Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 540–556
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

The uniform boundedness principles for γ-max-pseudo-norm-subadditive and quasi-homogeneous operators in F^* spaces

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Abstract

In this paper, we prove that every F^* space (i.e., Hausdorff topological vector space satisfying the first countable axiom) can be characterized by means of its "standard generating family of pseudo-norms". By using the standard generating family of pseudo-norms $\mathscr{P}$, the concepts of $\mathscr{P}$-bounded set and γ-max-pseudo-norm-subadditive operator in F^* space are introduced. The uniform boundedness principles for family of γ-max-pseudo-norm-subadditive and quasi-homogeneous operators in F^* spaces are established. As applications, we obtain the corresponding uniform boundedness principles in classical normed spaces and Menger probabilistic normed spaces.

Keywords: Uniform boundedness principle, γ-max-pseudo-norm-subadditive operator, quasi-homogeneous operator, second category, F^* space.
2010 MSC: 46A30, 46A16, 46S50.

1. Introduction

The uniform boundedness principle (or the resonance theorem) is one of the foundation stones of functional analysis. For its importance there has been a lot of work (see books [2, 14, 16]) on uniform boundedness principles since Banach-Steinhaus theorems were established in 1927. Especially, today we can find some new improvements for uniform boundedness principles in many different mathematical fields (for examples, see [1, 5, 6, 8, 9, 10, 11, 12, 15, 17, 18]).

Recently, in [4, 7], R. Li et al. gave the definition of quasi-homogeneous operator and showed the family of quasi-homogeneous operators included all linear and many more nonlinear operators. The introduction

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Received 2014-11-20

of quasi-homogeneous operators has strongly broadened our research scope of operators. After that, some new uniform boundedness principles for quasi-homogeneous operators have appeared in the literature (see [9, 12]).

On the other hand, it is well known that the subadditive functions have played a important role in the study of uniqueness of differential equations, convex solids and extension of linear functional. Then the uniform boundedness principles for family of γ-subadditive functionals in some topological vector groups are established in [5]. Inspired by the work of [4, 5, 7, 9, 12], the main work of this paper is to establish the uniform boundedness principles for family of γ-max-pseudo-norm-subadditive and quasi-homogeneous operators in F^* spaces.

In this paper, we use the terminology in [14], where a F^* space means a Hausdorff topological vector space satisfying the first countable axiom, and a complete F^* space is called a F space. In Section 2, we first prove that every F^* space can be characterized by means of its "standard generating family of pseudo-norms". Then by using the standard generating family of pseudo-norms $\mathscr{P}$, the concepts of $\mathscr{P}$-bounded set and γ-max-pseudo-norm-subadditive operator in F^* space are introduced. In Section 3, the uniform boundedness principles for family of pointwise bounded γ-max-pseudo-norm-subadditive and quasi-homogeneous operators in F^* spaces (or normed spaces) are established. In Section 4, we give some applications concerning our results. As example, we obtain the corresponding uniform boundedness principles for family of pointwise probabilistic bounded γ-max-probabilistic subadditive and quasi-homogeneous operators in Menger probabilistic normed spaces, and the elements of the space (l^p) form a subset of the first category in the space (c_0), where $p \geqslant 1$.

2. Preliminaries

In this section, we first introduce the concept of "standard generating family of pseudo-norms" of a F^* space, and prove that the linear topology on every F^* space can be determined by its standard generating family of pseudo-norms $\mathscr{P}$. Secondly, we use the standard generating family of pseudo-norms $\mathscr{P}$ to give the definitions of $\mathscr{P}$-bounded set and γ-max-pseudo-norm-subadditive operator in F^* spaces, and study their relevant properties.

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ be the set of all positive integers. In order to give a new characteristic description of F^* spaces, we need the following lemma.

Lemma 2.1. *(cf.[11]) Suppose X is a linear space on a number field $\mathbb{K}$ (real number field or complex field), $\mathscr{P} = \{p_\lambda \mid \lambda \in (0,1]\}$ is a family of pseudo-norms in X satisfying the following conditions:*

(P-1) $p_\lambda(x) = \inf_{0<\mu<\lambda} p_\mu(x)$ *for each $x \in X$ and $\lambda \in (0,1]$;*
(P-2) $p_\lambda(kx) = |k| p_\lambda(x)$ *for all $x \in X$, $\lambda \in (0,1]$ and $k \in \mathbb{K}$;*
(P-3) for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$, such that

$$p_\lambda(x+y) \leqslant p_\mu(x) + p_\mu(y) \quad \text{for all } x, y \in X;$$

(P-4) $p_\lambda(x) = 0$ for each $\lambda \in (0,1] \iff x = \theta$.

Then there exists a unique topology $\mathscr{T}$ on X, such that $(X, \mathscr{T})$ is a Hausdorff topological vector space, and

$$\mathscr{U} = \{U(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0,1]\}$$

is a $\mathscr{T}$-neighborhood base of θ, where $U(\varepsilon,\lambda) = \{x \in X \mid p_\lambda(x) < \varepsilon\}$.

Remark 2.2. Below, the topology $\mathscr{T}$ introduced in Lemma 2.1 on the linear space X, is called a topology generated by the family of pseudo-norms $\mathscr{P}$ in X.

Theorem 2.3. *Let $(X, \mathcal{T})$ be a F^* space, then there exists a family of pseudo-norms $\mathscr{P} = \{p_\lambda \mid \lambda \in (0,1]\}$ in X satisfying conditions (P-1)–(P-4), such that the topology $\mathscr{T}$ generated by $\mathscr{P}$ is equivalent to the original topology $\mathcal{T}$ on X.*

Proof. Since $(X, \mathcal{T})$ is a topological vector space satisfying the first countable axiom, there exists a balanced $\mathcal{T}$-neighborhood base $\{V_n \mid n \in \mathbb{N}\}$ of θ, with $V_{n+1} \subset V_n$ $(n = 1, 2, \cdots)$. For each $\lambda \in (0, 1]$, we define U_λ as follows:

$$\text{If } \lambda \in (\frac{1}{n+1}, \frac{1}{n}], \text{ then } U_\lambda = \lambda V_n \quad (n = 1, 2, \cdots). \tag{2.1}$$

Obviously, $\{U_\lambda \mid \lambda \in (0, 1]\}$ is also a balanced $\mathcal{T}$-neighborhood base of θ, and from (2.1) we know

(i) $0 < \mu < \lambda \leqslant 1 \implies U_\mu \subset U_\lambda$;
(ii) If $\lambda, \mu \in (\frac{1}{n+1}, \frac{1}{n}]$, then $U_\lambda = \frac{\lambda}{\mu} U_\mu$.

Assume $p_\lambda(\cdot)$ is a Minkowski functional on U_λ, that is,

$$p_\lambda(x) = \inf\{t > 0 \mid x \in tU_\lambda\}, \quad x \in X. \tag{2.2}$$

We now prove that $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ is a family of pseudo-norms in X satisfying conditions (P-1)$-$(P-5).

(P-1) From (i) and (2.2) we know, $0 < \mu < \lambda \Rightarrow p_\lambda(x) \leqslant p_\mu(x)$, and so $p_\lambda(x) \leqslant \inf_{0<\mu<\lambda} p_\mu(x)$. On the other hand, for any $\lambda \in (0, 1]$, we can assume $\lambda \in (\frac{1}{n+1}, \frac{1}{n}]$ without loss of generality. Taking $\mu \in (\frac{1}{n+1}, \lambda)$, by (ii) and (2.2) we have

$$\begin{aligned} p_\lambda(x) &= \inf\{t > 0 \mid x \in tU_\lambda\} = \inf\left\{t > 0 \,\middle|\, x \in \frac{t\lambda}{\mu} U_\mu\right\} \\ &= \frac{\mu}{\lambda} p_\mu(x) \geqslant \frac{\mu}{\lambda} \inf_{0<\alpha<\lambda} p_\alpha(x). \end{aligned}$$

Letting $\mu \to \lambda$, we get $p_\lambda(x) \geqslant \inf_{\alpha \in (0,\lambda)} p_\alpha(x)$. Hence $p_\lambda(x) = \inf_{\mu \in (0,\lambda)} p_\mu(x)$ and (P-1) holds.

(P-2) can be easily checked by (2.2) and the balance of U_λ.

(P-3) Note that $\{U_\lambda \mid \lambda \in (0, 1]\}$ is the $\mathcal{T}$-neighborhood base of θ satisfying (i), hence for each $\lambda \in (0, 1]$, there exists $\mu \in (0, \lambda]$ such that $U_\mu + U_\mu \subset U_\lambda$. Let $p_\mu(x) = a$, $p_\mu(y) = b$, then for any $\varepsilon > 0$, there exist $0 < s < a + \varepsilon$ and $0 < t < b + \varepsilon$, such that $x \in sU_\mu$, $y \in tU_\mu$. Thus

$$\begin{aligned} x + y &\in sU_\mu + tU_\mu = (s + t)\left(\frac{s}{s+t} U_\mu + \frac{t}{s+t} U_\mu\right) \\ &\subset (s + t)(U_\mu + U_\mu) \subset (s + t)U_\lambda, \end{aligned}$$

which means that $p_\lambda(x + y) \leqslant s + t < a + b + 2\varepsilon$. By the arbitrariness of ε, (P-3) holds.

(P-4) By (2.2), $p_\lambda(\theta) = 0$ for each $\lambda \in (0, 1]$. Conversely, if for each $\lambda \in (0, 1]$, $p_\lambda(x) = 0$, then from (2.2) and the balance of U_λ, we know $x \in \bigcap\{U_\lambda \mid \lambda \in (0, 1]\}$. Since $\{U_\lambda \mid \lambda \in (0, 1]\}$ is a $\mathcal{T}$-neighborhood base of θ in the Hausdorff topological vector space $(X, \mathcal{T})$, we have $x = \theta$. Therefore, (P-4) holds.

