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Existence of an optimal control for fractional stochastic partial neutral integro-differential equations with infinite delay

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Abstract

In this paper we study optimal control problems governed by fractional stochastic partial neutral functional integro-differential equations with infinite delay in Hilbert spaces. We prove an existence result of mild solutions by using the fractional calculus, stochastic analysis theory, and fixed point theorems with the properties of analytic α-resolvent operators. Next, we derive the existence conditions of optimal pairs of these systems. Finally an example of a nonlinear fractional stochastic parabolic optimal control system is worked out in detail.

Keywords: Fractional stochastic partial neutral functional integro-differential equations, optimal controls, infinite delay, analytic α-resolvent operator, fixed point theorem.
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1. Introduction

The optimal control is one of the important fundamental concepts in mathematical control theory and plays a vital role in both deterministic and stochastic control systems. Optimal control problems appear in many applications. For example, for biological reasons delays occur naturally in population dynamics models. Therefore, when dealing with optimal harvesting problem of biological systems, one is led to optimal control of systems with delay. In recent years, optimal control problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications (see [3], [7] and the references therein).

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The theory of stochastic differential equations has attracted great interest due to its applications in characterizing many problems in physics, biology, chemistry, mechanics, and so on. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems. It is well-known that the optimal control problems for stochastic differential equations have become a field of increasing interest (see [18] and references therein). In particular, there are several papers devoted to the existence of an optimal controls of systems governed by stochastic partial differential equations in abstract spaces (see [4], [5], [23]). Recently, Ahmed [6] considered a class of partially observed semilinear stochastic evolution equations on infinite dimensional Hilbert spaces. Zhu and Zhou [28] considered an infinite horizon optimal control problem in which the controlled state dynamics is governed by a stochastic delay evolution equation in Hilbert spaces. The existence of optimal controls for backward stochastic partial evolution differential systems in the abstract space; see Meng and Shi [16], Zhou and Liu [27]. Brzeźniak and Serrano [8] discussed the existence of optimal relaxed controls for a class of semilinear stochastic evolution equation on Banach spaces perturbed by multiplicative noise and driven by a cylindrical Wiener process.

Fractional differential equations have gained considerable importance due to their applications in various fields of the science such as physics, mechanics, chemistry engineering etc. Significant development has been made in ordinary and partial differential equations involving fractional derivatives; see [20]. Further, many authors investigated the existence of mild solutions of abstract fractional functional differential and integro-differential equations in Banach spaces by using fixed point techniques; see [2], [10], [11] and references therein. Optimal controls for system governed by fractional differential systems is studied; see Agrawal [1]. For semilinear fractional control systems including delay systems in Banach spaces, some papers discussed the existence of optimal controls of systems. For instance, Mophou [17] considered the optimal control of fractional diffusion equation by using the classical control theory. Wang et al. [25] discussed the optimal control problems for a class of fractional integrodifferential controlled systems. The authors [24] also studied the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay in Banach spaces by using Banach contraction principle.

More recently, the existence, uniqueness and other quantitative and qualitative properties of mild solutions to various semilinear fractional stochastic differential and integro-differential equations have been studied; see [12], [22], [26] and references therein. However, to the best of our knowledge, the optimal control problem for nonlinear fractional stochastic system in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we will study the optimal control problem for nonlinear fractional stochastic systems, which are natural generalizations of optimal control concepts well known in the theory of infinite dimensional deterministic control systems. Specifically, we will consider the Bolza problem of systems governed by fractional stochastic partial neutral functional integro-differentia equations with infinite delay in an ϑ-norm and the existence result of optimal controls will be presented. In fact, the results in this paper are motivated by the recent work of [5], [6] and the fractional differential equations discussed in [24], [25]. The main tools used in this paper are the fractional calculus, stochastic analysis theory, and the Sadovskii's fixed point theorem with the properties of analytic α-resolvent operators. Moreover, an example is given to demonstrate the applicability of our results.

2. Problem Formulation and Preliminaries

Throughout this paper, we use the following notations. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with probability measure P on Ω and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is the filtration is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H, \|\cdot\|_K$ their vector norms, respectively. $L(K, H)$ be the space of linear operators mapping K into H, and $L_b(K, H)$ be the space of bounded linear operators mapping K into H equipped with the usual norm $\|\cdot\|_H$ and $L_b(H)$ denotes the Hilbert space of bounded linear operators from H to H. Let $\{w(t) : t \geq 0\}$ denote an K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with covariance operator Q, that is $E\langle w(t), x\rangle_K \langle w(s), y\rangle_K = (t \wedge s)\langle Qx, y\rangle_K$, for all $x, y \in K$, where Q is a positive, self-adjoint, trace class operator on K. In particular, we denote $w(t)$ an

K-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $w(t)$, we introduce the subspace $K_0 = Q^{1/2}(K)$ of K which is endowed with the inner product $\langle \tilde{u}, \tilde{v}\rangle_{K_0} = \langle Q^{-1/2}\tilde{u}, Q^{-1/2}\tilde{v}\rangle_K$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $Qe_n = \lambda_n e_n$, and a sequence β_n of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t),\ e \in K, t \in [0,T],$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(K_0, H)$ be the space of all Hilbert-Schmidt operators from K_0 to H with the norm $\| \psi \|^2_{L_2^0} = \mathrm{Tr}((\psi Q^{1/2})(\psi Q^{1/2})^*)$ for any $\psi \in L_2^0$. Clearly for any bounded operators $\psi \in L_b(K,H)$ this norm reduces to $\| \psi \|^2_{L_2^0} = \mathrm{Tr}(\psi Q \psi^*)$.

In this article, we consider a mathematical model given by the following fractional stochastic partial neutral functional integro-differential equations with infinite delay

$$^cD^{\alpha}[x(t) - g(t, x_t)] = Ax(t) + \int_0^t R(t-s)x(s)ds + B(t)u(t) + h(t, x_t) + f(t, x_t)\frac{dw(t)}{dt},\quad t \in J = [0,T], \tag{2.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = 0, \tag{2.2}$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H, $^cD^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$; A, $(R(t))_{t\geq 0}$ are closed linear operators defined on a common domain $D(A)$ which is dense in $(H, \|\cdot\|_H)$, the control function u takes value from a separable reflexive Hilbert space Y, and B is a linear operator from Y into H, $p \geq 2$ be an integer. $D_t^{\alpha}\sigma(t)$ represents the Caputo derivative of order $\alpha > 0$ defined by

$$D_t^{\alpha}\sigma(t) = \int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}\sigma(s)ds,$$

where n is the smallest integer greater than or equal to α and $g_{\beta}(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. The time history $x_t : (-\infty, 0] \to H$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; and g, h, f are appropriate functions specified latter. The initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment.

In this paper, the notation $[D(A)]$ represents the domain of A endowed with the graph norm. Furthermore, for appropriate functions $\mathcal{K} : [0,\infty) \to H$ the notation $\widehat{\mathcal{K}}$ denotes the Laplace transform of $\mathcal{K}$, and $B_r(x, H)$ stands for the closed ball with center at x and radius $r > 0$ in H. We denote by $(-A)^{\vartheta}$ the fractional power of the operator $-A$ for $0 < \vartheta \leq 1$. The subspace $D((-A^{\vartheta}))$ is dense in H and the expression $\| x \|_{\vartheta} = \| (-A)^{\vartheta}x \|, x \in D((-A)^{\vartheta})$, defines a norm on $D((-A)^{\vartheta})$. Hereafter, we denote by H_{ϑ} be the Banach space $D((-A)^{\vartheta})$ endowed with the norm $\| x \|_{\vartheta}$, which is equivalent to the graph norm of $(-A)^{\vartheta}$. For more details about the above preliminaries, we refer to [19].

Let $L^p(\mathcal{F}_T, H)$ be the Banach space of all $\mathcal{F}_b$-measurable pth power integrable random variables with values in the Hilbert space H. Let $C([0,T]; L^p(\mathcal{F}, H))$ be the Banach space of continuous maps from $[0,T]$ into $L^p(\mathcal{F}, H)$ satisfying the condition $\sup_{t\in J} E \| x(t) \|^p < \infty$. In particular, we introduce the space $\mathcal{C}(J, H_{\vartheta})$ denote the closed subspace of $C([0,T]; L^p(\mathcal{F}, H_{\vartheta}))$ consisting of measurable and $\mathcal{F}_t$-adapted H_{ϑ}-valued stochastic processes $x \in C([0,T]; L^p(\mathcal{F}, H_{\vartheta}))$ endowed with the norm

$$\| x \|_{\mathcal{C}} = (\sup_{0\leq t\leq T} E \| x(t) \|^p_{\vartheta})^{\frac{1}{p}}$$

Then $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ is a Banach space.

Now, we give knowledge on the α-resolvent operator which appeared in [2].

Definition 2.1. A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on H is called an α-resolvent operator for

$$ {}^cD^\alpha x(t) = Ax(t) + \int_0^t R(t-s)x(s)ds, \tag{2.3} $$

$$ x_0 = \varphi \in H, \quad x'(0) = 0, \tag{2.4} $$

if the following conditions are verified.

(a) The function $\mathcal{R}_\alpha(\cdot) : [0,\infty) \to L_b(H)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in H$ and $\alpha \in (1,2)$.

(b) For $x \in D(A), \mathcal{R}_\alpha(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1((0,\infty),H)$, and

$$ D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t R(t-s)\mathcal{R}_\alpha(s)xds, \quad D_t^\alpha \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)R(s)xds $$

for every $t \geq 0$.

In this work we have considered the following conditions.

(P1) The operator $A : D(A) \subseteq H \to H$ is a closed linear operator with $[D(A)]$ dense in H. Let $\alpha \in (1,2)$. For some $\phi_0 \in (0, \frac{\pi}{2}]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that $\lambda \in \rho(A)$ for each

$$ \lambda \in \Sigma_{0,\alpha\vartheta} = \{\lambda \in \mathbb{C}, \lambda \neq 0, |\arg(\lambda)| < \alpha\vartheta\}, $$

where $\vartheta = \phi + \frac{\pi}{2}$ and $\| R(\lambda, A) \|_H \leq \frac{C_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,\alpha\vartheta}$.

(P2) For all $t \geq 0, R(t) : D(R(t)) \subseteq H \to H$ is a closed linear operator, $D(A) \subseteq D(R(t))$ and $R(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda) > 0$ and $\| R(t)x \|_H \leq b(t) \| x \|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{R} : \Sigma_{0,\pi/2} \to L_b([D(A)], H)$ has an analytical extension (still denoted by $\widehat{R}$) to $\Sigma_{0,\vartheta}$ such that $\| \widehat{R}(\lambda)x \|_H \leq \| \widehat{R}(\lambda) \|_H \| x \|_1$ for all $x \in D(A)$, and $\| \widehat{R}(\lambda) \|_H = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$.

