Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4226–4234
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

General mixed width-integral of convex bodies

Yibin Feng

School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China.

Communicated by Sh. Wu

Abstract

In this article, we introduce a new concept of general mixed width-integral of convex bodies, and establish some of its inequalities, such as isoperimetric inequality, Aleksandrov-Fenchel inequality, and cyclic inequality. We also consider the general width-integral of order i and show its related properties and inequalities.

Keywords: General mixed width-integral, mixed width-integral, general width-integral of order i.
2010 MSC: 52A20, 52A40.

1. Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [6, 25])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The study of width-integral has a long history. The notion of the classical width-integral was first considered by Blaschke (see [3]) and was further studied by Hardy, Littlewood and Pólya (see [12]). It was generalized to the mixed width-integral by Lutwak [19] in 1977. Many important results related to the mixed width-integral were obtained from these articles (see [13, 17, 18, 21]).

Email address: fengyibin001@163.com (Yibin Feng)

Received 2016-04-22

The mixed width-integral, $B(K_1,\cdots,K_n)$, of $K_1,\cdots,K_n\in\mathcal{K}^n$ was defined by (see [19])

$$B(K_1,\cdots,K_n)=\frac{1}{n}\int_{S^{n-1}}b(K_1,u)\cdots b(K_n,u)dS(u), \tag{1.1}$$

where $dS(u)$ is the $(n-1)$-dimensional volume element on S^{n-1} and $b(K,u)$ denotes the half width of K in the direction u, namely, $b(K,u)=\frac{1}{2}h(K,u)+\frac{1}{2}h(K,-u)$. If there exists a constant $\lambda>0$ such that $b(K,u)=\lambda b(L,u)$ for all $u\in S^{n-1}$, then K and L are said to have similar width.

The main aim of this article is to define a corresponding notion of mixed width-integral, and to extend Lutwak's inequalities to the entire family of this new mixed width-integral.

For $\tau\in(-1,1)$, the general mixed width-integral, $B^{(\tau)}(K_1,\cdots,K_n)$, of $K_1,\cdots,K_n\in\mathcal{K}^n$ is defined by

$$B^{(\tau)}(K_1,\cdots,K_n)=\frac{1}{n}\int_{S^{n-1}}b^{(\tau)}(K_1,u)\cdots b^{(\tau)}(K_n,u)dS(u), \tag{1.2}$$

where $b^{(\tau)}(K,u)=f_1(\tau)h(K,u)+f_2(\tau)h(K,-u)$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as follows

$$f_1(\tau)=\frac{(1+\tau)^2}{2(1+\tau^2)},\qquad f_2(\tau)=\frac{(1-\tau)^2}{2(1+\tau^2)}. \tag{1.3}$$

Clearly,

$$f_1(\tau)+f_2(\tau)=1, \tag{1.4}$$

$$f_1(-\tau)=f_2(\tau),\qquad f_2(-\tau)=f_1(\tau). \tag{1.5}$$

Together with (1.3), the case $\tau=0$ in definition (1.2) is just Lutwak's mixed width-integral $B(K_1,\cdots,K_n)$. Two convex bodies K and L are said to have similar general width if there exists a constant $\lambda>0$ such that $b^{(\tau)}(K,u)=\lambda b^{(\tau)}(L,u)$ for all $u\in S^{n-1}$. If $b^{(\tau)}(K,u)b^{(\tau)}(L,u)$ is a constant for all $u\in S^{n-1}$, then we call K and L with joint constant general width.

The general operator belongs to the asymmetric Brunn-Minkowski theory which has its starting point in the theory of valuations in connection with isoperimetric and analytic inequalities (see [1, 2, 4, 5, 7–11, 14–16, 22–24, 26–30]).

The main results are the following: We first establish the isoperimetric and Aleksandrov-Fenchel inequalities for the general mixed width-integral.

Theorem 1.1. *If $\tau\in(-1,1)$ and $K_1,\cdots,K_n\in\mathcal{K}_c^n$, then*

$$V(K_1)\cdots V(K_n)\le B^{(\tau)}(K_1,\cdots,K_n)^n, \tag{1.6}$$

with equality if and only if $K_1,\cdots,K_n$ are n-balls.