Since the family of pseudo-norms $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ satisfies conditions (P-1)$-$(P-4), then by Lemma 2.1, there exists a topology $\mathscr{T}$ on X, such that $(X, \mathscr{T})$ is a Hausdorff topological vector space, and $\mathscr{U} = \{U(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0, 1]\}$ is the $\mathscr{T}$-neighborhood base of θ, where

$$U(\varepsilon, \lambda) = \{x \in X \mid p_\lambda(x) < \varepsilon\}. \tag{2.3}$$

It is easy to show that for any $\varepsilon > 0$ and $\lambda \in (0, 1]$, we have the following two inclusion relations between sets

$$U(1, \lambda) \subset U_\lambda, \quad \frac{\varepsilon}{2} U_\lambda \subset U(\varepsilon, \lambda).$$

Therefore, the topology $\mathscr{T}$ generated by $\mathscr{P}$ is equivalent to the original topology $\mathcal{T}$ on X. $\square$

Remark 2.4. If the topology $\mathscr{T}$ generated by a family of pseudo-norms $\mathscr{P}$ on X is equivalent to the topology $\mathcal{T}$ in a F^* space $(X, \mathcal{T})$, then we call $\mathscr{P}$ a generating family of pseudo-norms of $\mathscr{T}$ on X, or a generating family of pseudo-norms of $(X, \mathscr{T})$ for short. If a family of pseudo-norms $\mathscr{P} = \{p_\lambda \mid \lambda \in (0,1]\}$ satisfies conditions (P-1)−(P-4), then we call $\mathscr{P}$ a standard generating family of pseudo-norms in the F^* space X. According to the proof of Theorem 2.3, we know that $\mathscr{U} = \{U(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0,1]\}$ is a neighborhood base of θ in $(X, \mathscr{T})$, where $U(\varepsilon, \lambda)$ is given by (2.3).

For the sake of neatness, below we will denote a F^* space $(X, \mathscr{T})$ by $(X, \mathscr{P})$ or $(X, \{p_\lambda\}_{\lambda\in(0,1]})$, where $\mathscr{P}$ or $\{p_\lambda\}_{\lambda\in(0,1]}$ is a standard generating family of pseudo-norms on $(X, \mathscr{T})$. It is easy to see that if $\{x_n\}_{n=1}^\infty$ is a sequence in X, and $x \in X$, then

(1) $x_n \xrightarrow{\mathscr{T}} x$ $(n \to \infty)$ iff for each $\lambda \in (0,1]$, $p_\lambda(x_n - x) \to 0$ $(n \to \infty)$.

(2) $\{x_n\}_{n=1}^\infty$ is a $\mathscr{T}$-Cauchy sequence iff for each $\varepsilon > 0$ and $\lambda \in (0,1]$, there exists $N \in \mathbb{N}$, such that $p_\lambda(x_m - x_n) < \varepsilon$, whenever $m, n \geqslant N$.

Notice that the definition of a family of pseudo-norms in Theorem 2.3, we can easily obtain the following lemma.

Lemma 2.5. *If $(X, \mathscr{P})$ be a locally convex F^* space, then there exists a standard generating family of semi-norms $\{p_\lambda\}_{\lambda\in(0,1]}$ in X, such that p_λ is a semi-norm for each $\lambda \in (0,1]$.*

Example 2.6. Let $X = C(\mathbb{R})$ and the family of pseudo-norms $\{p_\lambda\}_{\lambda\in(0,1]}$ be defined on X as follows

$$p_\lambda(x) = \sup_{s\in[1-\frac{1}{\lambda}, \frac{1}{\lambda}-1]} |x(s)| \ \text{ for all } \ \lambda \in (0,1].$$

Then $(X, \mathscr{P})$ (or $(C(\mathbb{R}), \{p_\lambda\}_{\lambda\in(0,1]})$) is a locally convex F^* space, and the family of semi-norms $\{p_\lambda\}_{\lambda\in(0,1]}$ satisfies conditions (P-1)−(P-4).

Obviously, $\{p_\lambda\}_{\lambda\in(0,1]}$ is a family of semi-norms, and (P-2)−(P-4) are easy to check. For any $0 < \mu < \lambda \leqslant 1$, we know $[1 - \frac{1}{\lambda}, \frac{1}{\lambda} - 1] \subset [1 - \frac{1}{\mu}, \frac{1}{\mu} - 1] \Rightarrow p_\lambda(x) \leqslant p_\mu(x)$, and so $p_\lambda(x) \leqslant \inf_{0<\mu<\lambda} p_\mu(x)$. On the other hand, note that $x(s) \in C(\mathbb{R})$, we have $p_\lambda(x) = \max_{s\in[1-\frac{1}{\lambda}, \frac{1}{\lambda}-1]} |x(s)|$. From the continuity of $x(s)$, we can know that for any $\varepsilon > 0$, there exist $\delta > 0$ such that

$$p_\lambda(x) = \max_{s\in[1-\frac{1}{\lambda}, \frac{1}{\lambda}-1]} |x(s)| \geqslant \max_{s\in[1-\frac{1}{\lambda}-\delta, \frac{1}{\lambda}-1+\delta]} |x(s)| - \varepsilon \geqslant \inf_{0<\mu<\lambda} p_\mu(x) - \varepsilon.$$

By the arbitrariness of ε, we have $p_\lambda(x) \geqslant \inf_{0<\mu<\lambda} p_\mu(x)$. Thus (P-1) holds. Hence the check is completed.

From the proof of Example 2.6 we have the following example.

Example 2.7. Let $X = C(\mathbb{R}^+)$ and the family of pseudo-norms $\{p_\lambda\}_{\lambda\in(0,1]}$ be defined on X as follows

$$p_\lambda(x) = \sup_{s\in[0, \frac{1}{\lambda}-1]} |x(s)| \ \text{ for all } \ \lambda \in (0,1].$$

Then $(C(\mathbb{R}^+), \{p_\lambda\}_{\lambda\in(0,1]})$ is a locally convex F^* space, and the family of semi-norms $\{p_\lambda\}_{\lambda\in(0,1]}$ satisfies conditions (P-1)−(P-4).

Now we use the standard generating family of pseudo-norms $\mathscr{P} = \{p_\lambda \mid \lambda \in (0,1]\}$ in a F^* space $(X, \mathscr{P})$ to give the definitions of $\mathscr{P}$-bounded set.

Definition 2.8. Let $(X, \mathscr{P})$ be a F^* space and $A \subset X$. A is said to be a $\mathscr{P}$-bounded set if

$$\sup_{x\in A} p_\lambda(x) < +\infty \text{for each} \lambda \in (0,1].$$

Proposition 2.9. *Let $(X, \mathscr{P})$ be a F^* space and $A \subset X$, then A is a $\mathscr{P}$-bounded set in $(X, \mathscr{P})$ iff A is a topology bounded set (for short, bounded set) in $(X, \mathscr{P})$, i.e., A can be absorbed by any $\mathscr{T}$-neighborhood of θ in $(X, \mathscr{P})$.*

Proof. Suppose A is a $\mathscr{P}$-bounded set, i.e., for each $\lambda \in (0, 1]$, $\sup_{x\in A} p_\lambda(x) < +\infty$. Suppose W is a $\mathscr{T}$-neighborhood of θ in $(X, \mathscr{P})$. By Remark 2.4, $\mathscr{U} = \{U(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0, 1]\}$ is a neighborhood base of θ, and so there exist $\varepsilon_0 > 0$ and $\lambda_0 \in (0, 1]$, such that $U(\varepsilon_0, \lambda_0) \subset W$. Evidently, there exists $M > 0$ such that $\sup_{x\in A} p_{\lambda_0}(x) < M$, which implies $A \subset U(M, \lambda_0)$. Applying (P-2), we have $U(M, \lambda_0) = (M/\varepsilon_0)U(\varepsilon_0, \lambda_0)$. Let $t_0 = M/\varepsilon_0$, then $A \subset t_0W$. Therefore, A is a bounded set.

Conversely, suppose A is a bounded set. Note that for any $\lambda \in (0, 1]$, $U(1, \lambda)$ is a $\mathscr{T}$-neighborhood of θ in $(X, \mathscr{P})$, hence there exists $M > 0$ such that $A \subset MU(1, \lambda) = U(M, \lambda)$, which means

$$\sup_{x\in A} p_\lambda(x) \leqslant M < +\infty.$$

Therefore, A is a $\mathscr{P}$-bounded set. □

Remark 2.10. The concept "bounded set" is well known in an ordinary topological vector spaces. As a consequence of the above proposition, in a F^* space, a "bounded set" is equivalent to a "$\mathscr{P}$-bounded set".

Lemma 2.11. *Let $(X, \|\cdot\|)$ be a classical normed space. Set $\|x\| = p_\lambda(x)$, $x \in X$ and $\lambda \in (0, 1]$. Then $(X, \mathscr{P})$ is a locally convex F^* space, and we call it the induced F^* space by the norm $\|\cdot\|$.*

Remark 2.12. By Lemma 2.11, it is easy to see that if $(X, \mathscr{P})$ be an induced F^* space by a norm $\|\cdot\|$, then $\|x\| = p_1(x) = p_\lambda(x)$ for all $\lambda \in (0, 1]$ and $x \in X$. Hence, we have
(1) A is a bounded set in $(X, \|\cdot\|)$ iff A is a $\mathscr{P}$-bounded set in $(X, \mathscr{P})$;
(2) $(X, \|\cdot\|)$ is complete iff $(X, \mathscr{P})$ is complete.

Definition 2.13. (cf. [4, 7]) Let X,Y be vector spaces. An operator T from X to Y is said to be quasi-homogeneous if there exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{t\to 0} \varphi(t) = 0 = \varphi(0)$, such that $T(tx) = \varphi(t)T(x)$ for all $t \in \mathbb{R}$ and $x \in X$.

Remark 2.14. Say that the function φ in Definition 2.13 is the eigenfunction of a quasi-homogeneous operator T. It is clear that , if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function such that $T : X \to Y$ is quasi-homogeneous, then φ satisfies $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in \mathbb{R}$. Let

$$C(0) = \left\{\varphi \in \mathbb{R}^{\mathbb{R}} : \lim_{t\to 0} \varphi(t) = \varphi(0) = 0,\ \varphi(st) = \varphi(s)\varphi(t),\ \forall\, s, t \in \mathbb{R}\right\}.$$

Obviously, if φ is the eigenfunction of a quasi-homogeneous operator T, then $\varphi \in C(0)$.

