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Coupled coincidence point theorems for mappings without mixed monotone property under c-distance in cone metric spaces

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Abstract

Fixed point theory in the field of partially ordered metric spaces has been an area of attraction since the appearance of Ran and Reurings theorem and Nieto and Rodríguez-López theorem. One of the most significant hypotheses of these theorems was the mixed monotone property which has been avoided and replaced by the notion of invariant set in recent years and many statements have been proved using the concept of invariant set. In this paper we show that the invariant condition guides us to prove many similar results to which we were exposed to using the mixed monotone property. We present some examples in support of applicability of our results.

Keywords: fixed point, coincidence point, cone metric space, c-distance, (F, g)-invariant set.
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1. Introduction

In [14], Ran and Reurings extended the Banach contraction principle to metric spaces endowed with a partial ordering and later in [13], Nieto and López extended the result of Ran and Reurings [14] for non-decreasing mappings and applied their results to get a unique solution for a first order differential equation. The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in the Banach spaces. The concept of cone metric space was introduced in the work of Huang and Zhang [7]. Then, several authors have studied fixed point problems in cone metric spaces. For some of the work on cone metric spaces, one may refer to [1, 5, 7, 8, 19]. Bhaskar and Lakshmikantham [4] introduced the notion of a coupled fixed point of a mapping F from $X \times X$ into X. Lakshmikantham and Ćirić [11] introduced the concept of coupled coincidence points and proved coupled coincidence and coupled common fixed point results for mappings F from $X \times X$ into X and g from X into X satisfying nonlinear contraction in ordered metric space. For more study on coupled fixed point theory see [1, 6, 10, 11, 12, 15, 16, 18]. Recently Cho *et al.* [5] introduced a new concept of c-distance in cone metric spaces which is a cone version of w-distance of Kada *et al.* In [2] Batra *et al.* established coupled fixed point theorems for weak contraction mappings by using the concept of (F, g)-invariant set and c-distance in partially ordered cone metric spaces. Further, in [3] Batra *et al.* proved some coupled fixed and coincidence results using functions taking values in $[0, 1)$ as a coefficient in different contractive conditions. In this paper we use the concept of an (F,g)-invariant set and extend the results of Batra *et al.* [2, 3] as we establish the existence of coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying nonlinear contraction under c-distance in cone metric spaces having an (F, g)-invariant subset with functions taking values in $[0, 1)$ as a coefficient in different contractive conditions .

2. Preliminaries

Throughout this paper, $(X, \sqsubseteq)$ denotes a partially ordered set with partial order $\sqsubseteq$.

Definition 2.1. [4] A mapping $F : X \times X \to X$ is said to have mixed monotone property if for any $x, y \in X, x_1, x_2 \in X, x_1 \sqsubseteq x_2 \Rightarrow F(x_1, y) \sqsubseteq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \sqsubseteq y_2 \Rightarrow F(x, y_1) \sqsupseteq F(x, y_2)$.

Definition 2.2. [11] A mapping $F : X \times X \to X$ is said to have mixed g-monotone property if for any $x, y \in X, x_1, x_2 \in X, gx_1 \sqsubseteq gx_2 \Rightarrow F(x_1, y) \sqsubseteq F(x_2, y)$ and $y_1, y_2 \in X, gy_1 \sqsubseteq gy_2 \Rightarrow F(x, y_1) \sqsupseteq F(x, y_2)$.

Definition 2.3. [16] Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a non empty subset of X^4. We say that M is an F-invariant subset of X^4 if and only if for all $x, y, z, w \in X$ we have

(a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and
(b) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Definition 2.4. [2] Let (X, d) be a metric space and $F : X \times X \to X$, $g : X \to X$ be given mappings. Let M be a non empty subset of X^4. We say that M is an (F, g)-invariant ssubset of X^4 if and only if for all $x, y, z, w \in X$ we have

(a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and
(b) $(gx, gy, gz, gw) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

We observe that

1. The set $M = X^4$ is trivially (F, g)-invariant.
2. Every F-invariant set is (F, I_X)-invariant. Here I_X denotes identity map on X.

Following example shows that we may have (F, g)-invariant set which is not F-invariant.

Example 2.5. [2] Let $X = \mathbb{R}$ and $F : X \times X \to X$ be defined by $F(x, y) = 1 - x^2$. Let $g : X \to X$ be given by $gx = 1 + x$. Then $M = \{(x, y, z, w) \in X^4 : y = z = 0\}$ is not F-invariant as $(1, 0, 0, 1) \in M$ but $(F(1, 0), F(0, 1), F(0, 1), F(1, 0)) = (0, 1, 1, 0)$ does not belong to M. It is easy to see that M is (F, g)-invariant.

Example 2.6. [2] Let (X, d) be a cone metric space endowed with a partial order $\sqsubseteq$. Let $F : X \times X \to X$ and $g : X \to X$ be any two mappings such that F satisfies mixed g-monotone property. Define a subset M of X^4 by $M = \{(a, b, c, d) : c \sqsubseteq a, b \sqsubseteq d\}$. Then M is (F, g)-invariant.

Definition 2.7. [4] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.8. [11] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.9. [11] Let $F : X \times X \to X$ and $g : X \to X$. The mappings F and g are said to commute if $gF(x, y) = F(gx, gy)$ for all x, $y \in X$.