Theorem 1.2. *If $\tau\in(-1,1)$, $K_1,\cdots,K_n\in\mathcal{K}^n$ and $1<m\le n$, then*

$$B^{(\tau)}(K_1,\cdots,K_n)^m\le\prod_{i=1}^{m}B^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},\cdots,K_{n-i+1}), \tag{1.7}$$

with equality if and only if $K_{n-m+1},\cdots,K_n$ are all of similar general width.

Moreover, we show a cyclic inequality for the general mixed width-integral.

Theorem 1.3. *If $\tau\in(-1,1)$ and $K,L\in\mathcal{K}^n$, then for $i<j<k$,*

$$B_i^{(\tau)}(K,L)^{k-j}B_k^{(\tau)}(K,L)^{j-i}\ge B_j^{(\tau)}(K,L)^{k-i}, \tag{1.8}$$

with equality if and only if K and L have similar general width.

Here $B_i^{(\tau)}(K,L)=B_i^{(\tau)}(K,n-i;L,i)$ in which K appears $n-i$ times and L appears i times.

The proofs of Theorems 1.1–1.3 will be given in the Section 3 of this paper. In Section 4, we consider the general width-integral of order i and establish its related properties and inequalities.

2. Preliminaries

The radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact star-shaped (about the origin) set K in $\mathbb{R}^n$ is defined, for $u \in S^{n-1}$, by (see [6, 25])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}. \tag{2.1}$$

The polar body, K^*, of $K \in \mathcal{K}^n$ is defined by (see [6, 25])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.2}$$

It is easy to check that for $K \in \mathcal{K}_o^n$,

$$(K^*)^* = K,$$

and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.$$

An extension of the well-known Blaschke-Santaló inequality is as follows (see [20]):

Theorem 2.1. *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid.

For $K \in \mathcal{K}^n$ and $i = 0, 1, \cdots, n-1$, the quermassintegrals, $W_i(K)$, of K is given by (see [6, 25])

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS_i(K, u), \tag{2.4}$$

where $S_i(K, \cdot)$ denotes the mixed surface area measure of K. Besides, we know that

$$W_0(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K, u) = V(K). \tag{2.5}$$

The polar coordinate formula for volume of a body K in $\mathbb{R}^n$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u). \tag{2.6}$$

3. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. It follows by Jensen's inequality (see [12]) that

$$\begin{aligned} B^{(\tau)}(K_1, \cdots, K_n) &= \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_n, u) dS(u) \\ &\geq n\omega_n^2 \left[\int_{S^{n-1}} b^{(\tau)}(K_1, u)^{-1} \cdots b^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-1}, \end{aligned} \tag{3.1}$$

with equality if and only if $K_1, \cdots, K_n$ have joint constant general width. Together with Hölder's inequality (see [12]), we have

$$\left[\int_{S^{n-1}} b^{(\tau)}(K_1, u)^{-1} \cdots b^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-n} \geq \prod_{i=1}^{n} \left[\int_{S^{n-1}} b^{(\tau)}(K_i, u)^{-n} dS(u) \right]^{-1}, \tag{3.2}$$

with equality if and only if $K_1, \cdots, K_n$ have similar general width. Using Minkowski's inequality (see [12]), we have

$$\begin{aligned}\left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K_i,u)^{-n}dS(u)\right]^{-\frac{1}{n}} &= \left[\frac{1}{n}\int_{S^{n-1}} (f_1(\tau)h(K_i,u)+f_2(\tau)h(K_i,-u))^{-n}\,dS(u)\right]^{-\frac{1}{n}} \\ &\geq \left[\frac{1}{n}\int_{S^{n-1}} h(K_i,u)^{-n}dS(u)\right]^{-\frac{1}{n}} = V(K_i^*)^{-\frac{1}{n}},\end{aligned} \tag{3.3}$$

with equality if and only if K_i is origin-symmetric. It follows from Theorem 2.1 that for inequality (3.3),

$$\left[\frac{1}{n\omega_n^2}\int_{S^{n-1}} b^{(\tau)}(K_i,u)^{-n}dS(u)\right]^{-1} \geq V(K_i), \tag{3.4}$$

with equality if and only if K_i is an n-dimensional ellipsoid. From inequalities (3.1), (3.2) and (3.4), this yields

$$V(K_1)\cdots V(K_n) \leq B^{(\tau)}(K_1,\cdots,K_n)^n.$$

By the equality conditions of inequalities (3.1), (3.2) and (3.4), equality holds in (1.6) if and only if $K_1,\cdots,K_n$ are n-balls. □