Below, for convenience, for each $\varphi \in C(0)$, let

$$\mathbf{QH}_\varphi(X, Y) = \left\{T \in Y^X : T(tx) = \varphi(t)T(x),\ \forall\ t \in \mathbb{R}, x \in X\right\}.$$

We define a function $\varphi_0 : \mathbb{R} \to \mathbb{R}$ by $\varphi_0(t) = t$ $(t \in \mathbb{R})$. Obviously, $\varphi_0 \in C(0)$, and we know that if T is an ordinary homogeneous operator from X to Y, then $T \in \mathbf{QH}_{\varphi_0}(X, Y)$ and $\mathbf{QH}_{\varphi_0}(X, Y)$ is just the family of all homogeneous operators from X to Y.

Definition 2.15. Let $(X, \mathscr{P})$ and $(Y, \tilde{\mathscr{P}})$ be F^* spaces, with $(Y, \tilde{\mathscr{P}})$ is locally convex.

(1) An operator $T : X \to Y$ is said to be γ-max-pseudo-norm-subadditive if there exists $\gamma > 0$, such that

$$\tilde{p}_\lambda(T(x + y)) \leqslant \gamma \cdot \max\{\tilde{p}_\lambda(Tx),\ \tilde{p}_\lambda(Ty)\} \tag{2.4}$$

for all $x, y \in X$ and $\lambda \in (0, 1]$;

(2) An operator $T : X \to Y$ is said to be γ-pseudo-norm-subadditive if there exists $\gamma > 0$, such that

$$\tilde{p}_\lambda(T(x + y)) \leqslant \gamma(\tilde{p}_\lambda(Tx) + \tilde{p}_\lambda(Ty)) \tag{2.5}$$

for all $x, y \in X$ and $\lambda \in (0, 1]$. In particular, T is pseudo-norm-subadditive if $\gamma = 1$.

Remark 2.16. It is easy to see that an additive operator must be γ-pseudo-norm-subadditive, and a γ-pseudo-norm-subadditive operator must be γ-max-pseudo-norm-subadditive since $\gamma(\tilde{p}_\lambda(Tx) + \tilde{p}_\lambda(Ty)) \leqslant 2\gamma \cdot \max\{\tilde{p}_\lambda(Tx),\ \tilde{p}_\lambda(Ty)\}$. But the converses are not always true. For example, we can refer to examples in [5] or Examples 2.17-2.20 in this paper.

If X and Y be two F^* spaces with Y is locally convex, by γ-$\mathbf{MSAQH}(X,Y)$ we denote the γ-max-pseudo-norm-subadditive and quasi-homogeneous operators from X to Y. By γ-$\mathbf{SAQH}(X,Y)$ we denote the γ-pseudo-norm-subadditive and quasi-homogeneous operators from X to Y. In particular, by γ-$\mathbf{MSAQH}_\varphi(X,Y)$ and γ-$\mathbf{SAQH}_\varphi(X,Y)$ we denote the γ-max-pseudo-norm-subadditive and γ-pseudo-norm-subadditive with φ-quasi-homogeneous operators from X to Y, respectively. Moreover, denote the continuous operators from X to Y by $\mathbf{C}(X,Y)$, the linear operators from X to Y by $\mathbf{L}(X,Y)$, and the bounded linear operators from X to Y by $\mathbf{BL}(X,Y)$. It is clear that

$$\mathbf{BL}(X,Y) \subset \gamma\text{-}\mathbf{SAQH}(X,Y) \cap \mathbf{C}(X,Y) \subset \gamma\text{-}\mathbf{MSAQH}(X,Y) \cap \mathbf{C}(X,Y),$$

but $\mathbf{BL}(X,Y)$ can be a proper subfamily of γ-$\mathbf{SAQH}(X,Y)\cap\mathbf{C}(X,Y)$, and γ-$\mathbf{SAQH}(X,Y)\cap\mathbf{C}(X,Y)$ can be a proper subfamily of γ-$\mathbf{MSAQH}(X,Y)\cap\mathbf{C}(X,Y)$.

Example 2.17. Let $(X,\mathscr{P}) = (C(\mathbb{R}^+), \{p_\lambda\}_{\lambda\in(0,1]})$ (see Example 2.7). For $1 \geqslant \alpha > 0$, we define the operator $T : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ as follows:

$$T(x)(s) = \textstyle\int_0^s |x(u)|^\alpha du \quad \text{for all } x \in X \quad \text{and } s \geqslant 0.$$

Then $T(x) \in X$ for all $x \in X$ and $T \in \mathbf{1}$-$\mathbf{SAQH}(X,X) \cap \mathbf{C}(X,X)$.

In fact, it is easy to see that $T(x) \in X$ for all $x \in X$, and $T \in \mathbf{QH}(X,X)$ with $\varphi(t) = |t|^\alpha \in C(0)$ is the eigenfunction of a quasi-homogeneous operator T. Note that $(a+b)^\alpha \leqslant a^\alpha + b^\alpha \quad (a,b \geqslant 0, 1 \geqslant \alpha > 0)$, for all $x, y \in X$ and $\lambda \in (0,1]$, we

$$\begin{aligned}
p_\lambda(T(x+y)) &= \sup_{s\in[0,\frac{1}{\lambda}-1]} \left|\int_0^s |x(u)+y(u)|^\alpha du\right| \\
&\leqslant \sup_{s\in[0,\frac{1}{\lambda}-1]} \left|\int_0^s (|x(u)|^\alpha + |y(u)|^\alpha) du\right| \\
&\leqslant \sup_{s\in[0,\frac{1}{\lambda}-1]} \int_0^s |x(u)|^\alpha du + \sup_{s\in[0,\frac{1}{\lambda}-1]} \int_0^s |y(u)|^\alpha du \\
&= p_\lambda(Tx) + p_\lambda(Ty),
\end{aligned}$$

which implies that the operator T is pseudo-norm-subadditive.

Furthermore, for any $\{x_n\}, x \in X$, if $x_n \to x \ (n\to\infty)$, then we have for each $\lambda \in (0,1]$, $p_\lambda(x_n - x) \to 0 \ (n \to \infty)$. Note that $|x_n(u)|^\alpha \leqslant (|x_n(u) - x(u)| + |x(u)|)^\alpha \leqslant |x_n(u) - x(u)|^\alpha + |x(u)|^\alpha$ for $1 \geqslant \alpha > 0$, we can obtain $||x_n(u)|^\alpha - |x(u)|^\alpha| \leqslant |x_n(u) - x(u)|^\alpha$. Then we have

$$\begin{aligned}
p_\lambda(Tx_n - Tx) &= \sup_{s\in[0,\frac{1}{\lambda}-1]} \left|\int_0^s |x_n(u)|^\alpha du - \int_0^s |x(u)|^\alpha du\right| \\
&\leqslant \sup_{s\in[0,\frac{1}{\lambda}-1]} \int_0^s \left||x_n(u)|^\alpha - |x(u)|^\alpha\right| du \\
&\leqslant \sup_{s\in[0,\frac{1}{\lambda}-1]} \int_0^s |x_n(u) - x(u)|^\alpha du \\
&\leqslant (\frac{1}{\lambda} - 1) \cdot (p_\lambda(x_n - x))^\alpha,
\end{aligned}$$

which implies that for each $\lambda \in (0,1]$, $p_\lambda(Tx_n - Tx) \to 0 \ (n \to \infty)$. This shows that $T \in \mathbf{C}(X,X)$. Hence $T \in \mathbf{1}$-$\mathbf{SAQH}(X,X) \cap \mathbf{C}(X,X)$.

Example 2.18. Let $(X,\mathscr{P})=(C(\mathbb{R}),\{p_\lambda\}_{\lambda\in(0,1]})$ (see Example 2.6). For $\alpha>1$, we define the operator $T:C(\mathbb{R})\to C(\mathbb{R})$ as follows:

$$T(x)(s)=|x(s)|^\alpha \ \text{ for all } x\in X \ \text{ and } s\in\mathbb{R}.$$

Then $T(x)\in X$ for all $x\in X$ and $T\in \mathbf{2}^\alpha\text{-}\mathbf{MSAQH}(X,X)\cap\mathbf{C}(X,X)$.

Obviously, $T(x)\in X$ for all $x\in X$, and $T\in\mathbf{QH}(X,X)$ with $\varphi(t)=|t|^\alpha\in C(0)$ is the eigenfunction of a quasi-homogeneous operator T. For all $x,y\in X$ and $\lambda\in(0,1]$, we have

$$\begin{aligned}
p_\lambda(T(x+y)) &= \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}|x(s)+y(s)|^\alpha\\
&\leqslant \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}(|x(s)|+|y(s)|)^\alpha\\
&\leqslant \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}(2^\alpha\max\{|x(s)|^\alpha,|y(s)|^\alpha\})\\
&\leqslant 2^\alpha\max\{p_\lambda(Tx),p_\lambda(Ty)\},
\end{aligned}$$

which implies that the operator T is 2^α-max-pseudo-norm-subadditive.

Furthermore, for any $\{x_n\},x\in X$, if $x_n\to x\ (n\to\infty)$, then we have for each $\lambda\in(0,1]$, $p_\lambda(x_n-x)\to 0\ (n\to\infty)$. Note that $\{x_n\}\cup\{x\}$ is a bounded set, by Definition 2.1, we know that for each $\lambda\in(0,1]$, there exists $M_\lambda>0$ such that $p_\lambda(y)\leqslant M_\lambda$ for all $y\in\{x_n\}\cup\{x\}$.

If $\alpha\in\{2,3,4,\cdots\}$, then we have

$$\begin{aligned}
p_\lambda(Tx_n-Tx) &= \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}||x_n(s)|^\alpha-|x(s)|^\alpha|\\
&= \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}||x_n(s)|-|x(s)||\cdot(|x_n(s)|^{\alpha-1}\\
&\quad +|x_n(s)|^{\alpha-2}|x(s)|+\cdots+|x(s)|^{\alpha-1})\\
&\leqslant \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}|x_n(s)-x(s)|\cdot(|x_n(s)|^{\alpha-1}\\
&\quad +|x_n(s)|^{\alpha-2}|x(s)|+\cdots+|x(s)|^{\alpha-1})\\
&\leqslant \alpha M_\lambda^{\alpha-1}\cdot p_\lambda(x_n-x),
\end{aligned}$$

which implies that for each $\lambda\in(0,1]$, $p_\lambda(Tx_n-Tx)\to 0\ (n\to\infty)$. This shows that $T\in\mathbf{C}(X,X)$.