(P2) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant C_1 such that $A(D) \subseteq D(A), \widehat{R}(\lambda)(D) \subseteq D(A)$, and $\| A\widehat{R}(\lambda)x \|_H \leq C_1 \| x \|_H$ for every $x \in D$ and all $\lambda \in \Sigma_{0,\vartheta}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \vartheta)$,

$$ \Sigma_{r,\theta} = \{\lambda \in \mathbb{C}, |\lambda| > r, |\arg(\lambda)| < \theta\}, $$

for $\Gamma_{r,\theta}, \Gamma^i_{r,\theta}, i = 1,2,3$, are the paths

$$ \Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \quad \Gamma^2_{r,\theta} = \{te^{i\xi} : |\xi| \leq \theta\}, \quad \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\}, $$

and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\rho_\alpha(G_\alpha)$ are the sets

$$ \rho_\alpha(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{Q}(\lambda))^{-1} \in L(H)\}. $$

We now define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$ \mathcal{R}_\alpha(t) := \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, & t > 0, \\ I, & t = 0. \end{cases} $$

Lemma 2.2 ([10])**.** *There exists $r_1 > 0$ such that $\Sigma_{r_1,\vartheta} \subseteq \rho_\alpha(G_\alpha)$ and the function $G_\alpha : \Sigma_{r_1,\vartheta} \to L_b(H)$ is analytic. Moreover,*

$$ G_\alpha(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha, A)[I - \widehat{Q}(\lambda) R(\lambda^\alpha, A)]^{-1}, $$

and there exist constants $\widetilde{M_i}$ for $i = 1,2$ such that

$$\| G_\alpha(\lambda) \|_H = \frac{\widetilde{M_1}}{|\lambda|},$$
$$\| AG_\alpha(\lambda)x \|_H = \frac{\widetilde{M_2}}{|\lambda|} \| x \|_1, \quad x \in D(A),$$
$$\| AG_\alpha(\lambda) \|_H = \frac{\widetilde{M_2}}{|\lambda|^{1-\alpha}}$$

for every $\lambda \in \Sigma_{r_1,\vartheta}$.

Lemma 2.3 ([2])**.** *Assume that conditions (P1)-(P3) are fulfilled. Then there exists a unique* α*-resolvent operator for problem* (2.3)-(2.4)*.*

Lemma 2.4 ([2])**.** *The function* $\mathcal{R}_\alpha : [0,\infty) \to L_b(H)$ *is strongly continuous and* $\mathcal{R}_\alpha : (0,\infty) \to L(H)$ *is uniformly continuous.*

Definition 2.5 ([2])**.** Let $\alpha \in (1,2)$, we define the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t)x := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)ds$$

for each $t \geq 0$.

Lemma 2.6 ([2])**.** *If the function* $\mathcal{R}_\alpha(\cdot)$ *is exponentially bounded in* $L_b(H)$*, then* $\mathcal{S}_\alpha(\cdot)$ *is exponentially bounded in* $L_b(H)$*.*

Lemma 2.7 ([2])**.** *If the function* $\mathcal{R}_\alpha(\cdot)$ *is exponentially bounded in* $L_b([D(A)])$*, then* $\mathcal{S}_\alpha(\cdot)$ *is exponentially bounded in* $L_b([D(A)])$*.*

Lemma 2.8 ([2])**.** *If* $R(\lambda_0^\alpha, A)$ *is compact for some* $\lambda_0^\alpha \in \rho(A)$*, then* $\mathcal{R}_\alpha(t)$ *and* $\mathcal{S}_\alpha(t)$ *are compact for all* $t > 0$.

Lemma 2.9 ([10])**.** *Suppose that the conditions (P1)-(P3) are satisfied. Let* $\alpha \in (1,2)$ *and* $\vartheta \in (0,1)$ *such that* $\alpha\vartheta \in (0,1)$*, then there exists positive number* M_ϑ *such that*

$$\| (-A)^\vartheta \mathcal{R}_\alpha(t) \|_H \leq M_\vartheta e^{rt} t^{-\alpha\vartheta}, \quad \| (-A)^\vartheta \mathcal{S}_\alpha(t) \|_H \leq M_\vartheta e^{rt} t^{\alpha(1-\vartheta)-1}$$

for all $t > 0$*. If* $x \in [D((-A)^\vartheta)]$*, then*

$$(-A)^\vartheta \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)(-A)^\vartheta x, \quad (-A)^\vartheta \mathcal{S}_\alpha(t)x = \mathcal{S}_\alpha(t)(-A)^\vartheta x.$$

In this paper, we assume that the phase space $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into H_ϑ, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [13]).

(A) If $x : (-\infty, \sigma+T] \to H_\vartheta$, $T > 0$, is such that $x|_{[\sigma,\sigma+T]} \in \mathcal{C}([\sigma,\sigma+T], H_\vartheta)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma+T]$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$;
  - (ii) $\| x(t) \|_\vartheta \leq \tilde{H} \| x_t \|_\mathcal{B}$;
  - (iii) $\| x_t \|_\mathcal{B} \leq K(t-\sigma) \sup\{E \| x(s) \|_\vartheta : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_\mathcal{B}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous and M is locally bounded; $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

In the following, let Y is a separable reflexive Hilbert space from which the controls u take the values. Operator $B \in L_\infty(J, L(Y, H))$, $\| B \|_\infty$ stands for the norm of operator B on Banach space $L_\infty(J, L(Y, H))$, where $L_\infty(J, L(Y, H))$ denote the space of operator valued functions which are measurable in the strong operator topology and uniformly bounded on the interval J. Let $L^p_{\mathcal{F}}(J, Y)$ is the closed subspace of $L^p_{\mathcal{F}}(J \times \Omega, Y)$, consisting of all measurable and $\mathcal{F}_t$-adapted, Y-valued stochastic processes satisfying the condition $E \int_0^T \| u(t) \|_Y^p \, dt < \infty$, and endowed with the norm

$$\| u \|_{L^p_{\mathcal{F}}(J,Y)} = \left(E \int_0^T \| u(t) \|_Y^p \, dt \right)^{\frac{1}{p}}.$$

Let U be a nonempty closed bounded convex subset of Y. We define the admissible control set

$$U_{ad} = \{ v(\cdot) \in L^p_{\mathcal{F}}(J, Y); v(t) \in U \text{ a.e. } t \in J \}.$$

Then, $Bu \in L^p(J, H)$ for all $u \in U_{ad}$.

Now we will derive the appropriate definition of mild solutions of (2.1)-(2.2).

Definition 2.10. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to H$ is called a mild solution of the system (2.1)-(2.2) with respect to u on $(-\infty, T]$, if $x_0 = \varphi \in \mathcal{B}$, $x|_J \in \mathcal{C}(J, H_\vartheta)$ for every $u \in U_{ad}$ there exists a $T = T(u) > 0$ and

(i) $x(t)$ is measurable and adapted to $\mathcal{F}_t, t \geq 0$.

(ii) $x(t) \in H$ has càdlàg paths on $t \in J$ a.s and for each $t \in J$, $x(t)$ satisfies

$$\begin{aligned} x(t) = {} & \mathcal{R}_\alpha(t)[\varphi(0) - g(0, \varphi)] + g(t, x_t) + \int_0^t A\mathcal{S}_\alpha(t-s) g(s, x_s) ds \\ & + \int_0^t \int_0^s R(s-\tau) \mathcal{S}_\alpha(t-s) g(\tau, x_\tau) d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s) B(s) u(s) ds \\ & + \int_0^t \mathcal{S}_\alpha(t-s) h(s, x_s) ds + \int_0^t \mathcal{S}_\alpha(t-s) f(s, x_s) dw(s), \quad t \in J. \end{aligned}$$

The next result is a consequence of the phase space axioms.

Lemma 2.11. *Let $x : (-\infty, T] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_J \in \mathcal{C}(J, H_\vartheta)$, then*

$$\| x_s \|_{\mathcal{B}} \leq M_T E \| \varphi \|_{\mathcal{B}} + K_T \sup_{0 \leq s \leq T} E \| x(s) \|_\vartheta,$$

where $M_T = \sup_{t \in J} M(t)$ and $K_T = \sup_{t \in J} K(t)$.

Lemma 2.12 ([9]). *For any $p \geq 1$ and for arbitrary $L_2^0(K, H)$-valued predictable process $\phi(\cdot)$ such that*

$$\sup_{s \in [0,t]} E \left\| \int_0^s \phi(v) dw(v) \right\|_H^{2p} \leq (p(2p-1))^p \left(\int_0^t (E \| \phi(s) \|_{L_2^0}^{2p})^{1/p} ds \right)^p, \ t \in [0, \infty).$$

Lemma 2.13 ([15]). *A measurable function $V : J \to H$ is Bochner integrable, if $\| V \|_H$ is Lebesgue integrable.*

Lemma 2.14 ([21]). *Let Φ be a condensing operator on a Banach space X, that is, Φ is continuous and takes bounded sets into bounded sets, and $\kappa(\Phi(D)) \leq \kappa(D)$ for every bounded set D of X with $\kappa(D) > 0$. If $\Phi(N) \subset N$ for a convex, closed and bounded set N of X, then Φ has a fixed point in X (where $\kappa(\cdot)$ denotes Kuratowski's measure of noncompactness.)*

3. Existence of solutions for fractional stochastic control system

In this section, we prove the existence of solutions for fractional stochastic control system (2.1)-(2.2). We make the following hypotheses:

(H1) The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exist constants M and M_1 such that $\| \mathcal{R}_\alpha(t) \|_{L_b(H)} \leq M$ and $\| \mathcal{S}_\alpha(t) \|_{L_b(H)} \leq M$ for every $t \in J$ and

$$\| (-A)^\vartheta \mathcal{S}_\alpha(t) \|_H \leq M_1 t^{\alpha(1-\vartheta)-1}, \quad 0 < t \leq T.$$

(H2) $R(\cdot)x \in C(J,H)$ for every $x \in [D((-A)^{1-\vartheta})]$, and there exist a constant M_2 and a positive function $\mu : J \to \mathbb{R}^+$ such that the function $\mu^p(\cdot) \in L^1(J,\mathbb{R}^+)$ and

$$\| R(s)\mathcal{S}_\alpha(t) \|_{L_b([D((-A)^\vartheta)],H)} \leq M_2 \mu(s) t^{\alpha\vartheta-1}, \ 0 \leq s < t \leq T.$$

(H3) There exists a constant $\beta \in (0,1)$ such that $g : J \times \mathcal{B} \to [D((-A)^{\beta+\vartheta})]$ satisfies the Lipschitz condition, i.e., there exists a constant $L_g > 0$ such that

$$E \| (-A)^{\beta+\vartheta} g(t_1,\psi_1) - (-A)^{\beta+\vartheta} g(t_2,\psi_2) \|_H^p \leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}^p$$

for any $0 \leq t_i \leq T, \psi_i \in \mathcal{B}, i = 1,2$, and

$$E \| (-A)^{\beta+\vartheta} g(t,\psi) \|_H^p \leq L_g(\| \psi \|_{\mathcal{B}}^p + 1)$$

for all $0 \leq t \leq T, \psi \in \mathcal{B}$.