Lemma 3.1 ([17]). *If $f_0, f_1, \cdots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1,\cdots,\lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u)f_1(u)\cdots f_m(u)dS(u) \leq \prod_{i=1}^m \left[\int_{S^{n-1}} f_0(u)f_i^{\lambda_i}(u)dS(u)\right]^{\frac{1}{\lambda_i}}, \tag{3.5}$$

with equality if and only if there exist positive constants $\alpha_1,\cdots,\alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.2. Let in Lemma 3.1

$$\begin{aligned}&\lambda_i = m \quad (1 \leq i \leq m),\\ &f_0 = b^{(\tau)}(K_1,u)\cdots b^{(\tau)}(K_{n-m},u) \qquad (f_0 = 1 \text{ if } m = n),\\ &f_i = b^{(\tau)}(K_{n-i+1},u) \quad (1 \leq i \leq m).\end{aligned}$$

Then

$$\begin{aligned}&\int_{S^{n-1}} b^{(\tau)}(K_1,u)\cdots b^{(\tau)}(K_n,u)dS(u)\\ &\qquad \leq \prod_{i=1}^m \left[\int_{S^{n-1}} b^{(\tau)}(K_1,u)\cdots b^{(\tau)}(K_{n-m},u)b^{(\tau)}(K_{n-i+1},u)^m dS(u)\right]^{\frac{1}{m}}.\end{aligned}$$

Combining with definition (1.2), we have

$$B^{(\tau)}(K_1,\cdots,K_n)^m \leq \prod_{i=1}^m B^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},\cdots,K_{n-i+1}).$$

The equality condition of inequality (3.5) implies that equality holds in (1.7) if and only if $K_{n-m+1},\cdots,K_n$ are all of similar general width. □

Proof of Theorem 1.3. It follows from Hölder's inequality (see [12]) that

$$\begin{aligned}B_i^{(\tau)}(K,L)^{\frac{k-j}{k-i}}B_k^{(\tau)}(K,L)^{\frac{j-i}{k-i}} &= \left(\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{n-i}b^{(\tau)}(L,u)^i dS(u)\right)^{\frac{k-j}{k-i}}\\ &\quad\times\left(\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{n-k}b^{(\tau)}(L,u)^k dS(u)\right)^{\frac{j-i}{k-i}}\\ &\geq \frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{n-j}b^{(\tau)}(L,u)^j dS(u) = B_j^{(\tau)}(K,L).\end{aligned}$$

This gives

$$B_i^{(\tau)}(K,L)^{k-j}B_k^{(\tau)}(K,L)^{j-i} \geq B_j^{(\tau)}(K,L)^{k-i}.$$

The equality condition of Hölder's inequality gets that equality holds in (1.8) if and only if K and L have similar general width. □

Taking $i = 0$, $j = i$ and $k = n$ in inequality (1.8), we have

Corollary 3.2. *If $\tau \in (-1, 1)$ and $K, L \in \mathcal{K}^n$, then for $0 \leq i \leq n$,*

$$B_i^{(\tau)}(K,L)^n \leq B^{(\tau)}(K)^{n-i}B^{(\tau)}(L)^i, \tag{3.6}$$

for $i < 0$ or $i > n$, inequality (3.6) is reversed, with equality in every inequality if and only if $i = n$ or, when $i \neq n$, K and L have similar general width.

Let $i = 1$ and $i = -1$ in Corollary 3.2, respectively. The dual Minkowski type inequalities for the general mixed width-integral are as follows:

Corollary 3.3. *If $\tau \in (-1, 1)$ and $K, L \in \mathcal{K}^n$, then*

$$B_1^{(\tau)}(K,L)^n \leq B^{(\tau)}(K)^{n-1}B^{(\tau)}(L),$$

with equality if and only if K and L have similar general width.

Corollary 3.4. *If $\tau \in (-1, 1)$ and $K, L \in \mathcal{K}^n$, then*

$$B_{-1}^{(\tau)}(K,L)^n \geq B^{(\tau)}(K)^{n+1}B^{(\tau)}(L)^{-1},$$

with equality if and only if K and L have similar general width.

4. General width-integral of order i

In this section, we consider the general width-integral of order i and show its related properties and inequalities.

Taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in (1.2), the general width-integral of order i, $B_i^{(\tau)}(K)$, of $K \in \mathcal{K}^n$ is given by

$$B_i^{(\tau)}(K) = \frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{n-i}dS(u). \tag{4.1}$$

Let $K_1 = \cdots = K_n = K$ in (1.2). We write $B^{(\tau)}(K)$ for $B^{(\tau)}(K, \cdots, K)$ called the general width-integral of $K \in \mathcal{K}^n$.

If $K_1, \cdots, K_m \in \mathcal{K}^n$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$, then the Minkowski linear combination is defined by (see [6, 25])

$$\lambda_1 K_1 + \cdots + \lambda_m K_m = \{\lambda_1 x_1 + \cdots + \lambda_m x_m : x_1 \in K_1, \cdots, x_m \in K_m\}.$$

It is easy to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_m K_m, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_m h(K_m, \cdot).$$

We now show that the general width-integral of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial of degree n in $\lambda_1, \cdots, \lambda_m$.

Theorem 4.1. *Suppose $\tau \in (-1, 1)$ and $K_1, \cdots, K_m \in \mathcal{K}^n$. If $K = \lambda_1 K_1 + \cdots + \lambda_m K_m$ then*

$$B^{(\tau)}(K) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \lambda_{j_1} \cdots \lambda_{j_n} B^{(\tau)}(K_{j_1}, \cdots, K_{j_n}). \tag{4.2}$$

The following is a direct consequence of Theorem 4.1.

Theorem 4.2. *Let $\tau \in (-1, 1)$ and $K \in \mathcal{K}^n$. If $K_\mu = K + \mu B$ $(\mu > 0)$ then for $j = 0, 1, \cdots, n$,*

$$B_j^{(\tau)}(K_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} B_{j+i}^{(\tau)}(K) \mu^i. \tag{4.3}$$

Further, we establish several inequalities for the general width-integral of order i.

Lemma 4.3. *If $\tau \in (-1, 1)$ and $K \in \mathcal{K}^n$, then*

$$B_{2n}^{(\tau)}(K) \leq V(K^*), \tag{4.4}$$

with equality if and only if K is origin-symmetric.

Proof. Using Minkowski's inequality (see [12]), we yield

$$\begin{aligned}
B_{2n}^{(\tau)}(K)^{-\frac{1}{n}} &= \left[\frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(K,u)^{-n} dS(u)\right]^{-\frac{1}{n}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)h(K,u) + f_2(\tau)h(K,-u)\right)^{-n} dS(u)\right]^{-\frac{1}{n}} \\
&\geq \left[\frac{1}{n}\int_{S^{n-1}} \left(f_1(\tau)h(K,u)\right)^{-n} dS(u)\right]^{-\frac{1}{n}} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} \left(f_2(\tau)h(K,-u)\right)^{-n} dS(u)\right]^{-\frac{1}{n}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} h(K,u)^{-n} dS(u)\right]^{-\frac{1}{n}}.
\end{aligned}$$

This implies

$$B_{2n}^{(\tau)}(K) \leq \frac{1}{n}\int_{S^{n-1}} h(K,u)^{-n} dS(u) = V(K^*).$$

The equality condition of Minkowski's inequality gives that equality holds in (4.4) if and only if K and $-K$ are dilated of one another, namely, K is origin-symmetric. □

Theorem 4.4. *If $\tau \in (-1, 1)$ and $K \in \mathcal{K}_c^n$, then for $n < i < 2n$,*

$$B_i^{(\tau)}(K) B_i^{(\tau)}(K^*) \leq \omega_n^2, \tag{4.5}$$

For $i < n$, inequality (4.5) is reversed, with equality in every inequality if and only if K is an ellipsoid centered at the origin.