If $1<\alpha\notin\{2,3,4,\cdots\}$, then there exists $m\in\{1,2,\cdots\}$ such that $\alpha-m\in(0,1)$. Similarly, we can obtain that

$$\begin{aligned}
p_\lambda(Tx_n-Tx) &= \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}||x_n(s)|^\alpha-|x(s)|^\alpha|\\
&= \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}\big||x_n(s)|^m\cdot(|x_n(s)|^{\alpha-m}-|x(s)|^{\alpha-m})\\
&\quad +|x(s)|^{\alpha-m}\cdot(|x_n(s)|^m-|x(s)|^m)\big|\\
&\leqslant \sup_{s\in[1-\frac{1}{\lambda},\frac{1}{\lambda}-1]}[\ |x_n(s)|^m\cdot\big||x_n(s)|^{\alpha-m}-|x(s)|^{\alpha-m}\big|\\
&\quad +|x(s)|^{\alpha-m}\cdot||x_n(s)|^m-|x(s)|^m|\ ]\\
&\leqslant M_\lambda^m\cdot(p_\lambda(x_n-x))^{\alpha-m}+mM_\lambda^{\alpha-1}\cdot p_\lambda(x_n-x),
\end{aligned}$$

which shows that for each $\lambda \in (0,1]$, $p_\lambda(Tx_n - Tx) \to 0$ $(n \to \infty)$; that is, $T \in \mathbf{C}(X,X)$. Hence $T \in \mathbf{2}^\alpha$-$\mathbf{MSAQH}(X,X) \cap \mathbf{C}(X,X)$.

Example 2.19. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces, and $f \in \mathbf{BL}(X,Y)$. For $0 < \alpha < 1$, define $T : X \to Y$ by

$$T_f(x) = \begin{cases} \frac{f(x)}{\|f(x)\|^\alpha}, & f(x) \neq \theta, \\ \theta, & f(x) = \theta, \end{cases} \quad \text{for all } x \in X.$$

Then the family $\{T_f\}_{f \in \mathbf{BL}(X,Y)} \subset \mathbf{1}$-$\mathbf{SAQH}(X,Y) \cap \mathbf{C}(X,Y)$.

For each $f \in \mathbf{BL}(X,Y)$, $T_f \in \mathbf{QH}(X,Y)$ is obvious, since $\varphi(t) \in C(0)$, where

$$\varphi(t) = \begin{cases} t/|t|^\alpha, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

For all $x, y \in X$, without loss of generality, suppose that $x \neq \theta$ and $y \neq \theta$. Note that $f \in \mathbf{BL}(X,Y)$ and $0 < \alpha < 1$, we have

$$\begin{aligned} \|T_f(x+y)\| &= \|f(x+y)\|^{1-\alpha} \leqslant (\|f(x)\| + \|f(y)\|)^{1-\alpha} \\ &\leqslant \|f(x)\|^{1-\alpha} + \|f(y)\|^{1-\alpha} = \|T_f(x)\| + \|T_f(y)\|, \end{aligned}$$

which implies that T_f is pseudo-norm-subadditive.

Moreover, for any $\{x_n\}, x \in X$, if $x_n \to x$ $(n \to \infty)$, by $f \in \mathbf{BL}(X,Y)$, we can know that $f(x_n) \to f(x)$ $(n \to \infty)$. If $x \neq \theta$, without loss of generality, we can suppose that $x_n \neq \theta$ for all $n = 1, 2, \cdots$. Note that $f \in \mathbf{BL}(X,Y)$ and $f(x_n) \to f(x)$ $(n \to \infty)$, we have

$$\begin{aligned} \|T_f(x_n) - T_f(x)\| &= \left\| \frac{f(x_n)}{\|f(x_n)\|^\alpha} - \frac{f(x)}{\|f(x)\|^\alpha} \right\| \\ &\leqslant \frac{\|f(x_n) - f(x)\|}{\|f(x_n)\|^\alpha} + \frac{\|f(x)\| \cdot |\|f(x)\|^\alpha - \|f(x_n)\|^\alpha|}{\|f(x_n)\|^\alpha \cdot \|f(x)\|^\alpha} \to 0 \quad (n \to \infty), \end{aligned}$$

which implies that $T_f \in \mathbf{C}(X,Y)$. If $x = \theta$, $T_f \in \mathbf{C}(X,Y)$ is clear. Thus $\{T_f\}_{f \in \mathbf{BL}(X,Y)} \subset \mathbf{1}$-$\mathbf{SAQH}(X,Y) \cap \mathbf{C}(X,Y)$.

Example 2.20. Let $X = \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x^2$ for all $x \in \mathbb{R}$.Then $T \in \mathbf{2}$-$\mathbf{SAQH}_\varphi(X,X) \cap \mathbf{C}(X,X)$, where $\varphi(t) = t^2 \in C(0)$.

3. Main results

For the sake of brevity, before we state the main results we introduce the following definitions.

Definition 3.1. Let $(X, \mathscr{P})$ and $(Y, \tilde{\mathscr{P}})$ be F^* spaces. A family $\{T_\beta\}_{\beta \in \Lambda}$ of quasi-homogeneous operators from X to Y is said to be uniformly quasi-homogeneous if $\lim_{t \to 0} \varphi_\beta(t) = 0$ uniformly for $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β.

Definition 3.2. Let $(X, \mathscr{P})$ and $(Y, \tilde{\mathscr{P}})$ be F^* spaces. A family $\{T_\beta\}_{\beta \in \Lambda}$ of γ-max-pseudo-norm-subadditive (or γ-pseudo-norm-subadditive) operators from X to Y is said to be uniformly γ-bounded if $\sup_{\beta \in \Lambda}\{\gamma_\beta\} < +\infty$.

Let $\delta > 0$ and $\omega \geqslant 1$ are two constants, then we denote by $C_{\delta,\omega}(0)$ the set

$$C_{\delta,\omega}(0) = \{\varphi \in C(0) : \delta \leqslant \varphi(t) \leqslant t^\omega \quad \text{for all} \quad t \in [1, +\infty)\}.$$

Theorem 3.3. *Let $(X,\mathscr{P})$ be a F^* space of second category and let $(Y,\tilde{\mathscr{P}})$ be a locally convex F^* space. Assume that $\{T_\beta\}_{\beta\in\Lambda}\subset\gamma$-$\mathbf{MSAQH}(X,Y)\cap\mathbf{C}(X,Y)$ satisfying:*

(a) *$\{T_\beta\}_{\beta\in\Lambda}$ is uniformly quasi-homogeneous, and $\varphi_\beta\in C_{\delta,\omega}(0)$ for each $\beta\in\Lambda$, where φ_β is the eigenfunction of T_β;*

(b) *$\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded;*

(c) *For each $x\in X$, the set $\{T_\beta(x):\ \beta\in\Lambda\}$ is $\tilde{\mathscr{P}}$-bounded in Y.*

Then $\bigcup_{\beta\in\Lambda}T_\beta(A)$ is a $\tilde{\mathscr{P}}$-bounded set in Y for each $\mathscr{P}$-bounded set A in X.

To prove Theorem 3.3, we need the following some lemmas:

Lemma 3.4. *Let $(X,\mathscr{P})$ be a locally convex F^* space. Then for each $\lambda\in(0,1]$, $V_\lambda=\{x\in X:p_\lambda(x)\leqslant\lambda\}$ is a closed, convex and absorbing set.*

Proof. Let $x,y\in V_\lambda$ and $t\in[0,1]$. By Lemma 2.1 and Lemma 2.5, we have

$$p_\lambda(tx+(1-t)y)\leqslant p_\lambda(tx)+p_\lambda((1-t)y)\leqslant t\lambda+(1-t)\lambda=\lambda,$$

and so $tx+(1-t)y\in V_\lambda$. Hence V_λ is convex.

Let $\{x_n\}\subset V_\lambda$ and $x_n\xrightarrow{\mathscr{T}}x\in X$ $(n\to\infty)$. Then we have

$$p_\lambda(x)\leqslant p_\lambda(x-x_n)+p_\lambda(x_n)\leqslant p_\lambda(x-x_n)+\lambda.$$

Letting $n\to\infty$, we obtain $p_\lambda(x)\leqslant\lambda$. So V_λ is closed. By Lemma 2.1 we know that V_λ is a neighborhood of θ in $(X,\mathscr{P})$. Therefore, V_λ is absorbing. □

Lemma 3.5. *(cf.[9, 12]) Let $\varphi\in C(0)$. If there exists a non-zero and continuous operator $T:X\to Y$ such that $T\in\mathbf{QH}_\varphi(X,Y)$, then φ has the following properties:*

(1) *$\varphi(1)=1=|\varphi(-1)|$;*

(2) *$\varphi(t)$ is continuous on $[0,+\infty)$;*

(3) *$\varphi(t)>0$ for all $t>0$;*

(4) *$\varphi(1/t)=1/\varphi(t)$ for all $t>0$.*

Proof. of Theorem 3.3. For each $\lambda\in(0,1]$, we now denote by

$$V_\lambda=\{y\in Y:\tilde{p}_\lambda(y)\leqslant 1\}\quad\text{and}\quad W_\lambda=\bigcap_{\beta\in\Lambda}T_\beta^{-1}(V_\lambda).$$

We shall first prove that W_λ is a closed and absorbing set for each $\lambda\in(0,1]$.

By Lemma 3.4 , we know that V_λ is a closed set in Y. Note that T_β is continuous, we have $T_\beta^{-1}(V_\lambda)$ is a closed set in X, and so W_λ is a closed set in X.