(H4) The function $h : J \times \mathcal{B} \to H$ satisfies the following conditions:
  (i) The function $h(t,\cdot) : \mathcal{B} \to H$ is continuous for each $t \in J$, and for every $\psi \in \mathcal{B}$, the function $t \to h(t,\psi)$ is strongly measurable.
  (ii) There exists a positive function $m_h \in L^p(J,\mathbb{R}^+)$ such that

$$E \| h(t,\psi) \|_H^p \leq m_h(t)$$

  for all $t \in J, \psi \in \mathcal{B}$.

(H5) The function $f : J \times \mathcal{B} \to L_b(K,H)$ satisfies the following conditions:
  (i) The function $f(t,\cdot) : \mathcal{B} \to L_b(K,H)$ is continuous for each $t \in J$, and for every $\psi \in \mathcal{B}$, the function $t \to f(t,\psi)$ is strongly measurable.
  (ii) There exists a positive function $m_f \in L^p(J,\mathbb{R}^+)$ such that

$$E \| f(t,\psi) \|_H^p \leq m_f(t)$$

  for all $t \in J, \psi \in \mathcal{B}$.

Theorem 3.1. *Let $x_0 \in L_2^0(\Omega, H_\alpha)$. If the assumptions (H1)-(H5) are satisfied, then for each $u \in U_{ad}$, the system* (2.1)-(2.2) *has at least one mild solution on J with respect to u, provided that $p^2(\alpha(1-\vartheta)-1)+p > 1$ and*

$$14^{p-1} K_T^p L_g \Big[\| (-A)^{-\beta} \|_H^p + M_1^p \frac{T^{p\alpha\beta}}{p(\alpha\beta-1)+1} + M_2^p \| \mu^p \|_{L^1} \frac{T^{p\alpha\beta+p-1}}{p(\alpha\beta-1)+1} \Big] < 1. \tag{3.1}$$

Proof. Consider the space $\mathcal{BC} = \{x \in \mathcal{C}(J,H_\vartheta) : x(0) = \varphi(0)\}$ endowed with the uniform convergence topology and define the operator $\Phi : \mathcal{BC} \to \mathcal{BC}$ by

$$\begin{aligned}(\Phi x)(t) =& \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t,\bar{x}_t) + \int_0^t A\mathcal{S}_\alpha(t-s) g(s,\bar{x}_s) ds \\ &+ \int_0^t \int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s) g(\tau,\bar{x}_\tau) d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s) B(s) u(s) ds \\ &+ \int_0^t \mathcal{S}_\alpha(t-s) h(s,\bar{x}_s) ds + \int_0^t \mathcal{S}_\alpha(t-s) f(s,\bar{x}_s) dw(s), \quad t \in J,\end{aligned}$$

where $\bar{x}(t) : (-\infty, 0] \to H_\vartheta$ is such that $\bar{x}(0) = \varphi$ and $\bar{x} = x$ on J. From axiom (A), the strong continuity of $\mathcal{R}_\alpha(t), \mathcal{S}_\alpha(t)$ and assumptions (H1)-(H5), we infer that $\Phi x \in \mathcal{BC}$. For $x \in B_r(0, \mathcal{BC})$, from Lemma 2.11, it follows that

$$\| \bar{x}_s \|_{\mathcal{B}}^p \le 2^{p-1}(M_T \| \varphi \|_{\mathcal{B}})^p + 2^{p-1} K_T^p r := r^*. \tag{3.2}$$

By (H1)-(H3) and (3.2), we have

$$\begin{aligned}
&E\left\| \int_0^t (-A)^{1-\beta} \mathcal{S}_\alpha(t-s)(-A)^{\beta+\vartheta} g(s, \bar{x}_s) ds \right\|_H^p \\
&\le M_1^p T^{p-1} \int_0^t (t-s)^{p(\alpha\beta-1)} E \| (-A)^{\beta+\vartheta} g(s, \bar{x}_s) \|_H^p ds \\
&\le M_1^p T^{p-1} \int_0^t (t-s)^{p(\alpha\beta-1)} L_g(\| \bar{x}_s \|_{\mathcal{B}}^p + 1) ds \\
&\le M_1^p T^{p-1} L_g(r^* + 1) \frac{1}{p(\alpha\beta-1)+1} T^{p(\alpha\beta-1)+1},
\end{aligned}$$

$$\begin{aligned}
&E\left\| \int_0^t \int_0^s R(s-\tau) \mathcal{S}_\alpha(t-s)(-A)^\vartheta g(\tau, \bar{x}_\tau) d\tau ds \right\|_H^p \\
&\le T^{2(p-1)} \int_0^t \int_0^s E \| R(s-\tau) \mathcal{S}_\alpha(t-s)(-A)^\vartheta g(\tau, \bar{x}_\tau) \|_H^p d\tau ds \\
&\le M_2^p T^{2(p-1)} \int_0^t \int_0^s \mu^p(t-\tau)(t-s)^{p(\alpha\beta-1)} L_g(\| \bar{x}_\tau \|_{\mathcal{B}}^p + 1) d\tau ds \\
&\le M_2^p T^{2(p-1)} \| \mu \|_{L^1}^p L_g(r^*+1) \frac{1}{p(\alpha\beta-1)+1} T^{p(\alpha\beta-1)+1},
\end{aligned}$$

and

$$\begin{aligned}
&E\left\| \int_0^t (-A)^\vartheta \mathcal{S}_\alpha(t-s) B(s) u(s) ds \right\|_H^p \\
&\le E\left[\int_0^t \| (-A)^\vartheta \mathcal{S}_\alpha(t-s) \|_H \| B(s)u(s) \|_H ds \right]^p \\
&\le M_1^p \| B \|_\infty^p E\left[\int_0^t (t-s)^{\alpha(1-\vartheta)-1} \| u(s) \|_Y ds \right]^p \\
&\le M_1^p \| B \|_\infty^p \left(\int_0^t (t-s)^{\frac{p(\alpha(1-\vartheta)-1)}{p-1}} ds \right)^{p-1} E \int_0^t \| u(s) \|_Y^p ds \\
&\le M_1^p \| B \|_\infty^p \left(\frac{p-1}{p\alpha(1-\vartheta)-1} \right)^{p-1} T^{p\alpha(1-\vartheta)-1} \| u \|_{L_{\mathcal{F}}^p(J,Y)}^p,
\end{aligned}$$

by $p^2(\alpha(1-\vartheta)-1)+p > 1$, we know that $p\alpha(1-\vartheta) > 1$. Then from Lemma 2.13, it follows that $A\mathcal{S}_\alpha(t-s)g(s,\bar{x}_s), \mathcal{S}_\alpha(t-s)B(s)u(s)$ are integrable on J. Therefore, Φ is well defined on $B_r(0, \mathcal{BC})$. In order to apply Lemma 2.14, we break the proof into a sequence of steps.

Step 1. There exists $r > 0$ such that $\Phi(B_r(0, \mathcal{BC})) \subset B_r(0, \mathcal{BC})$.

For each $r > 0$, $B_r(0, \mathcal{BC})$ is clearly a bounded closed convex subset in $\mathcal{BC}$. We claim that there exists $r > 0$ such that $\Phi(B_r(0, \mathcal{BC})) \subset B_r(0, \mathcal{BC})$. In fact, if this is not true, then for each $r > 0$ there exists $x^r \in B_r(0, \mathcal{BC})$ and $t^r \in J$ such that $r < E \| (-A)^\vartheta (\Phi x^r)(t^r) \|_H^p$. Then, by using (H1)-(H5), we have

$$\begin{aligned}
r &< E \| (-A)^\vartheta (\Phi x^r)(t^r) \|_H^p \\
&\le 7^{p-1} \| \mathcal{R}_\alpha(t^r)[(-A)^\vartheta \varphi(0) - (-A)^{-\beta}(-A)^{\beta+\vartheta} g(0,\varphi)] \|_H^p \\
&\quad + 7^{p-1} E \| (-A)^{-\beta}(-A)^{\beta+\vartheta} g(t^r, \overline{x^r}_{t^r}) \|_H^p \\
&\quad + 7^{p-1} E \left\| \int_0^{t^r} (-A)^{1-\beta} \mathcal{S}_\alpha(t^r-s)(-A)^{\beta+\vartheta} g(s, \overline{x^r}_s) ds \right\|_H^p
\end{aligned}$$

$$
\begin{aligned}
&+ 7^{p-1}E\left\| \int_0^{t^r}\int_0^s R(s-\tau)\mathcal{S}_\alpha(t^r-s)(-A)^\vartheta g(\tau,\overline{x^r}_\tau)d\tau ds\right\|_H^p \\
&+ 7^{p-1}E\left\| \int_0^{t^r}(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)B(s)u(s)ds\right\|_H^p \\
&+ 7^{p-1}E\left\| \int_0^{t^r}(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)h(s,\overline{x^r}_s)ds\right\|_H^p \\
&+ 7^{p-1}E\left\| \int_0^{t^r}(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)f(s,\overline{x^r}_s)dw(s)\right\|_H^p \\
\le{}& 14^{p-1}M^p[\| (-A)^\vartheta\varphi(0)\|_H^p + \|(-A)^{-\beta}\|_H^p E\|(-A)^{\beta+\vartheta}g(0,\varphi)\|_H^p] \\
&+ 7^{p-1}\|(-A)^{-\beta}\|_H^p E\|(-A)^{\beta+\vartheta}g(t,\overline{x^r}_{t^r})\|_H^p \\
&+ 7^{p-1}T^{p-1}\int_0^{t^r}\|(-A)^{1-\beta}\mathcal{S}_\alpha(t^r-s)\|_H^p E\|(-A)^{\beta+\vartheta}g(s,\overline{x^r}_s)\|\|_H^p ds \\
&+ 7^{p-1}T^{2(p-1)}\int_0^{t^r}\int_0^s E\|R(s-\tau)\mathcal{S}_\alpha(t^r-s)(-A)^\vartheta g(\tau,\overline{x^r}_\tau)\|_H^p d\tau ds \\
&+ 7^{p-1}E\left[\int_0^{t^r}\|(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)\|_H\|B(s)u(s)\|_H ds\right]^p \\
&+ 7^{p-1}T^{p-1}\int_0^{t^r}\|(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)\|_H^p E\|h(s,\overline{x^r}_s)\|_H^p ds \\
&+ 7^{p-1}C_p\left[\int_0^{t^r}[\|(-A)^\vartheta\mathcal{S}_\alpha(t^r-s)\|_H^p E\|f(s,\overline{x^r}_s)\|_H^p]^{2/p}ds\right]^{p/2} \\
\le{}& 14^{p-1}M^p[(\tilde{H}\|\varphi\|_{\mathcal{B}})^p + \|(-A)^{-\beta}\|_H^p L_g(\|\varphi\|_{\mathcal{B}}^p+1)] \\
&+ 7^{p-1}\|(-A)^{-\beta}\|_H^p L_g(\|\overline{x^r}_{t^r}\|_{\mathcal{B}}^p+1) \\
&+ 7^{p-1}M_1^pT^{p-1}\int_0^{t^r}(t^r-s)^{p(\alpha\beta-1)}L_g(\|\overline{x^r}_s\|_{\mathcal{B}}^p+1)ds \\
&+ 7^{p-1}M_2^pT^{2(p-1)}\int_0^{t^r}\int_0^s\mu^p(t^r-\tau)(t^r-s)^{p(\alpha\beta-1)}L_g(\|\overline{x^r}_\tau\|_{\mathcal{B}}^p+1)d\tau ds \\
&+ 7^{p-1}M_1^p\|B\|_\infty^p\left(\frac{p-1}{p\alpha(1-\vartheta)-1}\right)^{p-1}T^{p\alpha(1-\vartheta)-1}\|u\|_{L^p_{\mathcal{F}}(J,Y)}^p \\
&+ 7^{p-1}M_1^pT^{p-1}\left(\frac{p-1}{p^2(\alpha(1-\vartheta)-1)+p-1}\right)^{\frac{p-1}{p}}T^{\frac{p^2(\alpha(1-\vartheta)-1)+p-1}{p}}\left(\int_0^{t^r}(m_h(s))^pds\right)^{\frac1p} \\
&+ 7^{p-1}C_pM_1^pT^{p/2-1}\left(\frac{p-1}{p^2(\alpha(1-\vartheta)-1)+p-1}\right)^{\frac{p-1}{p}}T^{\frac{p^2(\alpha(1-\vartheta)-1)+p-1}{p}}\left(\int_0^{t^r}(m_f(s))^pds\right)^{\frac1p},
\end{aligned}
$$