Proof. Using Lemma 4.3 and Jensen's inequality (see [12]), we have for $i < 2n$ and $i \neq n$

$$\omega_n^{\frac{i-2n}{n(n-i)}} B_i^{(\tau)}(K)^{\frac{1}{n-i}} \geq B_{2n}^{(\tau)}(K)^{-\frac{1}{n}} \geq V(K^*)^{-\frac{1}{n}}. \tag{4.6}$$

Thus it follows from (4.6) that

$$\omega_n^{\frac{i-2n}{n(n-i)}} B_i^{(\tau)}(K^*)^{\frac{1}{n-i}} \geq V(K)^{-\frac{1}{n}}. \tag{4.7}$$

Together (4.6), (4.7) with Theorem 2.1, we get

$$\left[B_i^{(\tau)}(K) B_i^{(\tau)}(K^*)\right]^{\frac{1}{n-i}} \geq \omega_n^{\frac{2}{n-i}}. \tag{4.8}$$

If $n < i < 2n$ in inequality (4.8), then

$$B_i^{(\tau)}(K) B_i^{(\tau)}(K^*) \leq \omega_n^2.$$

If $i < n$ in inequality (4.8), then

$$B_i^{(\tau)}(K) B_i^{(\tau)}(K^*) \geq \omega_n^2.$$

By the equality conditions of inequality (4.4), inequality (2.3) and Jensen's inequality, we know that equality holds in every inequality if and only if K is an ellipsoid centered at the origin. □

Lemma 4.5 ([6]). *If $K \in \mathcal{K}^n$ and $0 \leq i < j < k \leq n$, then*

$$W_j(K)^{k-i} \geq W_i(K)^{k-j} W_k(K)^{j-i},$$

with equality if and only if K is an n-ball.

Taking $L = B$ in Theorem 1.3, the following is a direct result.

Lemma 4.6. *For $K \in \mathcal{K}^n$ and $\tau \in (-1, 1)$, if $i < j < k$ then*

$$B_j^{(\tau)}(K)^{k-i} \leq B_i^{(\tau)}(K)^{k-j} B_k^{(\tau)}(K)^{j-i},$$

with equality if and only if K is of similar general width.

Lemma 4.7. *If $\tau \in (-1, 1)$ and $K \in \mathcal{K}^n$, then*

$$B_{n-1}^{(\tau)}(K) = W_{n-1}(K).$$

Proof. It follows by definition (4.1) that

$$\begin{aligned} B_{n-1}^{(\tau)}(K) &= \frac{1}{n} \int_{S^{n-1}} [f_1(\tau) h(K, u) + f_2(\tau) h(K, -u)] dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(u) = W_{n-1}(K). \end{aligned}$$

□

Theorem 4.8. *For $\tau \in (-1, 1)$ and $K \in \mathcal{K}^n$, if $i < n-1$ then*

$$W_i(K) \leq B_i^{(\tau)}(K), \tag{4.9}$$

with equality if and only if K is an n-ball centered at the origin.

Proof. Using Lemma 4.5, it follows that

$$W_i(K) \leq \omega_n^{i+1-n} W_{n-1}^{n-i}(K), \tag{4.10}$$

with equality if and only if K is an n-ball. By Lemma 4.6, we have

$$\omega_n^{i+1-n} B_{n-1}^{(\tau)}(K)^{n-i} \leq B_i^{(\tau)}(K), \tag{4.11}$$

with equality if and only if K is of similar general width. Together (4.10), (4.11) with Lemma 4.7, this gives

$$W_i(K) \leq B_i^{(\tau)}(K).$$

From the equality conditions of inequalities (4.10) and (4.11), we obtain that equality holds in (4.9) if and only if K is an n-ball centered at the origin. □

Theorem 4.9. *For $\tau \in (-1, 1)$ and $K \in \mathcal{K}^n$, if $0 < i < n$ then*

$$B_{n+i}^{(\tau)}(K) \leq W_{n-i}(K^*), \tag{4.12}$$

with equality if and only if K is an n-ball centered at the origin.

Proof. By Lemma 4.2, we get

$$\omega_n^{n-i} V^i(K^*) \leq W_{n-i}^n(K^*), \tag{4.13}$$

with equality if and only if K^* is an n-ball. It follows from Lemma 4.6 that

$$B_{n+i}^{(\tau)}(K)^n \leq \omega_n^{n-i} B_{2n}^{(\tau)}(K)^i, \tag{4.14}$$

with equality if and only if K is of similar general width. By (4.13), (4.14) and Lemma 4.3, we have

$$B_{n+i}^{(\tau)}(K) \leq W_{n-i}(K^*).$$

The equality conditions of inequalities (4.13), (4.14) and (4.4) imply that equality holds in (4.12) if and only if K is an n-ball centered at the origin. □

Acknowledgment

The author is indebted to the editors and the anonymous referees for many valuable suggestions and comments. This work was supported by the National Natural Science Foundations of China (Grant No.11561020 and No.11371224) and the Young Foundation of Hexi University (Grant No.QN2015-02).