Moreover, since T_β is quasi-homogeneous, $T_\beta(\theta)=T_\beta(0\cdot x)=\varphi_\beta(0)T_\beta(x)=\tilde{\theta}\in V_\lambda$, and so $\theta\in W_\lambda$. Note that condition (c), $\{T_\beta(x):\beta\in\Lambda\}$ is a $\tilde{\mathscr{P}}$-bounded set in Y for each $x\in X$. Hence, from Definition 2.8, we know that for each $\lambda\in(0,1]$ there exists $M_\lambda>0$ such that $\tilde{p}_\lambda(T_\beta(x))\leqslant M_\lambda$ for all $\beta\in\Lambda$. Further, applying (3) of Lemma 3.5, we have $\varphi_\beta(t)>0$ for each $\beta\in\Lambda$ and $t>0$. By condition (a), $\lim_{t\to0}\varphi_\beta(t)=0$ uniformly for $\beta\in\Lambda$. Hence for the above $M_\lambda>0$ there exists $m_\lambda\in\mathbb{N}$ such that $0<\varphi_\beta(1/m_\lambda)\leqslant 1/M_\lambda$ for all $\beta\in\Lambda$. And then it is easy to obtain that

$$\begin{aligned}\tilde{p}_\lambda(T_\beta(\frac{1}{m_\lambda}x)) &= \tilde{p}_\lambda(\varphi_\beta(\frac{1}{m_\lambda})T_\beta(x))=\varphi_\beta(\frac{1}{m_\lambda})\tilde{p}_\lambda(T_\beta(x))\\ &\leqslant \varphi_\beta(\frac{1}{m_\lambda})M_\lambda\leqslant 1,\end{aligned}$$

which implies that $\frac{1}{m_\lambda}x\in\bigcap_{\beta\in\Lambda}T_\beta^{-1}(V_\lambda)=W_\lambda$, i.e., $x\in m_\lambda W_\lambda$. These show that W_λ is a absorbing set for each $\lambda\in(0,1]$.

In the next step we shall estimate the upper bounded of $\{\tilde{p}_\lambda(T_\beta(x)):\ \beta\in\Lambda\}$ for every $x\in X$.

Note that absorptivity of W_λ, we have $X = \bigcup_{k=1}^{\infty} kW_\lambda$. By $(X, \mathscr{P})$ is of second category , W_λ is closed and $\{U(\lambda,\lambda) : \ \lambda \in (0,1]\}$ is also a neighborhood base of θ in $(X, \mathscr{P})$, it is not difficult to see that there exist $k_0 \in \mathbb{N}$, $x_0 \in X$ and $\lambda_0 \in (0,1]$ such that

$$x_0 + U(\lambda_0, \lambda_0) \subset \overline{k_0 W_\lambda} = k_0 W_\lambda.$$

For any $x \in X$ and $\varepsilon > 0$, taking $z = x/(p_{\lambda_0}(x) + \varepsilon)$. It is clear that

$$x_1 = x_0 + \frac{\lambda_0 z}{2} \in x_0 + U(\lambda_0, \lambda_0) \subset k_0 W_\lambda,$$
$$x_2 = x_0 - \frac{\lambda_0 z}{2} \in x_0 + U(\lambda_0, \lambda_0) \subset k_0 W_\lambda,$$

and so $x_1/k_0, x_2/k_0 \in W_\lambda \subset T_\beta^{-1}(V_\lambda)$. This shows that $\tilde{p}_\lambda(T_\beta(x_1/k_0)) \leqslant 1$ and $\tilde{p}_\lambda(T_\beta(x_2/k_0)) \leqslant 1$. Note that T_β is γ_β-max-pseudo-norm-subadditve, applying (1) of Lemma 3.5, it is not difficult to obtain that

$$\tilde{p}_\lambda(T_\beta(\frac{x_1 - x_2}{k_0})) \leqslant \gamma_\beta \cdot \max\{\tilde{p}_\lambda(T_\beta(\frac{x_1}{k_0})), \tilde{p}_\lambda(T_\beta(-\frac{x_2}{k_0}))\} \leqslant \gamma_\beta. \tag{3.1}$$

Obviously $\frac{x_1 - x_2}{k_0} = \frac{\lambda_0 x}{k_0(p_{\lambda_0}(x)+\varepsilon)}$. By using (4) of Lemma 3.5, from (3.1) we can obtain that

$$\tilde{p}_\lambda(T_\beta(x)) \leqslant \gamma_\beta \varphi_\beta(k_0(p_{\lambda_0}(x) + \varepsilon)/\lambda_0) \ \text{ for all } \ \beta \in \Lambda.$$

From (2) of Lemma 3.5, we know that φ_β is continuous on $[0, +\infty)$. Hence, letting $\varepsilon \to 0$ in the above inequality, we can obtain that

$$\tilde{p}_\lambda(T_\beta(x)) \leqslant \gamma_\beta \varphi_\beta(k_0 p_{\lambda_0}(x)/\lambda_0) \ \text{ for all } \ \beta \in \Lambda.$$

Note that condition (b), $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded. Hence there exists $M > 0$ such that $\gamma_\beta \leqslant M$ for all $\beta \in \Lambda$. We have

$$\tilde{p}_\lambda(T_\beta(x)) \leqslant M \varphi_\beta(k_0 p_{\lambda_0}(x)/\lambda_0) \ \text{ for all } \ \beta \in \Lambda. \tag{3.2}$$

Indeed, for (3.2) we have three cases to consider:

Case 1: $p_{\lambda_0}(x) = 0$. By Definition 2.13, we have $\varphi_\beta(0) = 0$. Then it follows from (3.2) that $\sup_{\beta\in\Lambda} \tilde{p}_\lambda(T_\beta(x)) = 0$.

Case 2: $0 < p_{\lambda_0}(x) \leqslant \frac{\lambda_0}{k_0}$. This implies that $\frac{\lambda_0}{k_0 p_{\lambda_0}(x)} \geqslant 1$. Note that $\varphi_\beta \in C_{\delta,w}(0)$, we have $\varphi_\beta\left(\frac{\lambda_0}{k_0 p_{\lambda_0}(x)}\right) \geqslant \delta$ for all $\beta \in \Lambda$. Applying (4) of Lemma 3.5, we can obtain that $\varphi_\beta\left(\frac{k_0 p_{\lambda_0}(x)}{\lambda_0}\right) \leqslant \frac{1}{\delta}$ for all $\beta \in \Lambda$. So, from (3.2) it is not difficult to see that $\sup_{\beta\in\Lambda} \tilde{p}_\lambda(T_\beta(x)) \leqslant M/\delta$.

Case 3: $p_{\lambda_0}(x) > \frac{\lambda_0}{k_0}$. This shows that $\frac{k_0 p_{\lambda_0}(x)}{\lambda_0} > 1$. Since $\varphi_\beta \in C_{\delta,w}(0)$, we can obtain that

$$\varphi_\beta\left(\frac{k_0 p_{\lambda_0}(x)}{\lambda_0}\right) \leqslant \left(\frac{k_0 p_{\lambda_0}(x)}{\lambda_0}\right)^w,$$

and so from (3.2) we have

$$\tilde{p}_\lambda(T_\beta(x)) \leqslant M\left(\frac{k_0 p_{\lambda_0}(x)}{\lambda_0}\right)^w \ \text{ for all } \ \beta \in \Lambda.$$

In view of the above discussions, we can claim that

$$\sup_{\beta\in\Lambda} \tilde{p}_\lambda(T_\beta(x)) \leqslant \max\left\{\frac{M}{\delta}, M\left(\frac{k_0 p_{\lambda_0}(x)}{\lambda_0}\right)^w\right\}. \tag{3.3}$$

Finally, We shall prove that $\sup_{x\in A}\sup_{\beta\in\Lambda}\tilde{p}_\lambda(T_\beta(x)) < +\infty$ for each $\mathscr{P}$-bounded set A in X. Note that A is a $\mathscr{P}$-bounded set in X, by Definition 2.8 we can find that for the above λ_0 there exists $M_0 = M_0(\lambda_0) > 0$ such that $p_{\lambda_0}(x) \leqslant M_0$ for all $x \in A$. Thus, from (3.3) we have

$$\sup_{x\in A}\sup_{\beta\in\Lambda}\tilde{p}_\lambda(T_\beta(x)) \leqslant \max\left\{\frac{M}{\delta}, M\left(\frac{k_0M_0}{\lambda_0}\right)^w\right\} < +\infty$$

for each $\lambda \in (0, 1]$. This implies that $\bigcup_{\beta\in\Lambda} T_\beta(A)$ is a $\tilde{\mathscr{P}}$-bounded set in Y. This makes end to the proof. □

Lemma 3.6. *(cf. [14, 16]) Each F space (i.e. complete F^* space) $(X, \mathscr{P})$ is of the second category.*

Form Lemma 3.6 we can immediately deduce the following theorem.

Theorem 3.7. *If in Theorem 3.3 we replace the F^* space of second category $(X, \mathscr{P})$ by the F space $(X, \mathscr{P})$, then the conclusion of Theorem 3.3 remains true.*

Note that $\{T_\beta\}_{\beta\in\Lambda} \subset \mathbf{SAQH}(X, Y) \subset \gamma\text{-}\mathbf{SAQH}(X, Y) \subset \gamma\text{-}\mathbf{MSAQH}(X, Y)$, by Lemma 3.6, it is easy to obtain the following corollaries.