where $C_p = (p(p-1)/2)^{p/2}$. Using (3.2), it follows that

$$
\begin{aligned}
r^* <{}& 2^{p-1}(M_T\|\varphi\|_{\mathcal{B}})^p + 28^{p-1}M^pK_T^p[(\tilde{H}\|\varphi\|_{\mathcal{B}})^p + \|(-A)^{-\beta}\|_H^p L_g\|\varphi\|_{\mathcal{B}}^p+1)] \\
&+ 14^{p-1}K_T^p\|(-A)^{-\beta}\|_H^p L_g(r^*+1) + 14^{p-1}K_T^pM_1^pT^{p-1}\frac{T^{p(\alpha\beta-1)+1}}{p(\alpha\beta-1)+1}L_g(r^*+1) \\
&+ 14^{p-1}K_T^pM_2^pT^{2(p-1)}\|\mu^p\|_{L^1}\frac{T^{p(\alpha\beta-1)+1}}{p(\alpha\beta-1)+1}L_g(r^*+1) + 2^{p-1}K_T^p\widetilde{M},
\end{aligned}
$$

where

$$
\widetilde{M} = 7^{p-1}M_1^p\|B\|_\infty^p\left(\frac{p-1}{p\alpha(1-\vartheta)-1}\right)^{p-1}T^{p\alpha(1-\vartheta)-1}\|u\|_{L^p_{\mathcal{F}}(J,Y)}^p
$$

$$+ 7^{p-1}M_1^pT^{p-1}\left(\frac{p-1}{p^2(\alpha(1-\vartheta)-1)+p-1}\right)^{\frac{p-1}{p}}T^{\frac{p^2(\alpha(1-\vartheta)-1)+p-1}{p}}\left(\int_0^T(m_h(s))^pds\right)^{\frac{1}{p}}$$
$$+ 7^{p-1}C_pM_1^pT^{p/2-1}\left(\frac{p-1}{p^2(\alpha(1-\vartheta)-1)+p-1}\right)^{\frac{p-1}{p}}T^{\frac{p^2(\alpha(1-\vartheta)-1)+p-1}{p}}\left(\int_0^T(m_f(s))^pds\right)^{\frac{1}{p}}.$$

Dividing both sides by r^* and taking the lower limit, we have

$$1 \leq 14^{p-1}K_T^pL_g\Big[\parallel (-A)^{-\beta}\parallel_H^p + M_1^p\frac{T^{p\alpha\beta}}{p(\alpha\beta-1)+1} + M_2^p\parallel \mu^p\parallel_{L^1}\frac{T^{p\alpha\beta+p-1}}{p(\alpha\beta-1)+1}\Big],$$

which contradicts (3.1). Hence, there exists $r > 0$ such that $\Phi(B_r(0,\mathcal{BC})) \subset B_r(0,\mathcal{BC})$. In what follows, we aim to show that the operator Φ has a fixed point on $B_r(0,\mathcal{BC})$, which implies that (2.1)-(2.2) has a mild solution. To this end, we decompose Φ as $\Phi_1 + \Phi_2$ where

$$(\Phi_1x)(t) = -\mathcal{R}_\alpha(t)g(0,\varphi) + g(t,\bar{x}_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s,\bar{x}_s)ds$$
$$+ \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\bar{x}_\tau)d\tau ds, \quad t\in J,$$

$$(\Phi_2x)(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t\mathcal{S}_\alpha(t-s)B(s)u(s)ds$$
$$+ \int_0^t\mathcal{S}_\alpha(t-s)h(s,\bar{x}_s)ds + \int_0^t\mathcal{S}_\alpha(t-s)f(s,\bar{x}_s)dw(s), \quad t\in J.$$

We will verify that Φ_1 is a contraction while Φ_2 is a completely continuous operator.

Step 2. Φ_1 is a contraction.

Let $t \in [0,T]$ and $x^*, x^{**} \in B_r(0,\mathcal{BC})$. From (H3), we have

$$\begin{aligned}
&E\parallel (-A)^\vartheta(\Phi_1x^*)(t) - (-A)^\vartheta(\Phi_1x^{**})(t)\parallel_H^p\\
&\leq 3^{p-1}E\parallel (-A)^{-\beta}[(-A)^{\beta+\vartheta}g(t,\overline{x^*}_t) - (-A)^{\beta+\vartheta}g(t,\overline{x^{**}}_t)]\parallel_H^p\\
&\quad + 3^{p-1}E\left\|\int_0^t(-A)^{1-\beta}\mathcal{S}_\alpha(t-s)[(-A)^{\beta+\vartheta}g(s,\overline{x^*}_s) - (-A)^{\beta+\vartheta}g(s,\overline{x^{**}}_s)]ds\right\|_H^p\\
&\quad + 3^{p-1}E\left\|\int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)[(-A)^\vartheta g(\tau,\overline{x^*}_\tau) - (-A)^\vartheta g(\tau,\overline{x^{**}}_\tau)]d\tau ds\right\|_H^p\\
&\leq 3^{p-1}\parallel (-A)^{-\beta}\parallel_H^p L_g\parallel \overline{x^*}_t - \overline{x^{**}}_t\parallel_{\mathcal{B}}^p\\
&\quad + 3^{p-1}M_1^pT^{p-1}\int_0^t(t-s)^{p(\alpha\beta-1)}L_g\parallel \overline{x^*}_s - \overline{x^{**}}_s\parallel_{\mathcal{B}}^p ds\\
&\quad + 3^{p-1}M_2^pT^{2(p-1)}\int_0^t\int_0^s\mu^p(t-\tau)(t-s)^{p(\alpha\beta-1)}L_g\parallel \overline{x^*}_\tau - \overline{x^{**}}_\tau\parallel_{\mathcal{B}}^p d\tau ds\\
&\leq 6^{p-1}\parallel (-A)^{-\beta}\parallel_H^p K_T^pL_g\sup_{s\in[0,T]}E\parallel \overline{x^*}(s) - \overline{x^{**}}(s)\parallel_\vartheta^p\\
&\quad + 6^{p-1}M_1^pK_T^pL_gT^{p-1}\int_0^t(t-s)^{p(\alpha\beta-1)}ds\sup_{s\in[0,T]}E\parallel \overline{x^*}(s) - \overline{x^{**}}(s)\parallel_\vartheta^p\\
&\quad + 6^{p-1}M_2^pK_T^pL_gT^{2(p-1)}\int_0^t\int_0^s\mu^p(t-\tau)(t-s)^{p(\alpha\beta-1)}d\tau ds\sup_{s\in[0,T]}E\parallel \overline{x^*}(s) - \overline{x^{**}}(s)\parallel_\vartheta^p\\
&\leq 6^{p-1}K_T^pL_g\Big[\parallel (-A)^{-\beta}\parallel_H^p + M_1^pT^{p-1}\frac{T^{p(\alpha\beta-1)+1}}{p(\alpha\beta-1)+1} + M_2^p\parallel \mu^p\parallel_{L^1}T^{2(p-1)}
\end{aligned}$$

$$\times \frac{T^{p(\alpha\beta-1)+1}}{p(\alpha\beta-1)+1}\Bigg] \sup_{s\in[0,T]} E \parallel x^*(s) - x^{**}(s) \parallel_\vartheta^p \quad (\text{since } \bar{x} = x \text{ on } J)$$

$$= 6^{p-1}K_T^p L_g\Bigg[\parallel (-A)^{-\beta} \parallel_H^p + M_1^p \frac{T^{p\alpha\beta}}{p(\alpha\beta-1)+1} + M_2^p \parallel \mu^p \parallel_{L^1} \frac{T^{p\alpha\beta+p-1}}{p(\alpha\beta-1)+1}\Bigg] \parallel x^* - x^{**} \parallel_{\mathcal{C}}^p .$$

Taking supremum over t,

$$\parallel \Phi_1 x^* - \Phi_1 x^{**} \parallel_{\mathcal{C}}^p \leq L_0 \parallel x^* - x^{**} \parallel_{\mathcal{C}}^p,$$

where $L_0 = 14^{p-1}K_T^p L_g[\parallel (-A)^{-\beta} \parallel_H^p + M_1^p \frac{T^{p\alpha\beta}}{p(\alpha\beta-1)+1} + M_2^p \parallel \mu^p \parallel_{L^1} \frac{T^{p\alpha\beta+p-1}}{p(\alpha\beta-1)+1}] < 1$. Thus Φ_1 is a contraction on $B_r(0, \mathcal{BC})$.

Step 3. Φ_2 is completely continuous on $B_r(0, \mathcal{BC})$.

(1) Φ_2 is continuous on $B_r(0, \mathcal{BC})$.