References

[1] J. Abardia, A. Bernig, *Projection bodies in complex vector space*, Adv. Math., **227** (2011), 830–846. 1

[2] S. Alesker, A. Bernig, F. E. Schuster, *Harmonic analysis of translation invariant valuations*, Geom. Funct. Anal., **21** (2011), 751–773. 1

[3] W. Blaschke, *Vorlesungen über integralgeometric I, II*, Teubner, Leipzig, 1936, 1937; reprint, chelsea, New York, (1949). 1

[4] Y. B. Feng, W.-D. Wang, *General L_p-harmonic Blaschke bodies*, P. Indian Math. Soc., **124** (2014), 109–119. 1

[5] Y. B. Feng, W.-D. Wang, F. H. Lu, *Some inequalities on general L_p-centroid bodies*, Math. Inequal. Appl., **18** (2015), 39–49. 1

[6] R. J. Gardner, *Geometric Tomography: Second ed.*, Cambridge Univ. Press, Cambridge, (2006). 1, 2, 2, 2, 4, 4.5

[7] C. Haberl, *L_p-intersection bodies*, Adv. Math., **217** (2008), 2599–2624. 1

[8] C. Haberl, *Minkowski valuations intertunning the special linear group*, J. Eur. Math. Soc., **14** (2012), 1565–1597.

[9] C. Haberl, M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not., **2006** (2006), 29 pages.

[10] C. Haberl, F. E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal., **257** (2009), 641–658.

[11] C. Haberl, F. E. Schuster, J. Xiao, *An asymmetric affine Pólya-Szegö principle*, Math. Ann., **352** (2012), 517–542. 1

[12] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, (1934). 1, 3, 3, 3, 3, 4, 4

[13] G. Leng, C. Zhao, B. He, X. Li, *Inequalities for polars of mixed projection bodies*, Sci. China Ser., **47** (2004), 175–186. 1

[14] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc., **357** (2005), 4191–4213. 1

[15] M. Ludwig, *Valuations in the affine geometry of convex bodies: Proc. Conf. "Integral Geometry and Convexity"*, World Sci. Publ., **2006** (2006), 49–65.

[16] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom., **86** (2010), 133–161. 1

[17] E. Lutwak, *Dual mixed volumes*, Pacific J. Math., **58** (1975), 531–538. 1, 3.1

[18] E. Lutwak, *A general Bieberbach inequality*, Math. Proc. Cambridge Philos. Soc., **78** (1975), 493–495. 1

[19] E. Lutwak, *Mixed width-integrals of convex bodies*, Israel J. Math., **28** (1977), 249–253. 1

[20] E. Lutwak, *Extended affine surface area*, Adv. Math., **85** (1991), 39–68. 2

[21] E. Lutwak, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc., **339** (1993), 901–916. 1

[22] E. Lutwak, D. Yang, G. Y. Zhang, *L_p-affine isoperimetric inequalities*, J. Differential Geom., **56** (2000), 111–132. 1

[23] E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies*, Adv. Math., **223** (2010), 220–242.

[24] E. Lutwak, D. Yang, G. Zhang, *Orlicz centroid bodies*, J. Differential Geom., **84** (2010), 365–387. 1

[25] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory, Second ed.*, Cambridge Univ. Press, Cambridge, (2014). 1, 2, 2, 2, 4

[26] F. E. Schuster, *Convolutions and multiplier transformations of convex bodies*, Trans. Amer. Math. Soc., **359** (2007), 5567–5591. 1

[27] F. E. Schuster, *Crofton measures and Minkowski valuations*, Duke Math. J., **154** (2010), 1–30.

[28] W. D. Wang, Y. B. Feng, *A general L_p-version of Petty's affine projection inequality*, Taiwan J. Math., **17** (2013), 517–528.

[29] W. D. Wang, T. Y. Ma, *Asymmetric L_p-difference bodies*, Proc. Amer. Math. Soc., **142** (2014), 2517–2527.

[30] W. D. Wang, X. Y. Wan, *Shephard type problems for general L_p-projection bodies*, Taiwan J. Math., **16** (2012), 1749–1762. 1