Corollary 3.8. *If in Theorem 3.3 we replace $\{T_\beta\}_{\beta\in\Lambda} \subset \gamma\text{-}\mathbf{MSAQH}(X, Y)$ by $\{T_\beta\}_{\beta\in\Lambda} \subset \gamma\text{-}\mathbf{SAQH}(X, Y)$, then the conclusion of Theorem 3.3 remains true.*

Corollary 3.9. *Let $(X, \mathscr{P})$ be a F^* space of second category (or a F space) and let $(Y, \tilde{\mathscr{P}})$ be a locally convex F^* space. Assume that $\{T_\beta\}_{\beta\in\Lambda} \subset \mathbf{SAQH}(X, Y) \cap \mathbf{C}(X, Y)$, with $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly quasi-homogeneous, and $\varphi_\beta \in C_{\delta,\omega}(0)$ for each $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β. If for each $x \in X$, the set $\{T_\beta(x) : \beta \in \Lambda\}$ is $\tilde{\mathscr{P}}$-bounded in Y, then the conclusion of Theorem 3.3 remains true.*

Proof. Since $\{T_\beta\}_{\beta\in\Lambda} \subset \mathbf{SAQH}(X, Y)$, $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded by Definition 3.2. This shows that condition (b) in Theorem 3.3 is satisfied. In addition, by the assumption, we know that conditions (a) and (c) in Theorem 3.3 are also satisfied. Therefore the conclusion follows from Theorem 3.3 immediately. □

Theorem 3.10. *Let $(X, \mathscr{P})$ be a F^* space of second category (or a F space) and let $(Y, \tilde{\mathscr{P}})$ be a locally convex F^* space. Assume that $T : X \to Y$ be a γ-max-pseudo-norm-subadditive, quasi-homogeneous and continuous operator. Then T is a bounded operator, i.e., $T(A)$ is a $\tilde{\mathscr{P}}$-bounded set in Y for each $\mathscr{P}$-bounded set A in X.*

Proof. Obviously, $\{T(x)\}$ is a $\tilde{\mathscr{P}}$-bounded set in Y for each $x \in X$. As in the proof of Theorem 3.3, we can prove that there exist $k_0 \in \mathbb{N}$ and $\lambda_0 \in (0, 1]$ such that

$$\tilde{p}_\lambda(T(x)) \leqslant \gamma\varphi(k_0p_{\lambda_0}(x)/\lambda_0) \quad \text{for each} \quad \lambda \in (0, 1]. \tag{3.4}$$

Note that A is a $\mathscr{P}$-bounded set in X, by Definition 2.8 we know that for the above λ_0 there exists $M_0 = M_0(\lambda_0) > 0$ such that $p_{\lambda_0}(x) \leqslant M_0$ for all $x \in A$. Thus, from (3.4) and (2) of Lemma 3.5 we have

$$\sup_{x\in A}\tilde{p}_\lambda(T(x)) \leqslant \sup_{x\in A}\gamma\varphi(k_0p_{\lambda_0}(x)/\lambda_0) \leqslant \gamma \max_{t\in[0,\frac{k_0M_0}{\lambda_0}]}\{\varphi(t)\} < +\infty$$

for each $\lambda \in (0, 1]$. This implies that $T(A)$ is a $\tilde{\mathscr{P}}$-bounded set in Y. □

Applying Theorem 3.3 (or Theorem (3.7)) , it follows from Lemma 2.11 and Remark 2.12 that:

Theorem 3.11. *Let $(X, \|\cdot\|_X)$ be a normed space of second category (or a Banach space) and let $(Y, \|\cdot\|_Y)$ be a normed space. Let $\{T_\beta\}_{\beta\in\Lambda} \subset \gamma\text{-}\mathbf{MSAQH}(X, Y) \cap \mathbf{C}(X, Y)$, $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded, uniformly quasi-homogeneous and $\varphi_\beta \in C_{\delta,\omega}(0)$ for each $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β. If for each $x \in X$, the set $\{T_\beta(x) : \beta \in \Lambda\}$ is bounded in Y, then $\bigcup_{\beta\in\Lambda} T_\beta(A)$ is bounded in Y for each bounded set A in X.*

Remark 3.12. Note that $\mathbf{SAQH}(X,Y) \subset \gamma\text{-}\mathbf{SAQH}(X,Y) \subset \gamma\text{-}\mathbf{MSAQH}(X,Y)$, we easily obtain Theorem 3.3 in [12]. In addition, it is easy to see that Theorem 3.11 is the extension of classical uniform boundedness principle on normed spaces.

Theorem 3.13. *Let $(X, \|\cdot\|_X)$ be a normed space of second category (or a Banach space) and let $(Y, \|\cdot\|_Y)$ be a normed space. Assume that $\{T_\beta\}_{\beta\in\Lambda} \subset \gamma\text{-}\mathbf{MSAQH}(X,Y) \cap \mathbf{C}(X,Y)$ satisfying:*

(a) *$\varphi_\beta \in C_{\delta,\omega}(0)$ for each $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β;*

(b) *$\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded;*

(c) *There exists some subset E of second category in X such that for each $x \in E$, the set $\{T_\beta(x) : \beta \in \Lambda\}$ is bounded in Y.*

Then $\bigcup_{\beta\in\Lambda} T_\beta(A)$ is bounded in Y for each bounded set A in X.

Proof. Obviously, for each bounded set A in X, there exists $r > 0$ such that

$$A \subset B(\theta, r) = \{x \in X : \|x\|_X < r\}.$$

Thus we need only to prove that $\sup_{\beta\in\Lambda} \sup_{x\in B(\theta,r)}\{\|T_\beta(x)\|_Y\} < +\infty$ for each $r > 0$.

Let $p(x) = \sup_{\beta\in\Lambda}\{\|T_\beta(x)\|_Y\}$ for all $x \in X$, where $p(x) = +\infty$ is admissible. For each $k \in \mathbb{N}$, set

$$W_k = \{x : p(x) < k, x \in E\}.$$

Note that condition (c), by Definition 2.8 and Remark 2.12, we have $E = \bigcup_{k=1}^{\infty} W_k$. By E is of second category, we can see that there exist $k_0 \in \mathbb{N}, \delta_0 > 0$ and $x_0 \in W_{k_0}$ such that

$$B(x_0, \delta_0) = \{x \in X : \|x - x_0\|_X < \delta_0\} \subset \overline{W}_{k_0}.$$

Below, we shall prove that $p(x)$ is upper bounded on $B(\theta, \delta_0)$. In fact, for each $\beta \in \Lambda$ and $x \in B(\theta, \delta_0)$, by $T_\beta \in \mathbf{C}(X,Y)$, we have that for a given $\varepsilon_0 > 0$ there exists $\delta_{(x,\beta)} > 0$ such that

$$\|T_\beta(x)\|_Y \leqslant \varepsilon_0 + \|T_\beta(y)\|_Y \quad \text{for all} \quad y \in B(x, \delta_{(x,\beta)}). \tag{3.5}$$

It is clear that $x_0 + x \in B(x_0, \delta_0)$. Since W_{k_0} is dense in $B(x_0, \delta_0)$, there exists a $y_0 \in W_{k_0}$ such that $\|y_0 - x_0 - x\|_X = \|y_0 - (x_0 + x)\|_X < \delta_{(x,\beta)}$. By $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded, we know that there exists a $M > 0$ such that $\sup_{\beta\in\Lambda}\{\gamma_\beta\} < M$. Note that $T_\beta \in \gamma\text{-}\mathbf{MSAQH}(X,Y)$, from (1) of Lemma 3.5 and (3.5), we have

$$\begin{aligned}\|T_\beta(x)\|_Y &\leqslant \varepsilon_0 + \|T_\beta(y_0 - x_0)\|_Y \leqslant \varepsilon_0 + \gamma_\beta \max\{\|T_\beta(y_0)\|_Y, \|T_\beta(-x_0)\|_Y\}\\ &= \varepsilon_0 + \gamma_\beta \max\{\|T_\beta(y_0)\|_Y, |\varphi_\beta(-1)| \cdot \|T_\beta(x_0)\|_Y\},\end{aligned}$$

i,e.,

$$\|T_\beta(x)\|_Y \leqslant \varepsilon_0 + Mk_0. \tag{3.6}$$

From (3.6) we can see that $\varepsilon_0 + Mk_0$ is independent of $\beta \in \Lambda$ and $x \in B(\theta, \delta_0)$. This shows that $p(x)$ is upper bounded on $B(\theta, \delta_0)$.

Moreover, we know that for each $r > 0$, there exists a $t_r > 0$ such that $B(\theta, r) \subset t_r B(\theta, \delta_0)$. Thus for each $x \in B(\theta, r)$, there exists a $\overline{x} \in B(\theta, \delta_0)$ such that $x = t_r \overline{x}$. Note that $\varphi_\beta \in C_{\delta,\omega}(0)$, by using (4) of Lemma 3.5 and 3.6, we can obtain that for each $x \in B(\theta, r)$ such that

$$\begin{aligned}p(x) &= \sup_{\beta\in\Lambda} \|T_\beta(x)\|_Y = \sup_{\beta\in\Lambda} \|T_\beta(t_r\overline{x})\|_Y = \sup_{\beta\in\Lambda} |\varphi_\beta(t_r)| \cdot \|T_\beta(\overline{x})\|_Y\\ &\leqslant (\varepsilon_0 + Mk_0) \sup_{\beta\in\Lambda} |\varphi_\beta(t_r)|\\ &\leqslant \begin{cases} (\varepsilon_0 + Mk_0) \cdot t_r^\omega, & \text{if } t_r \in [1, +\infty),\\ (\varepsilon_0 + Mk_0) \cdot \frac{1}{\delta}, & \text{if } t_r \in (0, 1)\end{cases}\\ &< +\infty,\end{aligned}$$

which implies that $\sup_{\beta\in\Lambda} \sup_{x\in B(\theta,r)}\{\|T_\beta(x)\|_Y\} < +\infty$ for each $r > 0$. Hence the conclusion of theorem holds. □

4. Some applications

In this section, we shall give some applications concerning our results. As example, first we shall apply Theorem 3.3 to establish uniform boundedness principles for family of pointwise probabilistic bounded γ-max-probabilistic subadditive and quasi-homogeneous operators in Menger probabilistic normed spaces. To complete the results, we need state some basic concepts and results which will be used in Menger probabilistic normed spaces (cf. [3] or [13]).

We denote by $\mathscr{D}$ the set of all (left-continuous) distribution functions F satisfying $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$. $\mathscr{D}^+ = \{F \in \mathscr{D} : F(0) = 0\}$ and H is a specific distribution function defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t > 0. \end{cases}$$

A function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:
for any $a, b, c, d \in [0, 1], \Delta(a, 1) = a$; $\Delta(a, b) = \Delta(b, a)$; $a \geqslant b,\ c \geqslant d \Rightarrow \Delta(a, c) \geqslant \Delta(b, d)$;
$\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 4.1. (cf.[3, 13]) A triplet $(X, \mathscr{F}, \Delta)$ is a Menger probabilistic normed space (for short, Menger PN-space) if X is a real linear space, Δ is a t-norm and $\mathscr{F} : X \to \mathscr{D}^+$ is a mapping satisfying the following conditions (in the sequel, $\mathscr{F}(x)$ is denoted by F_x):
(PN-1) $F_x(t) = H(t)$ for all $t \in \mathbb{R}$ iff $x = \theta$;
(PN-2) $F_{\alpha x}(t) = F_x(t/|\alpha|)$, for $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(PN-3) $F_{x+y}(s + t) \geqslant \Delta(F_x(s), F_y(t))$ for all $x,\ y \in X$ and $s,\ t \geqslant 0$.