Let $\{x^n\} \subseteq B_r(0, \mathcal{B}C)$ with $x^n \to x(n \to \infty)$ in $\mathcal{B}C$. From axiom (A), it is easy to see that $(\overline{x^n})_s \to \bar{x}_s$ uniformly for $s \in (-\infty, T]$ as $n \to \infty$. By the assumptions (H4)-(H5), we have

$$h(s, \overline{x^n}_s) \to h(s, \bar{x}_s) \text{ as } n \to \infty,$$

$$f(s, \overline{x^n}_s) \to f(s, \bar{x}_s) \text{ as } n \to \infty$$

for each $s \in [0, t]$, and since

$$E \parallel h(s, \overline{x^n}_s) - h(s, \bar{x}_s) \parallel_H^p \leq 2^{p-1} m_h(s),$$

$$E \parallel f(s, \overline{x^n}_s) - f(s, \bar{x}_s) \parallel_H^p \leq 2^{p-1} m_f(s),$$

and $(\int_0^t (m_h(s))^p ds)^{\frac{1}{p}} < +\infty, (\int_0^t (m_f(s))^p ds)^{\frac{1}{p}} < +\infty$. Then, by the dominated convergence theorem, we have

$$\begin{aligned}
&E \parallel (-A)^\vartheta (\Phi_2 x^n)(t) - (-A)^\vartheta (\Phi_2 x)(t) \parallel_H^p \\
&\leq \sup_{t\in J} E \Bigg\| \int_0^t (-A)^\vartheta \mathcal{S}_\alpha(t-s)[h(s, \overline{x^n}_s) - h(s, \bar{x}_s)]ds \\
&\quad + \int_0^t (-A)^\vartheta \mathcal{S}_\alpha(t-s)[f(s, \overline{x^n}_s) - f(s, \bar{x}_s)]dw(s) \Bigg\|_H^p \\
&\leq 2^{p-1}T^{p-1} \int_0^t \parallel (-A)^\vartheta \mathcal{S}_\alpha(t-s) \parallel_H^p E \parallel h(s, \overline{x^n}_s) - h(s, \bar{x}_s) \parallel_H^p ds \\
&\quad + 2^{p-1} \Bigg[\int_0^t [\parallel (-A)^\vartheta \mathcal{S}_\alpha(t-s) \parallel_H^p E \parallel f(s, \overline{x^n}_s) - f(s, \bar{x}_s) \parallel_H^p]^{2/p} ds \Bigg]^{p/2} \\
&\leq 2^{p-1} M_1^p T^{p-1} \Bigg(\int_0^t (t-s)^{\frac{p^2(\alpha(1-\vartheta)-1)}{p-1}} ds \Bigg)^{\frac{p-1}{p}} \Bigg(\int_0^t [E \parallel h(s, \overline{x^n}_s) - h(s, \bar{x}_s) \parallel_H^p]^p ds \Bigg)^{\frac{1}{p}} \\
&\quad + 2^{p-1} M_1^p C_p T^{p/2-1} \Bigg(\int_0^t (t-s)^{\frac{p^2(\alpha(1-\vartheta)-1)}{p-1}} ds \Bigg)^{\frac{p-1}{p}} \Bigg(\int_0^t [E \parallel f(s, \overline{x^n}_s) - f(s, \bar{x}_s) \parallel_H^p]^p ds \Bigg)^{\frac{1}{p}} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, Φ_2 is continuous.

(2) $\Phi_2(B_r(0, \mathcal{B}C)) = \{\Phi_2 x : x \in B_r(0, \mathcal{B}C)\}$ is clearly bounded.

(3) $\Phi_2(B_r(0, \mathcal{B}C)) = \{\Phi_2 x : x \in B_r(0, \mathcal{B}C)\}$ is equicontinuous.

Let $0 < t_1 < t_2 \leq T$. For each $x \in B_r(0, \mathcal{B}C)$, we have

$$\begin{aligned}
&E \parallel (-A)^\vartheta (\Phi_2 x)(t_2) - (-A)^\vartheta (\Phi_2 x)(t_1) \parallel_H^p \\
&\leq 4^{p-1} E \parallel [\mathcal{R}_\alpha(t_2) - \mathcal{R}_\alpha(t_1)](-A)^\vartheta \varphi(0) \parallel_H^p \\
&\quad + 4^{p-1} E \Bigg\| \int_0^{t_2} (-A)^\vartheta \mathcal{S}_\alpha(t_2-s)B(s)u(s)ds - \int_0^{t_1} (-A)^\vartheta \mathcal{S}_\alpha(t_1-s)B(s)u(s)ds \Bigg\|_H^p
\end{aligned}$$

$$
\begin{aligned}
&+ 4^{p-1}E\Big\| \int_0^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)B(s)u(s)ds - \int_0^{t_1}(-A)^{\vartheta}\mathcal{S}_\alpha(t_1-s)B(s)u(s)ds\Big\|_H^p \\
&+ 4^{p-1}E\Big\| \int_0^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)B(s)u(s)ds - \int_0^{t_1}(-A)^{\vartheta}\mathcal{S}_\alpha(t_1-s)B(s)u(s)ds\Big\|_H^p \\
&+ 4^{p-1}E\Big\| \int_0^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)h(s,\bar{x}_s)ds - \int_0^{t_1}(-A)^{\vartheta}\mathcal{S}_\alpha(t_1-s)h(s,\bar{x}_s)ds\Big\|_H^p \\
&+ 4^{p-1}E\Big\| \int_0^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)f(s,\bar{x}_s)dw(s) - \int_0^{t_1}(-A)^{\vartheta}\mathcal{S}_\alpha(t_1-s)f(s,\bar{x}_s)dw(s)\Big\|_H^p \\
&= \sum_{i=1}^{4} I_i.
\end{aligned}
$$

In view of (H1),(H2), (H4) and (H5) and Hölder's inequality, it follows that

$$
I_1 = 4^{p-1}E \parallel [\mathcal{R}_\alpha(t_2)-\mathcal{R}_\alpha(t_1)](-A)^{\vartheta}\varphi(0) \parallel_H^p,
$$

$$
\begin{aligned}
I_2 &\leq 2^{p-1}E\Big\| \int_0^{t_1}(-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)]B(s)u(s)ds\Big\|_H^p \\
&\quad + 2^{p-1}E\Big\| \int_{t_1}^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)B(s)u(s)ds\Big\|_H^p \\
&\leq 2^{p-1} \parallel B \parallel_\infty^p \left(\int_0^{t_1} \parallel (-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)] \parallel_H^{\frac{p}{p-1}} ds\right)^{p-1} E\int_0^{t_1} \parallel u(s) \parallel_Y^p ds \\
&\quad + 2^{p-1} \parallel B \parallel_\infty^p M_1^p\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p(\alpha(1-\vartheta)-1)}{p-1}}ds\right)^{p-1} E\int_{t_1}^{t_2} \parallel u(s) \parallel_Y^p ds,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\leq 2^{p-1}E\Big\| \int_0^{t_1}(-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)]h(s,\bar{x}_s)ds\Big\|_H^p \\
&\quad + 2^{p-1}E\Big\| \int_{t_1}^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)h(s,\bar{x}_s)ds\Big\|_H^p \\
&\leq 2^{p-1}(t_1)^{p-1}\int_0^{t_1} \parallel (-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)] \parallel_H^p E \parallel h(s,\bar{x}_s) \parallel_H^p ds \\
&\quad + 2^{p-1}(t_2-t_1)^{p-1}\int_{t_1}^{t_2} \parallel (-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s) \parallel_H^p E \parallel h(s,\bar{x}_s) \parallel_H^p ds \\
&\leq 2^{p-1}(t_1)^{p-1}\left(\int_0^{t_1} \parallel (-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)] \parallel_H^{\frac{p^2}{p-1}} ds\right)^{\frac{p-1}{p}}\left(\int_0^{t_1}(m_h(s))^p ds\right)^{\frac{1}{p}} \\
&\quad + 2^{p-1}M_1^p(t_2-t_1)^{p-1}\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p^2(\alpha(1-\vartheta)-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_1}^{t_2}(m_h(s))^p ds\right)^{\frac{1}{p}},
\end{aligned}
$$

$$
\begin{aligned}
I_4 &\leq 2^{p-1}E\Big\| \int_0^{t_1}(-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)]f(s,\bar{x}_s)dw(s)\Big\|_H^p \\
&\quad + 2^{p-1}E\Big\| \int_{t_1}^{t_2}(-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s)f(s,\bar{x}_s)dw(s)\Big\|_H^p \\
&\leq 2^{p-1}\left[\int_0^{t_1}[\parallel (-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)] \parallel_H^p E \parallel f(s,\bar{x}_s) \parallel_H^p]^{2/p}ds\right]^{p/2} \\
&\quad + 2^{p-1}\left[\int_{t_1}^{t_2}[\parallel (-A)^{\vartheta}\mathcal{S}_\alpha(t_2-s) \parallel_H^p E \parallel f(s,\bar{x}_s)]^{2/p}ds\right]^{p/2}
\end{aligned}
$$

$$\leq 2^{p-1}C_p(t_1)^{p/2-1}\bigg(\int_0^{t_1} \| (-A)^{\vartheta}[\mathcal{S}_\alpha(t_2-s)-\mathcal{S}_\alpha(t_1-s)] \|_H^{\frac{p}{p-1}} ds\bigg)^{\frac{p-1}{p}}\bigg(\int_0^{t_1}(m_f(s))^p ds\bigg)^{\frac{1}{p}}$$
$$+ 2^{p-1}C_pM_1^p(t_2-t_1)^{p/2-1}\bigg(\int_{t_1}^{t_2}(t_2-s)^{\frac{p^2(\alpha(1-\vartheta)-1)}{p-1}}ds\bigg)^{\frac{p-1}{p}}\bigg(\int_{t_1}^{t_2}(m_f(s))^p ds\bigg)^{\frac{1}{p}}.$$

We see that $E \parallel (-A)^{\vartheta}(\Phi_2 x)(t_2) - (-A)^{\vartheta}(\Phi_2 x)(t_1) \parallel_H^p$ tends to zero independently of $x \in B_r(0,\mathcal{BC})$ as $t_2 - t_1 \to 0$, since the compactness of $\mathcal{R}_\alpha(t)$, $\mathcal{S}_\alpha(t)$ for $t > 0$ implies imply the continuity in the uniform operator topology The equicontinuities for the cases $t_1 < t_2 \leq 0$ or $t_1 \leq 0 \leq t_2 \leq T$ are very simple. Thus the set $\{\Phi_2 x : x \in B_r(0,\mathcal{BC})\}$ is equicontinuous.

(4) $(\Phi_2(B_r(0,\mathcal{BC}))(t) = \{(\Phi_2 x)(t) : x \in B_r(0,\mathcal{BC})\}$ is relatively compact for each $t \in J$.

