gamma_3} \int_{-\alpha}^{\beta} a^{-3} \,\mathrm{d}a \,\mathrm{d}s \,\mathrm{d}t \\ &\quad = 4\gamma_1\gamma_2\gamma_3 [\frac{1}{\alpha^2} - \frac{1}{\beta^2}] \|f\|_2 \|f^*\|_2 C_\varphi^{-1} \|\varphi\|_2^2. \end{aligned} \tag{2.5}$$

Now we compute

$$\begin{aligned} \|\varphi_{a,s,t}\|_2^2 = \int_{\mathbb{R}^2} |\varphi_{a,s,t}(x)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^2} \varphi(A_a^{-1} S_s^{-1}(x-t) \overline{\varphi}(A_a^{-1} S_s^{-1}(x-t) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} a^{\frac{3}{2}} \hat{\varphi}_1(aw_1) \hat{\varphi}_2 \left(a^{-\frac{1}{2}} \left(\frac{w_2}{w_1} + s \right) \right) e^{-2\pi i <w,t>} \\ &\quad \overline{\times \hat{\varphi}_1(aw_1) \hat{\varphi}_2 \left(a^{-\frac{1}{2}} \left(\frac{w_2}{w_1} + s \right) \right)} e^{2\pi i <w,t>} \,\mathrm{d}w \,\mathrm{d}s \,\mathrm{d}a \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \hat{\varphi}_1(aw_1) \hat{\varphi}_2 \left(a^{-\frac{1}{2}} \left(\frac{w_2}{w_1} + s \right) \right) \right|^2 \,\mathrm{d}w \,\mathrm{d}s \,\mathrm{d}a \\ &= a^3 \int_{\mathbb{R}^2} \frac{\hat{\varphi}(\zeta_1, \zeta_2)}{|\zeta_1|^2} \,\mathrm{d}\zeta_1 \,\mathrm{d}\zeta_2 = a^3 C_\varphi. \end{aligned}$$

Substituting the value of $\|\varphi\|_2^2$ in (2.5) we get the required result. □

Corollary 2.6.

$$\lim_{\alpha\to 0,\beta,\gamma_1,\gamma_2,\gamma_3\to\infty}\left\| f - C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t\right\|_2 = 0.$$

Proof. We see that the integral

$$C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2),$$

given by

$$\begin{aligned}
&C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t(f^*)\\
&=< C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t, f^* >, f^*\in L^2(\mathbb{R}^2).\\
&= C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t) < \varphi_{a,s,t}, f^* > a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

In view of Theorem 2.3, we get

$$\begin{aligned}
&\left\| f - C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t\right\|_2\\
&= \sup_{\|f^*\|=1}\left| < f - C_\varphi^{-1}\int_{-\gamma_1}^{\gamma_1}\int_{-\gamma_2}^{\gamma_2}\int_{-\gamma_3}^{\gamma_3}\int_{-\alpha}^{\beta} SH_\varphi(f)(a,s,t)\varphi_{a,s,t}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t, f^* >\right|\\
&\le \sup_{\|f^*\|=1}\left| C_\varphi^{-1}\int\int\int_{(|a|\ge\beta)or(|a|\le\alpha)or(|s|\ge\gamma_3)or(|t|\ge\gamma_1,\gamma_2)} SH_\varphi(f)(a,s,t)\overline{SH_\varphi(f^*)(a,s,t)}a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t\right|\\
&\le \sup_{\|f^*\|=1}\left[C_\varphi^{-1}\int\int\int_{(|a|\ge\beta)or(|a|\le\alpha)or(|s|\ge\gamma_2)or(|t|\ge\gamma_1,\gamma_2)} |SH_\varphi(f)(a,s,t)|^2a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t\right]^{\frac{1}{2}}\\
&\quad\times\left[C_\varphi^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} |SH_\varphi(f^*)(a,s,t)|^2a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t\right]^{\frac{1}{2}}.
\end{aligned}$$

The expression in the first bracket approaches zero as $\alpha\to 0$ and $\beta,\gamma_1,\gamma_2,\gamma_3\to\infty$ and the expression in the second bracket= $\|f^*\| = 1$. Hence the proof is completed. □

3. Conclusions and applications

Since the traditional $n-$dimensional continuous wavelet transform does not provide the information about the geometry of the set of singularities, in order to achieve this additional flexibility in this paper we consider the generalized wavelet transform namely continuous shearlet transform and derive some basic results including reconstruction formula similar to traditional wavelet transform. The cone-adapted shearlet system has been used to cover whole $\mathbb{R}^2$ and for giving equal treatment of all directions.

In the last few years we have seen an explosion of activity in machine learning, data analysis and search, implying that similar ideas and concepts, inspired by signal processing weight carry as much power in the context of the orchestration of massive high dimensional data sets. This digital data, medical records, music, sensor data, financial data etc., can be structured into geometries that result in new organizations of language. As an application this work is useful in the oil exploration and mining industry, in which one needs to decide where to drill or mine to greatest advantage for finding oil, gas, copper or other minerals. In medical diagnostics, important information can be learned from the analysis of data obtained from radiological, histological, chemical tests and this is important for arriving at an early detection of potentially dangerous tumors and other pathologies (see [3]).

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