Lemma 4.2. *(cf.[3, 13]) Let $(X, \mathscr{F}, \Delta)$ be a Menger PN-space, Δ satisfies the condition:*

$$\sup_{0<t<1} \Delta(t, t) = 1, \tag{4.1}$$

then $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological vector space in the topology $\mathscr{T}$ induced by the neighborhood base of θ

$$\mathscr{U} = \{N(\varepsilon, \lambda) : \varepsilon > 0,\ \lambda \in (0, 1]\},$$

where

$$N(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}. \tag{4.2}$$

Remark 4.3. The above topology $\mathscr{T}$ on $(X, \mathscr{F}, \Delta)$ is called its (ε, λ)-topology. In addition, it is not difficult to see that $\{U(1/n, 1/n) : n \in \mathbb{N}\}$ is also a neighborhood base of θ in $(X, \mathscr{F}, \Delta)$. Hence $(X, \mathscr{F}, \Delta)$ satisfies the first countability axiom.

In the following, we always assume that a t-norm Δ satisfies condition (4.1), unless otherwise stated.

Definition 4.4. (cf.[3]) Let $(X, \mathscr{F}, \Delta)$ be a Menger PN-space and $A \subset X$. A is said to be a probabilistic bounded set if $\sup_{t>0} \inf_{x\in A} F_x(t) = 1$.

Lemma 4.5. *Let $(X, \mathscr{F}, \Delta)$ be a Menger PN-space. Define a functional $p_\lambda(\cdot)$ on X by*

$$p_\lambda(x) = \inf\{t > 0 \mid x \in N(t, \lambda)\}, \tag{4.3}$$

for each $\lambda \in (0, 1]$ and $x \in X$. Then $(X, \mathscr{P})$ is a F^* space, called the induced F^* space by Menger PN-space $(X, \mathscr{F}, \Delta)$, and $p_\lambda(\cdot)$ is called the induced pseudo-norm by $\mathscr{F}$, where $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ is its standard generating family of pseudo-norms satisfying conditions (P-1)- (P-4).

Proof. Firstly, according to Lemma 4.2 and the fact that $\{N(1/n, 1/n) \mid n \in \mathbb{N}\}$ is also a $\mathscr{T}$-neighborhood base of θ, $(X, \mathscr{F}, \Delta)$ is a F^* space. By (4.3) and the non-increasing and left continuity of the distribution function $F_x(\cdot)$, we have

$$p_\lambda(x) < t \iff x \in N(t, \lambda) \iff F_x(t) > 1 - \lambda. \tag{4.4}$$

We now prove that $\mathscr{P}$ satisfies conditions (P-1)−(P-4).

(P-1) Obviously, $0 < \mu < \lambda \Rightarrow N(t, \mu) \subset N(t, \lambda)$ $(\forall t > 0) \Rightarrow p_\lambda(x) \leqslant p_\mu(x)$. Hence $p_\lambda(x) \leqslant \inf_{0<\mu<\lambda} p_\mu(x)$. On the other hand, by (4.3), for any $\varepsilon > 0$, there exists $0 < t < p_\lambda(x) + \varepsilon$ such that $x \in N(t, \lambda)$, i.e., $F_x(t) > 1-\lambda$. Evidently, there exists $\mu_0 \in (0, \lambda)$ such that $F_x(t) > 1-\mu_0$, i.e., $x \in N(t, \mu_0)$. By (4.4), we have $p_{\mu_0}(x) < t < p_\lambda(x)+\varepsilon$. By the arbitrariness of ε, $\inf_{0<\mu<\lambda} p_\mu(x) \leqslant p_{\mu_0}(x) \leqslant p_\lambda(x)$. Therefore $p_\lambda(x) = \inf_{0<\mu<\lambda} p_\mu(x)$, and (P-1) is satisfied.

(P-2) For any $k \in \mathbb{K}$ with $k \neq 0$, note that

$$\begin{aligned} p_\lambda(kx) < t \iff kx \in N(t, \lambda) \iff F_{kx}(t) = F_x\big(\frac{t}{|k|}\big) > 1 - \lambda \\ \iff x \in N\big(\frac{t}{|k|}, \lambda\big) \iff p_\lambda(x) < \frac{t}{|k|} \\ \iff |k| p_\lambda(x) < t. \end{aligned}$$

Thus $p_\lambda(kx) = |k|p_\lambda(x)$, which also holds for $k = 0$. Therefore, (P-2) holds.

(P-3) Since $\sup_{0<t<1} \Delta(t, t) = 1$, for each $\lambda \in (0, 1]$, there exists $\mu \in (0, \lambda)$ such that $\Delta(1-\mu, 1-\mu) > 1-\lambda$. This implies that for any $t, s > 0$, $N(t, \mu) + N(s, \mu) \subset N(t + s, \lambda)$. Then by (4.4), we have $p_\lambda(x + y) \leq p_\mu(x) + p_\mu(y)$, and (P-3) is satisfied.

(P-4) Since $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological vector space with $\mathscr{U} = \{N(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0, 1]\}$ a $\mathscr{T}$-neighborhood base of θ, we have $\bigcap\{N(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0, 1]\} = \{\theta\}$. Note that

$$\begin{aligned} p_\lambda(x) = 0\ (\forall\, \lambda \in (0, 1]) &\iff x \in N(\varepsilon, \lambda)\ (\forall\, \varepsilon > 0,\ \lambda \in (0, 1]) \\ &\iff x \in \bigcap\{N(\varepsilon, \lambda) \mid \varepsilon > 0,\ \lambda \in (0, 1]\}. \end{aligned}$$

Therefore, (P-4) holds. This shows that $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ is the standard generating family of pseudo-norms in X. □

Remark 4.6. If we assume $\Delta = \min$ in Lemma 4.5, then $(X, \mathscr{F}, \Delta)$ is a locally convex F^* space with respect to its (ε, λ)-topology $\mathscr{T}$, and $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ defined by (4.3) is its family of standard generating semi-norms satisfying conditions (P-1)−(P-4). In fact, it suffices to show that for any $\varepsilon > 0$, $\lambda \in (0, 1]$, $N(\varepsilon, \lambda)$ is convex set in X. Taking any $x, y \in N(\varepsilon, \lambda)$ and $t \in [0, 1]$, by (4.2) in Lemma 4.2 and (PN-2), (PN-3) in Definition 4.1, we have

$$F_{tx+(1+t)y}(\varepsilon) \geqslant \min\{F_{tx}(t\varepsilon), F_{(1-t)y}((1 - t)\varepsilon)\} = \min\{F_x(\varepsilon), F_y(\varepsilon)\} > 1 - \lambda,$$

which implies $tx + (1 + t)y \in N(\varepsilon, \lambda)$, i.e., $N(\varepsilon, \lambda)$ is a convex set in X.

Lemma 4.7. *Let $(X, \mathscr{F}, \Delta)$ be a Menger PN-space. Let $\mathscr{P} = \{p_\lambda \mid \lambda \in (0, 1]\}$ be its standard generating family of pseudo-norms, where $p_\lambda(\cdot)$ is defined by* (4.3)*, and $A \subset X$. Then A is a probabilistic bounded set iff A is a $\mathscr{P}$-bounded set.*

Proof. Denote $D(A) = \sup_{t>0} \inf_{x\in A} F_x(t)$. Suppose A is a probabilistic bounded set, i.e., $D(A) = 1$, then for each $\lambda \in (0, 1]$, there exists $t_0 > 0$ such that $F_x(t_0) > 1 - \lambda$ for all $x \in A$. Then from (4.4) we get $\sup_{x\in A} p_\lambda(x) \leqslant t_0 < +\infty$, i.e., A is a $\mathscr{P}$-bounded set.

Conversely, suppose A is a $\mathscr{P}$-bounded set, i.e., for each $\lambda \in (0, 1]$, there exists $t_0 = t_0(\lambda) > 0$ such that $p_\lambda(x) < t_0$ $(\forall\, x \in A)$. According to (4.4), $D(A) \geqslant \inf_{x\in A} F_x(t_0) \geq 1-\lambda$. By the randomness of λ, $D(A) = 1$. Therefore, A is a probabilistic bounded set. □

Definition 4.8. Let $(X, \mathscr{F}, \Delta_1)$ and $(Y, \tilde{\mathscr{F}}, \Delta_2)$ be Menger PN-spaces.

(1) An operator from X to Y is said to be γ-max-probabilistic subadditive if there exists $\gamma > 0$ such that

$$\tilde{F}_{T(x+y)}(\gamma \max\{s,t\}) \geqslant \Delta_2\{\tilde{F}_{Tx}(s), \tilde{F}_{Ty}(t)\} \text{ for all } x,y \in X \text{ and } s,t > 0; \tag{4.5}$$

(2) An operator from X to Y is said to be probabilistic subadditive if

$$\tilde{F}_{T(x+y)}(s+t) \geqslant \Delta_2\{\tilde{F}_{Tx}(s), \tilde{F}_{Ty}(t)\} \text{ for all } x,y \in X \quad \text{and} \quad s,t > 0. \tag{4.6}$$

Lemma 4.9. *Let $(X, \mathscr{F}, \Delta_1)$ and $(Y, \tilde{\mathscr{F}}, \Delta_2)$ be Menger PN-spaces, and let $(X, \mathscr{P})$ and $(Y, \tilde{\mathscr{P}})$ be the induced F^* spaces by $(X, \mathscr{F}, \Delta_1)$ and $(Y, \tilde{\mathscr{F}}, \Delta_2)$ respectively.*

(1) *If $T : X \to Y$ is a γ-max-probabilistic subadditive operator and $\Delta_2 = \min$, then it is also γ-max-pseudo-norm-subadditive.*

(2) *If $T : X \to Y$ is a probabilistic subadditive operator and $\Delta_2 = \min$, then it is also pseudo-norm-subadditive.*