We note that $(\Phi_2(B_r(0,\mathcal{BC}))(t)$ is relatively compact in $\mathcal{BC}$ for $t = 0$. Let $0 < t \leq s \leq T$ be fixed and ε a real number satisfying $0 < \varepsilon < t$ for $x \in B_r(0,\mathcal{BC})$. We define

$$(\Phi_2^{\varepsilon}x)(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)B(s)u(s)ds$$
$$+ \int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)h(s,\bar{x}_s)ds + \int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)f(s,\bar{x}_s)dw(s).$$

Using the compactness of $\mathcal{R}_\alpha(t), \mathcal{S}_\alpha(t)$ for $t > 0$, we deduce that the set $U_\varepsilon(t) = \{(\Phi_2^{\varepsilon}x)(t) : x \in B_r(0,\mathcal{BC})\}$ is relatively compact in H_ϑ for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in B_r(0,\mathcal{BC})$ we have

$$\begin{aligned}
&E \parallel (-A)^{\vartheta}(\Phi_2 x)(t) - (-A)^{\vartheta}(\Phi_2^{\varepsilon}x)(t) \parallel_H^p \\
&\leq 3^{p-1}E\bigg\| \int_{t-\varepsilon}^{t}(-A)^{\vartheta}\mathcal{S}_\alpha(t-s)B(s)u(s)ds\bigg\|_H^p \\
&\quad + 3^{p-1}E\bigg\| \int_{t-\varepsilon}^{t}(-A)^{\vartheta}\mathcal{S}_\alpha(t-s)h(s,\bar{x}_s)ds\bigg\|_H^p \\
&\quad + 3^{p-1}E\bigg\| \int_{t-\varepsilon}^{t}(-A)^{\vartheta}\mathcal{S}_\alpha(t-s)f(s,\bar{x}_s)dw(s)\bigg\|_H^p \\
&\leq 3^{p-1}E\bigg[\int_{t-\varepsilon}^{t} \| (-A)^{\vartheta}\mathcal{S}_\alpha(t-s) \|_H \| B(s)u(s) \|_H ds\bigg]^p \\
&\quad + 3^{p-1}T^{p-1}\int_{t-\varepsilon}^{t} \| (-A)^{\vartheta}\mathcal{S}_\alpha(t-s) \|_H^p E \| h(s,\bar{x}_s) \|_H^p ds \\
&\quad + 3^{p-1}C_p\bigg[\int_{t-\varepsilon}^{t} [\| (-A)^{\vartheta}\mathcal{S}_\alpha(t-s) \|_H^p E \| f(s,\bar{x}_s) \|_H^p]^{2/p}ds\bigg]^{p/2} \\
&\leq 3^{p-1} \| B \|_\infty^p M_1^p\bigg(\int_{t-\varepsilon}^{t}(t-s)^{\frac{p(\alpha(1-\vartheta)-1)}{p-1}}ds\bigg)^{p-1}E\int_{t-\varepsilon}^{t} \| u(s) \|_H^p ds \\
&\quad + 3^{p-1}M_1^pT^{p-1}\bigg(\int_{t-\varepsilon}^{t}(t-s)^{\frac{p^p(\alpha(1-\vartheta)-1)}{p-1}}ds\bigg)^{\frac{p-1}{p}}\bigg(\int_{t-\varepsilon}^{t}(m_h(s))^p ds\bigg)^{\frac{1}{p}} \\
&\quad + 3^{p-1}M_1^pT^{p/2-1}\bigg(\int_{t-\varepsilon}^{t}(t-s)^{\frac{p^p(\alpha(1-\vartheta)-1)}{p-1}}ds\bigg)^{\frac{p-1}{p}}\bigg(\int_{t-\varepsilon}^{t}(m_f(s))^p ds\bigg)^{\frac{1}{p}},
\end{aligned}$$

and there are relatively compact sets arbitrarily close to the set $\{(\Phi_2 x)(t) : x \in B_r(0,\mathcal{BC})\}$, and $(\Phi_2(B_r(0,\mathcal{BC}))(t)$ is a relatively compact in H_ϑ. By the Arzelá-Ascoli theorem, we can conclude that Φ_2 is a completely continuous map.

Therefore, $\Phi = \Phi_1 + \Phi_2$ is a condensing map from $B_r(0,\mathcal{BC})$ into $B_r(0,\mathcal{BC})$. Consequently, by Lemma 2.14, we deduce that Φ has a fixed point $x \in B_r(0,\mathcal{BC})$, which is a mild solution of problem (2.1)-(2.2). The proof is complete. $\square$

4. Existence of fractional stochastic optimal controls

In this section we consider a control problem and present a result on the existence of fractional stochastic optimal controls.

Let x^u denote the mild solution of system (2.1)-(2.2) corresponding to the control $u \in U_{ad}$. We consider the Bolza problem (P): find an optimal pair $(x^0, u^0) \in \mathcal{BC} \times U_{ad}$ such that

$$\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x^u, u) \text{ for all } u \in U_{ad},$$

where the cost function

$$\mathcal{J}(x^u, u) = E\int_0^T l(t, x_t^u, x^u(t), u(t))dt + E\Psi(x^u(T)).$$

We introduce the following assumption on l and Ψ.

(B1) The functional $l : J \times \mathcal{B} \times H \times Y \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.
(B2) $l(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathcal{B} \times H \times Y$ for almost all $t \in J$.
(B2) $l(t, x, y, \cdot)$ is convex on Y for each $x \in \mathcal{B}, y \in H$ and almost all $t \in J$.
(B3) There exist constants $d_1, d_2 \geq 0, d_3 > 0$, μ is nonnegative and $\mu \in L^1(J, \mathbb{R})$ such that $l(t, x, y, u) \geq \mu(t) + d_1 \parallel x \parallel_{\mathcal{B}} + d_2 \parallel x \parallel_H + d_2 \parallel u \parallel_Y^p$.
(B4) The functional $\Psi : H \to \mathbb{R}$ is continuous and nonnegative.

To prove the existence of solution for problem (P), we need the following important lemma.

Lemma 4.1. *Operator $\mathcal{Q} : L^p(J, Y) \to \mathcal{BC}$ for some $p\alpha(1 - \vartheta) > 1$ given by*

$$(\mathcal{Q}u)(\cdot) = \int_0^{\cdot} S_\alpha(\cdot - s)B(s)u(s)ds$$

is completely continuous.

Proof. Suppose that $u^n \subseteq L^p_{\mathcal{F}}(J, Y)$ is bounded, we define $\Theta_n(t) = (\mathcal{Q}u^n)(t), t \in J$. Similar to the proof of Theorem 3.1, one can know that for any fixed $t \in J$ and, $E \parallel \Theta_n(t) \parallel_\vartheta^p$ is bounded. By using (H1)-(H5), it is ease to verify that $\Theta_n(t)$ is relatively compact in H_ϑ and is also equicontinuous. Due to Ascoli-Arzela Theorem again, $\{\Theta_n(t)\}$ is compact in H_ϑ. Obviously, $\mathcal{Q}$ is linear and continuous. Hence, $\mathcal{Q}$ is a completely continuous operator. The proof is complete. □

Next we can give the following result on existence of optimal controls for problem (P).

Theorem 4.2. *Let $x_0 \in L_2^0(\Omega, H_\alpha)$. If the assumptions (B1)-(B4) and the assumptions of Theorem 3.1 hold. Then the Bolza problem (P) admits at least one optimal pair on $\mathcal{BC} \times U_{ad}$.*

Proof. Without loss of generality, we assume that $\inf\{\mathcal{J}(x^u, u) | u \in U_{ad}\} = \varepsilon < +\infty$. Otherwise, there is nothing to prove. Using assumptions (B1)-(B4), we have

$$\begin{aligned}\mathcal{J}(x^u, u) \geq &\int_0^T \mu(t)dt + d_1\int_0^T \parallel x_t^u(t) \parallel_{\mathcal{B}} dt + d_2\int_0^T \parallel x^u(t) \parallel_H dt \\ &+ d_3\int_0^T \parallel u(t) \parallel_Y^p dt + \Psi(x^u(T)) \geq -\eta > -\infty,\end{aligned}$$

where $\eta > 0$ is a constant. Hence, $\varepsilon \geq -\eta > -\infty$. On the other hand, by using definition of infimum there exists a minimizing sequence of feasible pair $\{(x^m, u^m)\} \subset A_{ad}$, where $\mathcal{A}_{ad} = \{(x, u) | x$ is a mild solution of system (2.1)-(2.2) corresponding to $u \in U_{ad}\}$, such that $\mathcal{J}(x^m, u^m) \to \varepsilon$ as $m \to +\infty$.

For $\{u^m\} \subseteq U_{ad}$, $\{u^m\}$ is bounded in $L^p_{\mathcal{F}}(J,Y)$, so there exists a subsequence, relabeled as $\{u^m\}$, and $u^0 \in L^p_{\mathcal{F}}(J,Y)$ such that

$$u^m \xrightarrow{w} u^0 \quad \text{in} \quad L^p_{\mathcal{F}}(J,Y) \text{ as } m \to \infty.$$

Since U_{ad} is closed and convex, by Marzur Lemma, we conclude that $u^0 \in U_{ad}$.

Now we suppose that x^m are the mild solutions of system (2.1)-(2.2) corresponding to $u^m (m = 0,1,2,\ldots)$, and x^m satisfied the following integral equation

$$\begin{aligned} x^m(t) &= \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t, \overline{x^m}_t) + \int_0^t A\mathcal{S}_\alpha(t-s) g(s, \overline{x^m}_s) ds \\ &+ \int_0^t \int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s) g(\tau, \overline{x^m}_\tau) d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s) B(s) u^m(s) ds \\ &+ \int_0^t \mathcal{S}_\alpha(t-s) h(s, \overline{x^m}_s) ds + \int_0^t \mathcal{S}_\alpha(t-s) f(s, \overline{x^m}_s) dw(s), \quad t \in J. \end{aligned}$$

Let $h_m(s) \equiv h(s, \overline{x^m}_s)$, $f_m(s) \equiv f(s, \overline{x^m}_s)$. Then by (H4) and (H5), we obtain that

$$\begin{aligned} \| h_m \|^p_{L^p(J,H)} &= E\Big(\int_0^T \| h_m(s) \|^p_H \, ds \Big) = \int_0^T E \| h_m(s) \|^p_H \, ds \\ &\leq \int_0^T m_h(s) ds \leq T^{\frac{p-1}{p}} \Big(\int_0^T (m_h(s))^p ds \Big)^{\frac{1}{p}}, \end{aligned}$$

$$\begin{aligned} \| f_m \|^p_{L^p(J,L_b(K,H))} &= E\Big(\int_0^T \| f_m(s) \|^p_{L_b(K,H)} \, ds \Big) = \int_0^T E \| f_m(s) \|^p_{L_b(K,H)} \, ds \\ &\leq \int_0^T m_f(s) ds \leq T^{\frac{p-1}{p}} \Big(\int_0^T (m_f(s))^p ds \Big)^{\frac{1}{p}}. \end{aligned}$$

That is to say, $h_m : J \to H$ and $f_m : J \to L_b(K,H)$ are bounded continuous operators. Hence, $h_m(\cdot) \in L^p(J,H)$, $f_m(\cdot) \in L^p(J, L_b(K,H))$. Furthermore, $\{h_m(\cdot)\}$, $\{f_m(\cdot)\}$ is bounded in $L^p(J,H)$ and in $L^p(J, L_b(K,H))$, and there are subsequences, relabeled as $\{h_m(\cdot)\}, \{f_m(\cdot)\}$, and $\widehat{h}(\cdot) \in L^p(J,H)$,$\widehat{f}(\cdot) \in L^p(J, L_b(K,H))$ such that

$$h_m(\cdot) \xrightarrow{w} \widehat{h}(\cdot) \quad \text{in} \quad L^p(J,H) \text{ as } m \to \infty,$$
$$f_m(\cdot) \xrightarrow{w} \widehat{f}(\cdot) \quad \text{in} \quad L^p(J, L_b(K,H)) \text{ as } m \to \infty.$$