Proof. (1) Suppose that $\tilde{p}_\lambda(Tx) = a, \tilde{p}_\lambda(Ty) = b$ for any $x, y \in X$ and $\lambda \in (0,1]$. It follows from (4.3) that

$$\tilde{F}_{Tx}(a+\varepsilon) > 1-\lambda, \quad \tilde{F}_{Ty}(b+\varepsilon) > 1-\lambda$$

for all $\varepsilon > 0$. Since T is a γ-max-probabilistic subadditive and $\Delta_2 = \min$, we have

$$\tilde{F}_{T(x+y)}(\gamma \max\{a+\varepsilon, b+\varepsilon\}) \geqslant \min\Big(\tilde{F}_{Tx}(a+\varepsilon),\ \tilde{F}_{Ty}(b+\varepsilon)\Big) > 1-\lambda,$$

and so

$$\tilde{p}_\lambda(T(x+y)) < \gamma \max\{a+\varepsilon,\ b+\varepsilon\}$$

Hence, by the arbitrariness of ε, we have

$$\tilde{p}_\lambda(T(x+y)) \leqslant \gamma \max\{\tilde{p}_\lambda(Tx),\ \tilde{p}_\lambda(Ty)\}$$

for any $x, y \in X$ and $\lambda \in (0,1]$, i.e., T is γ-max-pseudo-norm-subadditive.
(2) In a similar way, we can show that conclusion (2) holds. □

Theorem 4.10. *Let $(X, \mathscr{F}, \Delta_1)$ be a complete Menger PN-space and let $(Y, \tilde{\mathscr{F}}, \Delta_2)$ be a Menger PN-space with $\Delta_2 = \min$. Assume that $\{T_\beta \in Y^X : \beta \in \Lambda\}$ be a family of γ-max-probabilistic subadditive and continuous operators satisfying :*

(a) *$\{T_\beta\}_{\beta\in\Lambda}$ is uniformly quasi-homogeneous, and $\varphi_\beta \in C_{\delta,\omega}(0)$ for each $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β;*

(b) *$\{T_\beta\}_{\beta\in\Lambda}$ is uniformly γ-bounded, i.e., $\sup_{\beta\in\Lambda}\{\gamma_\beta\} < +\infty$;*

(c) *For each $x \in X$, $\{T_\beta(x) : \beta \in \Lambda\}$ is probabilistic bounded in Y.*

Then $\bigcup_{\beta\in\Lambda} T_\beta(A)$ is probabilistic bounded in Y for each probabilistic bounded set A in X.

Proof. Suppose that $(X, \mathscr{P})$ and $(Y, \tilde{\mathscr{P}})$ are the induced F^* spaces by $(X, \mathscr{F}, \Delta_1)$ and $(Y, \tilde{\mathscr{F}}, \Delta_2)$ respectively. By $(X, \mathscr{F}, \Delta_1)$ is complete, we can prove that $(X, \mathscr{P})$ is also complete, i.e., $(X, \mathscr{P})$ is a F space. By Remark 4.6, $\Delta_2 = \min$ implies that $(Y, \tilde{\mathscr{P}})$ is a locally convex F^* space. By Lemma 4.7 and (1) of Lemma 4.9, we can see that $\{T_\beta\}_{\beta\in\Lambda}$ in this theorem also satisfies that $\{T_\beta\}_{\beta\in\Lambda} \subset \gamma$-$\mathbf{MSAQH}(X,Y) \cap \mathbf{C}(X,Y)$ and conditions (a), (b) and (c) in Theorem 3.3. Therefore from Theorem 3.3 we can prove that the conclusion holds. □

As the proof of Theorem 4.10, we can prove the following corollary.

Corollary 4.11. *Let $(X, \mathscr{F}, \Delta_1)$ and $(Y, \tilde{\mathscr{F}}, \Delta_2)$ be Menger PN-spaces, with $(X, \mathscr{F}, \Delta_1)$ is complete and $\Delta_2 = \min$. Assume that $\{T_\beta \in Y^X : \beta \in \Lambda\}$ be a family of probabilistic subadditive and continuous operators, with $\{T_\beta\}_{\beta\in\Lambda}$ is uniformly quasi-homogeneous, and $\varphi_\beta \in C_{\delta,\omega}(0)$ for each $\beta \in \Lambda$, where φ_β is the eigenfunction of T_β. If for each $x \in X$, $\{T_\beta(x) : \beta \in \Lambda\}$ is probabilistic bounded in Y, then $\bigcup_{\beta\in\Lambda} T_\beta(A)$ is probabilistic bounded in Y for each probabilistic bounded set A in X.*

Remark 4.12. It is easy to see that the probabilistic subadditive operators of Theorem 4.1 in [12] (and the linear operators of Theorem 2 in [17]) are necessarily γ-max-probabilistic subadditive operators in this paper. Therefore, the conclusions in this paper unite and generalize the corresponding conclusions in [12, 17].

Finally, we shall give an application of Theorem 3.13 in normed spaces.

Let $(c_0) = \{\{\xi_n\} : \xi_n \to 0\ (n \to \infty), \xi_n \in \mathbb{R}, n \in \mathbb{N}\}$. Define the norm $\|\cdot\|_c$ by $\|x\|_c = \sup_{n\in\mathbb{N}} |\xi_n|$ for all $x \in (c_0)$, then $((c_0), \|\cdot\|_c)$ is a complete normed space.

For $p \geqslant 1$, Let $(l^p) = \{\{\xi_n\} : \sum\limits_{n=1}^{\infty} |\xi_n|^p < +\infty, \xi_n \in \mathbb{R}, n \in \mathbb{N}\}$. Define the norm $\|\cdot\|_l$ by $\|x\|_l = (\sum\limits_{n=1}^{\infty} |\xi_n|^p)^{\frac{1}{p}}$ for all $x \in (l^p)$, then $((l^p), \|\cdot\|_l)$ is a complete normed space. It is clear that $(l^p) \subset (c_0)$.

Theorem 4.13. *For each $p \geqslant 1$, the elements of the space (l^p) form a subset of the first category in the space (c_0). Further, it constitutes by itself a normed subspace of the first category in the norm of the space (c_0).*

Proof. Obviously, $((c_0), \|\cdot\|_c)$ is a Banach space and $((l^p), \|\cdot\|_l)$ is a normed space. For $x \in (c_0), m \in \mathbb{N}$, define the operator T_m as follows $T_m(x) = (|\xi_1|, \cdots, |\xi_m|, 0, \cdots)$. We have $T_m(x) \in (l^p)$, since $\sum\limits_{n=1}^{\infty} |\xi_n|^p = \sum\limits_{n=1}^{m} |\xi_n|^p < +\infty$. This shows that we can define the family of operators $\{T_m\}_{m\in\mathbb{N}}$ from $((c_0), \|\cdot\|_c)$ to $((l^p), \|\cdot\|_l)$.

It is clear that $\{T_m\}_{m\in\mathbb{N}} \subset \mathbf{QH}_\varphi((c_0), (l^p))$, where $\varphi(t) = |t| \in C_{1,1}(0)$.

For all $x = \{\xi_n\}, y = \{\eta_n\} \in (c_0)$, by $(a+b)^p \leqslant 2^p(a^p + b^p)$ $(a, b \geqslant 0, p \geqslant 1)$, we have

$$\begin{aligned}
\|T_m(x+y)\|_l &= (\sum_{n=1}^{m} |\xi_n + \eta_n|^p)^{\frac{1}{p}} \leqslant (\sum_{n=1}^{m} 2^p(|\xi_n|^p + |\eta_n|^p))^{\frac{1}{p}} \\
&= 2(\sum_{n=1}^{m} |\xi_n|^p + \sum_{n=1}^{m} |\eta_n|^p)^{\frac{1}{p}} \leqslant 2((\sum_{n=1}^{m} |\xi_n|^p)^{\frac{1}{p}} + (\sum_{n=1}^{m} |\eta_n|^p)^{\frac{1}{p}}) \\
&\leqslant 4\max\{\|T_m(x)\|_l, \|T_m(y)\|_l\},
\end{aligned}$$

which implies that $\{T_m\}_{m\in\mathbb{N}} \subset \gamma\text{-}\mathbf{MSA}((c_0), (l^p))$ with uniformly γ-bounded.

Moreover, for any $\{x_n\}, x \in (c_0)$, if $x_k \to x (k \to \infty)$, then $\|x_k - x\|_c = \sup_{n\in\mathbb{N}} |\xi_n^k - \xi_n| \to 0 (k \to \infty)$. Then we have

$$\begin{aligned}
\|T_m(x_k) - T_m(x)\|_l &= (\sum_{n=1}^{m} |(|\xi_n^k| - |\xi_n|)|^p)^{\frac{1}{p}} \leqslant (\sum_{n=1}^{m} |\xi_n^k - \xi_n|^p)^{\frac{1}{p}} \\
&\leqslant (m)^{\frac{1}{p}} \|x_k - x\|_c^{\frac{1}{p}} \to 0 (k \to \infty),
\end{aligned}$$

which shows that $\{T_m\}_{m\in\mathbb{N}} \subset \mathbf{C}((c_0), (l^p))$. Furthermore, for each $x \in (l^p)$, we have

$$\sup_{in\mathbb{N}}\{\|T_m(x)\|_l\} = \sup_{m\in\mathbb{N}}\{(\sum_{n=1}^{m} |\xi_n|^p)^{\frac{1}{p}}\} = (\sum_{n=1}^{\infty} |\xi_n|^p)^{\frac{1}{p}} < +\infty.$$

Now we can claim that for each $p \geqslant 1$, the elements of the space (l^p) form a subset of the first category in the space (c_0). In fact, suppose that the elements of the space (l^p) form a subset of the second category in the space (c_0), from Theorem 3.13, we can know that the set $\bigcup_{m\in\mathbb{N}} T_m(B(\theta,2))$ is bounded in $((l^p),\|\cdot\|_l)$, where $B(\theta,2)\subset(c_0)$, i.e.,

$$\sup_{m\in\mathbb{N}}\sup_{x\in B(\theta,2)}\{\|T_m(x)\|_l\}<+\infty. \tag{4.7}$$

If we take $x=(1,\frac{1}{2},\cdots,\frac{1}{n},\cdots)\in B(\theta,2)\subset(c_0)$, then it is clear that,

$$\sup_{m\in\mathbb{N}}\{\|T_m(x)\|_l\}=\sum_{n=1}^{\infty}\frac{1}{n}=+\infty. \tag{4.8}$$

This is a contradiction with (4.7).

In addition, note that the elements of the space (l^p) is dense in the norm of the space (c_0), we can know that the conclusion of theorem holds. □

Acknowledgements

This work was supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 13KJB110004).

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