By Lemma 4.1, we have

$$\mathcal{Q}h_m \to \mathcal{Q}\widehat{h} \quad \text{in} \quad \mathcal{B}C \text{ as } m \to \infty,$$
$$\mathcal{Q}f_m \to \mathcal{Q}\widehat{f} \quad \text{in} \quad \mathcal{B}C \text{ as } m \to \infty.$$

Next we turn to consider the following controlled system

$$^cD^\alpha[x(t) - g(t,x_t)] = Ax(t) + \int_0^t R(t-s)x(s)ds + B(t)u^0(t) + \widehat{h}(t) + \widehat{f}(t)\frac{dw(t)}{dt}, \tag{4.1}$$
$$t \in J = [0,T], u^0 \in U_{ad},$$
$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = 0. \tag{4.2}$$

By Theorem 3.1, it is easy to see that system (4.1)-(4.2) has a mild solution

$$\begin{aligned} \widehat{x}(t) &= \mathcal{R}_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t, \overline{\widehat{x}}_t) + \int_0^t A\mathcal{S}_\alpha(t-s) g(s, \overline{\widehat{x}}_s) ds \\ &+ \int_0^t \int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s) g(\tau, \overline{\widehat{x}}_\tau) d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s) B(s) u^0(s) ds \end{aligned}$$

$$+\int_0^t \mathcal{S}_\alpha(t-s)\widehat{h}(s)ds+\int_0^t \mathcal{S}_\alpha(t-s)\widehat{f}(s)dw(s), \quad t\in J.$$

For each $t\in J, x_m(\cdot),\widehat{x}(\cdot)\in\mathcal{B}C$, we have

$$E\parallel x^m(t)-\widehat{x}(t)\parallel_\vartheta^p\leq \nu_m^{(1)}(t)+\nu_m^{(2)}(t)+\nu_m^{(3)}(t)+\nu_m^{(4)}(t)+\nu_m^{(5)}(t)+\nu_m^{(6)}(t),$$

where

$$\nu_m^{(1)}(t)=6^{p-1}E\parallel(-A)^{-\beta}S_\alpha(t-s)(-A)^{\beta+\vartheta}[g(t,\overline{x^m}_t)-g(t,\overline{\widehat{x}}_t)]\parallel_H^p,$$

$$\nu_m^{(2)}(t)=6^{p-1}E\left\|\int_0^t(-A)^{1-\beta}S_\alpha(t-s)(-A)^{\beta+\vartheta}[g(s,\overline{x^m}_s)-g(s,\overline{\widehat{x}}_s)]ds\right\|_H^p,$$

$$\nu_m^{(3)}(t)=6^{p-1}E\left\|\int_0^t\int_0^s R(s-\tau)(-A)^{\vartheta}\mathcal{S}_\alpha(t-s)[g(\tau,\overline{x^m}_\tau)-g(\tau,\overline{\widehat{x}}_\tau)]d\tau ds\right\|_H^p,$$

$$\nu_m^{(4)}(t)=6^{p-1}E\left\|\int_0^t(-A)^{\vartheta}S_\alpha(t-s)B(s)[u^m(s)-u^0(s)]ds\right\|_H^p,$$

$$\nu_m^{(5)}(t)=6^{p-1}E\left\|\int_0^t(-A)^{\vartheta}S_\alpha(t-s)[h_m(s)-\widehat{h}(s)]ds\right\|_H^p,$$

$$\nu_m^{(6)}(t)=6^{p-1}E\left\|\int_0^t(-A)^{\vartheta}S_\alpha(t-s)[f_m(s)-\widehat{f}(s)]dw(s)\right\|_H^p.$$

By (H1)-(H3), we can obtain

$$\begin{aligned}
&\nu_m^{(1)}(t)+\nu_m^{(2)}(t)+\nu_m^{(3)}(t)\\
&\leq 6^{p-1}\parallel(-A)^{-\beta}\parallel_H^p L_g\parallel\overline{x^m}_t-\overline{\widehat{x}}_t\parallel_{\mathcal{B}}^p+6^{p-1}M_1^pT^{p-1}\int_0^t(t-s)^{p(\alpha\beta-1)}L_g\parallel\overline{x^m}_s-\overline{\widehat{x}}_s\parallel_{\mathcal{B}}^p ds\\
&\quad+6^{p-1}M_2^pT^{2(p-1)}\int_0^t\int_0^s\mu^p(t-\tau)(t-s)^{p(\alpha\beta-1)}L_g\parallel\overline{x^m}_\tau-\overline{\widehat{x}}_\tau\parallel_{\mathcal{B}}^p d\tau ds\\
&\leq 6^{p-1}\parallel(-A)^{-\beta}\parallel_H^p K_T^pL_g\sup_{s\in[0,T]}\parallel\overline{x^m}(s)-\overline{\widehat{x}}(s)\parallel_\vartheta^p\\
&\quad+6^{p-1}M_1^pK_T^pL_gT^{p-1}\int_0^t(t-s)^{p(\alpha\beta-1)}ds\sup_{s\in[0,T]}\parallel\overline{x^m}(s)-\overline{\widehat{x}}(s)\parallel_\vartheta^p\\
&\quad+6^{p-1}M_2^pK_T^pL_gT^{2(p-1)}\int_0^t\int_0^s\mu^p(t-\tau)(t-s)^{\alpha\beta-1}d\tau ds\sup_{s\in[0,T]}\parallel\overline{x^m}(s)-\overline{\widehat{x}}(s)\parallel_\vartheta^p\\
&\leq 6^{p-1}K_T^pL_g\Big[\parallel(-A)^{-\beta}\parallel_H^p+M_1^pT^{p-1}\frac{T^{p(\alpha\beta-1)}}{p(\alpha\beta-1)}+M_2^p\parallel\mu^p\parallel_{L^1}T^{2(p-1)}\frac{T^{p(\alpha\beta-1)}}{p(\alpha\beta-1)}\Big]\\
&\quad\times\sup_{s\in[0,T]}\parallel x^m(s)-\widehat{x}(s)\parallel_\vartheta^p\quad(\text{since } \bar{x}=x \text{ on } J)\\
&\leq L_0\parallel x^m-\widehat{x}\parallel_{\mathcal{C}}^p.
\end{aligned}$$

Using the Hölder inequality again, we have

$$\begin{aligned}
\nu_m^{(4)}(t)&\leq 6^{p-1}E\Big[\int_0^t\parallel B(s)(-A)^{\vartheta}S_\alpha(t-s)[u^m(s)-u^0(s)]\parallel_H ds\Big]^p\\
&\leq 6^{p-1}\parallel B\parallel_\infty^p T^{p-1}\int_0^t E\parallel(-A)^{\vartheta}S_\alpha(t-s)[u^m(s)-u^0(s)]\parallel_H^p ds,
\end{aligned}$$

$$\begin{aligned}\nu_m^{(5)}(t) &\le 6^{p-1}E\Big[\int_0^t \| (-A)^{\vartheta} S_\alpha(t-s)[h_m(s)-\widehat{h}(s)] \|_H \, ds\Big]^p \\ &\le 6^{p-1}T^{p-1}\int_0^t E \| (-A)^{\vartheta} S_\alpha(t-s)[h_m(s)-\widehat{h}(s)] \|_H^p \, ds,\end{aligned}$$

$$\begin{aligned}\nu_m^{(6)}(t) &\le 6^{p-1}C_p\Big[\int_0^t [E \| (-A)^{\vartheta} S_\alpha(t-s)[f_m(s)-\widehat{f}(s)] \|_H^p]^{2/p} ds\Big]^{p/2} \\ &\le 6^{p-1}C_pT^{p/2-1}\int_0^t E \| (-A)^{\vartheta} S_\alpha(t-s)[f_m(s)-\widehat{f}(s)] \|_H^p \, ds.\end{aligned}$$

By virtue of Lemma 4.1 and Lebesgue's dominated convergence theorem,

$$\int_0^t E \| (-A)^{\vartheta} S_\alpha(t-s)[u_m(s)-u^0(s)] \|_H^p \, ds \to 0 \text{ as } m\to\infty,$$

$$\int_0^t E \| (-A)^{\vartheta} S_\alpha(t-s)[h_m(s)-\widehat{h}(s)] \|_H^p \, ds \to 0 \text{ as } m\to\infty,$$

$$\int_0^t E \| (-A)^{\vartheta} S_\alpha(t-s)[f_m(s)-\widehat{f}(s)] \|_H^p \, ds \to 0 \text{ as } m\to\infty.$$

Thus

$$\nu_m^{(4)}(t), \nu_m^{(5)}(t), \nu_m^{(6)}(t) \to 0 \text{ as } m\to\infty.$$

Then we have

$$E \| x^m(t)-\widehat{x}(t) \|_{\vartheta}^p \le L_0 \| x^m-\widehat{x} \|_{\mathcal{C}}^p + \nu_m^{(4)}(t)+\nu_m^{(5)}(t)+\nu_m^{(6)}(t),$$

which implies that

$$\| x^m-\hat{x}(t) \|_{\mathcal{C}}^p \le \frac{\nu_m^{(4)}(t)+\nu_m^{(5)}(t)+\nu_m^{(6)}(t)}{1-L_0},$$

and we can infer that

$$x_m \to \widehat{x} \text{ in } \mathcal{B}C \text{ as } m\to\infty.$$

Further, by (H4) and (H5), we can obtain

$$h_m(\cdot) \to h(\cdot,\bar{x}_\cdot) \ \mathcal{B}C \text{ as } m\to\infty,$$

$$f_m(\cdot) \to f(\cdot,\bar{x}_\cdot) \ \mathcal{B}C \text{ as } m\to\infty.$$

Using the uniqueness of limit, we have

$$\widehat{h}(t) = h(t,\bar{x}_t), \ \widehat{f}(t) = f(t,\bar{x}_t).$$

Therefore, $\widehat{x}$ can be given by

$$\begin{aligned}\widehat{x}(t) &= \mathcal{R}_\alpha(t)[\varphi(0)-g(0,\varphi)] + g(t,\overline{\widehat{x}}_t) + \int_0^t A\mathcal{S}_\alpha(t-s)g(s,\overline{\widehat{x}}_s)ds \\ &+ \int_0^t\int_0^s R(s-\tau)\mathcal{S}_\alpha(t-s)g(\tau,\overline{\widehat{x}}_\tau)d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)B(s)u^0(s)ds \\ &+ \int_0^t \mathcal{S}_\alpha(t-s)\widehat{h}(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\widehat{f}(s)dw(s), \quad t\in J,\end{aligned}$$

which is just a mild solution of system (2.1)-(2.2) corresponding to u^0. Since $\mathcal{B}C \hookrightarrow L^1(J,H)$, using (B1)-(B5) and Balder's theorem, we can obtain

$$\begin{aligned}\varepsilon &= \lim_{m\to\infty} E\int_0^T l(t, x_t^m x^m(t), u^m(t))dt + \lim_{m\to\infty} E\Psi(x^m(T))\\ &\geq E\int_0^T l(t, \widehat{x}_t, \widehat{x}(t), u^0(t))dt + E\Psi(\widehat{x}(T)) = \mathcal{J}(\widehat{x}, u^0) \geq \varepsilon.\end{aligned}$$

This shows that $\mathcal{J}$ attains its minimum at $(\widehat{x}, u^0) \in \mathcal{BC} \times U_{ad}$ and the proof is complete. □

5. Application

Consider the following fractional stochastic partial neutral functional integro-differential system of the form

$$\begin{aligned} D_t^\alpha\Big[z(t,x) - \int_{-\infty}^t \int_0^\pi \tilde{b}_1(t-s,\tau,x)z(s,\tau)d\tau ds\Big]\\ = \frac{\partial^2}{\partial x^2}z(t,x) + \int_0^t (t-s)^\zeta e^{-\varsigma(t-s)}\frac{\partial^2}{\partial x^2}z(s,x)ds + \int_{[0,\pi]}\int_0^T q(t,x)u(s,\tau))dsd\tau\\ + \int_{-\infty}^t \tilde{b}_2(t,t-s,x,z(s,x))ds + \int_{-\infty}^t \tilde{b}_3(t,t-s,x,z(s,x))ds\frac{w(t)}{dt},\\ 0\leq t\leq T, 0\leq x\leq \pi, u\in U_{ad}, \end{aligned} \tag{5.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad 0\leq t\leq T, \tag{5.2}$$

$$z_t(0,x) = 0, \quad 0\leq x\leq \pi, \tag{5.3}$$

$$z(t,x) = \varphi(t,x), \quad t\leq 0, 0\leq x\leq \pi, \tag{5.4}$$

where D_t^α is a Caputo fractional partial derivative of order $\alpha \in (1,2)$, ζ and ς are positive numbers, and $q : [0,T]\times[0,\pi] \to \mathbb{R}$ is continuous. $w(t)$ denotes a one-dimensional standard Wiener process in H defined on a stochastic space $(\Omega, \mathcal{F}, P)$. Let $H = Y = L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A : D(A) \subseteq H \to H$ by $A\omega = \omega''$ with the domain

$$D(A) := \{\omega \in H : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in H, \omega(0) = \omega(\pi) = 0\}.$$

Then A generates a compact, analytic semigroup $T(\cdot)$ of uniformly bounded linear operator. Moreover, the eigenvalues of A are $n^2\pi^2$ and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, \cdots$. Let $H_{\frac{1}{2}} := (D((-A)^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$, where $\|\cdot\|_{\frac{1}{2}} := \| (-A)^{\frac{1}{2}}x \|$ for each $x \in D((-A)^{\frac{1}{2}})$. The operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}}\omega = \sum_{n=1}^\infty n\langle\omega, e_n\rangle e_n$$

on the space $D((-A)^{\frac{1}{2}}) = \{\omega(\cdot) \in H, \sum_{n=1}^\infty n\langle\omega, e_n\rangle e_n \in H\}$ and $\| (-A)^{-\frac{1}{2}} \| = 1$. Hence, A is sectorial of type and (P1) is satisfied. The operator $R(t) : D(A) \subseteq H \to H, t \geq 0, R(t)x = t^\zeta e^{-\varsigma t}x''$ for $x \in D(A)$. Moreover, it is easy to see that conditions (P2) and (P3) in Section 2 are satisfied with $b(t) = t^\zeta e^{-\varsigma t}$ and $D(A) = C_0^\infty([0,\pi])$, where $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $x = 0$ and $x = \pi$.

For $d > 0$, we define the admissible control set $U_{ad} = \{u(\cdot, y)|[0,T] \to Y$ measurable, $\mathcal{F}_t$-adapted stochastic processes, and $\| u \|_{L^p_{\mathcal{F}}([0,T],Y)} \leq d\}$. Next we consider that the following the phase space.

Let $r \geq 0, 1 \leq p < 1$ and let $\tilde{h} : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al. [14]. Briefly, this means that h is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\tilde{h}(\xi + \tau) \leq \gamma(\xi)h(\tau)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero. We denote by

$\mathcal{C}_r \times L^2(\tilde{h}, H_{\frac{1}{2}})$ the set consists of all classes of functions $\varphi : (-\infty, 0] \to H_{\frac{1}{2}}$ such that $\varphi_{|_{[-r,0]}} \in \mathcal{C}([-r, 0], H_{\frac{1}{2}})$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$, and $\tilde{h} \parallel \varphi \parallel^p_{\frac{1}{2}}$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm is given by

$$\parallel \varphi \parallel_{\mathcal{B}} = \sup_{-r\leq\theta\leq 0} \parallel \varphi(\theta) \parallel_{\frac{1}{2}} + \left(\int_{-\infty}^{-r} \tilde{h}(\theta) \parallel \varphi(\theta) \parallel^p_{\frac{1}{2}} d\theta \right)^{1/p}.$$

The space $\mathcal{B} = \mathcal{C}_r \times L^p(\tilde{h}, H_{\frac{1}{2}})$ satisfies axioms (A)-(C). Moreover, when $r = 0$ and $p = 2$, we can take $\tilde{H} = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + (\int_{-t}^{0} \tilde{h}(\theta) d\theta)^{\frac{1}{2}}$, for $t \geq 0$ (see [14], Theorem 1.3.8 for details).

Additionally, we choose $\beta = \frac{1}{2}$ and assume that the following conditions hold:

(i) The functions $\tilde{b}_1(s, \tau, x), \frac{\partial \tilde{b}_1(s,\tau,x)}{\partial x}, \frac{\partial^2 \tilde{b}_1(s,\tau,x)}{\partial^2 x}$ are measurable, $\tilde{b}_1(s, \tau, \pi) = \tilde{b}_1(s, \tau, 0) = 0$ for every $(s, \tau) \in (-\infty, 0] \times [0, \pi]$ and

$$\tilde{L}_g = \max \left\{ \left(\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{\tilde{h}(s)} \left(\frac{\partial^i \tilde{b}_1(s, \tau, x)}{\partial x^i} \right)^2 d\tau ds dx \right)^{\frac{1}{2}} : i = 0, 1, 2 \right\} < \infty.$$

(i) The function $\tilde{b}_2 : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there is continuous function $\mu_1 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|\tilde{b}_2(t, s, x, y)| \leq \mu_1(t, s) \cos |y|, \quad (t, s, x, y) \in \mathbb{R}^4.$$

(i) The function $\tilde{b}_3 : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there is continuous function $\mu_2 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$|\tilde{b}_3(t, s, x, y)| \leq \mu_2(t, s) \cos |y|, \quad (t, s, x, y) \in \mathbb{R}^4.$$

Take $\varphi \in \mathcal{B} = \mathcal{C}_0 \times L^2(\tilde{h}, H_{\frac{1}{2}})$ with $\varphi(s)(\tau) = \varphi(s, \tau)$.
Let $g : [0, T] \times \mathcal{B} \to H$, $h : [0, T] \times \mathcal{B} \to H$, $f : [0, T] \times \mathcal{B} \to L_b(K, H)$ be the operators defined by

$$g(t, \psi)(x) = \int_{-\infty}^0 \int_0^\pi \tilde{b}_1(-s, \upsilon, x)\psi(s, \upsilon) d\upsilon ds,$$

$$h(t, \psi)(x) = \int_{-\infty}^0 \tilde{b}_2(t, -s, x, \psi(s, x)) ds,$$

$$f(t, \psi)(x) = \int_{-\infty}^0 \tilde{b}_3(t, -s, x, \psi(s, x)) ds,$$

and $$B(t)u(t)(x) = \int_{[0,\pi]} \int_0^T q(t, x)u(s, \tau)) ds d\tau.$$

Using these definitions, we can represent the system (5.1)-(5.4) in the abstract form (2.1)-(2.2) with the cost function

$$\begin{aligned}\mathcal{J}(z, u) = {} & E \int_{[0,\pi]} \int_{-\infty}^0 \parallel z^u(t + s, x) \parallel^2_{\frac{1}{2}} ds dx + E \int_{[0,\pi]} \parallel z^u(t, x) \parallel^2_{\frac{1}{2}} dx \\ & + E \int_{[0,\pi]} \parallel u(x, t) \parallel^2_Y dx + E \parallel z(T) \parallel_{\frac{1}{2}} .\end{aligned}$$

Moreover, using (i) we can prove that g is $D(A)$-valued and

$$\begin{aligned} E \parallel Ag(t, \psi) \parallel^p &\leq \left[\tilde{L}_g \left[\parallel \psi(0) \parallel + \left(\int_{-\infty}^0 h(\theta) \parallel \psi(\theta) \parallel^2 d\theta \right)^{\frac{1}{2}} \right] \right]^p \\ &\leq \Bigg[\tilde{L}_g \Bigg[\parallel (-A)^{-\frac{1}{2}} \parallel \parallel (-A)^{\frac{1}{2}} \psi(0) \parallel \\ &\quad + \left(\int_{-\infty}^0 h(\theta) \parallel (-A)^{-\frac{1}{2}} \parallel^2 \parallel (-A)^{\frac{1}{2}} \psi(\theta) \parallel^2 d\theta \right)^{\frac{1}{2}} \Bigg] \Bigg]^p \end{aligned}$$

$$= \left[\tilde{L}_g\left[\parallel \psi(0) \parallel_{\frac{1}{2}} + \left(\int_{-\infty}^{0} h(\theta) \parallel \psi(\theta) \parallel_{\frac{1}{2}}^{2} d\theta\right)^{\frac{1}{2}}\right]\right]^p = (\tilde{L}_g)^p \parallel \psi \parallel_{\mathcal{B}}^p .$$

It follows by assumptions (ii) and (iii) that

$$E \parallel g(t,\psi) \parallel^p \leq (a_1(t))^p,\ E \parallel f(t,\psi) \parallel^p \leq (a_2(t))^p$$

for all $(t,\psi) \in [0,T] \times \mathcal{B}$, where $a_1(t) = (\int_{-\infty}^{0} \frac{(\mu_1(t,s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$,$a_2(t) = (\int_{-\infty}^{0} \frac{(\mu_2(t,s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$. Moreover, suppose that conditions $p^2(\frac{1}{2}\alpha - 1) + p > 1$ and (3.1) hold. Then it satisfies all the assumptions given in Theorem 4.2. Therefore, we can conclude that problem (5.1)-(5.4) has at least one optimal pair.

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