In [7], cone metric space was introduced in the following manner: Let $(E, \|.\|)$ be a real Banach space and θ denote the zero element in E. Assume that P is a subset of E. Then P is called a cone if and only if

(i) P is non empty, closed and $P \neq \{\theta\}$,

(ii) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subseteq E$ and $x, y \in E$, the partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in intP$. It can be easily shown that $\lambda.intP \subseteq intP$ for all $\lambda > 0$ and $intP + intP \subseteq intP$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying above is called the normal constant of P. In the following we always suppose E is a real Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.10. [7] Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following condition:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.11. [7] Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

1. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$ then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.
2. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X.
3. A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.12. *[7] Let (X, d) be a cone metric space, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then*

(i) *the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $\|d(x_n, x)\| \to 0$),*

(ii) *the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\|d(x_n, x_m)\| \to 0$).*

(iii) *the sequence* $\{x_n\}$ *(respectively,* $\{y_n\}$*) converges to* x *(respectively,* y*) then* $d(x_n, y_n) \to d(x, y)$.

Lemma 2.13. *[19] Every cone metric space* (X, d) *is a topological space. For* $c \gg 0,\ c \in E,\ x \in X$, *let* $B(x, c) = \{y \in X : d(y, x) \ll c\}$ *and* $\beta = \{B(x, c) : x \in X,\ c \gg 0\}$. *Then* $\tau_c = \{U \subseteq X :$ *for all* $x \in U$, *there exists* $B_x \in \beta$, *with* $x \in B_x \subseteq U\}$ *is a topology on* X.

Definition 2.14. [19] Let (X, d) be a cone metric space. A map $T : (X, d) \to (X, d)$ is called sequentially continuous if $x_n \in X$, $x_n \to x$ implies $Tx_n \to Tx$.

Lemma 2.15. *[19] Let* (X, d) *be a cone metric space, and* $T : (X, d) \to (X, d)$ *be any map. Then,* T *is continuous if and only if* T *is sequentially continuous.*

Let (X, d) be a cone metric space and $X^2 = X \times X$. Define a function $\rho : X^2 \times X^2 \to E$ by $\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ for all (x_1, y_1) and $(x_2, y_2) \in X^2$.Then (X^2, ρ) is a cone metric space [10].

Lemma 2.16. *[10] Let* $z_n = (x_n, y_n) \in X^2$ *be a sequence and* $z = (x, y) \in X^2$. *Then* $z_n \to z$ *if and only if* $x_n \to x$ *and* $y_n \to y$.

Next we give the notation of c-distance on a cone metric space which is generalization of w-distance of Kada et. al. [9] with some properties.

Definition 2.17. [5] Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
(q3) for each $x \in X$ and $n \in \mathbb{N}$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
(q4) For all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.18. The c-distance q is a w-distance on X if we let (X, d) be a metric space, $E = \mathbb{R}$, $P = [0, \infty)$ and q3 is replaced by the following condition: for any $x \in X$, $q(x, .) : X \to \mathbb{R}$ is lower semi continuous. Moreover, q3 holds whenever $q(x, .)$ is lower semi-continuous. Thus, if (X, d) is a metric space, $E = \mathbb{R}$, and $P = [0, \infty)$, then every w-distance is a c-distance. But the converse is not true in the general case. Therefore, the c-distance is a generalization of the w-distance.

Example 2.19. [18] Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = \|x - y\|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Example 2.20. [18] Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to P$ by $q(x, y) = d(x, y)$ for all $x, y \in X$. Then, q is c-distance.

Example 2.21. [18] Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0,\ t \in [0, 1]\}$. Let $X = [0, +\infty)$(with usual order), and $d(x, y)(t) = \|x - y\|\varphi(t)$ where $\varphi : [0, 1] \to \mathbb{R}$ is given by $\varphi(t) = e^t$ for all $t \in [0, 1]$. Then (X, d) is an ordered cone metric space(see [5] Example 2.9). This cone is not normal. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\varphi$ for all $x, y \in X$. Then q is a c-distance.

Example 2.22. [18] Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to P$ by $q(x, y) = d(u, y)$ for all $x, y \in X$, where u is a fixed point in X. Then q is a c-distance.

Lemma 2.23. *[5] Let* (X, d) *be a cone metric space and* q *be a* c*-distance on* X. *Let*$\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *and* $y, z \in X$. *Suppose that* u_n *is a sequence in* P *converging to* θ. *Then the following hold:*

1. *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*
2. *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then y_n converges to z.*
3. *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
4. *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.24. *[17] Let (X, d) be a cone metric space, and let q be a c-distance on X. Let $\{x_n\}$ be a sequence in X. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in P converging to θ. If $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$, then $y = z$.*

Remark 2.25. [5]

(i) $q(x, y) = q(y, x)$ may not be true for all $x, y \in X$.
(ii) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main Results

Theorem 3.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) be invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let $k : X \times X \to [0, 1)$ be any given function such that*

(i) *$k(F(x, y), F(y, x)) \leq k(gx, gy)$ for all $x, y \in X$ and*
(ii) *$q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$.*

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Choose $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$. Since $F(X \times X) \subseteq g(X)$, one can find $x_1, y_1 \in X$ in a way that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Repeating the same argument one can find $x_2, y_2 \in X$ in a way that $gx_2 = F(x_1, y_1)$ and $F(y_1, x_1) = gy_2$. Continuing this process one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X that satisfy $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. It is asserted that

$$(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M \tag{3.1}$$

for all $n \geq 0$. For $n = 0$, (3.1) follows by the choice of x_0 and y_0. Let us assume that (3.1) holds good for $n = k$, $k \geq 0$. So we have $(gx_{k+1}, gy_{k+1}, gx_k, gy_k) \in M$. (F, g) invariance of M now implies that

$$(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1}), F(x_k, y_k), F(y_k, x_k)) \in M$$

That is, $(gx_{k+2}, gy_{k+2}, gx_{k+1}, gy_{k+1}) \in M$. Thus (3.1) follows for $k + 1$. Hence, by induction, our assertion follows. Now for all $n \in \mathbb{N}$

$$\begin{aligned}
&q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\
&= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\preceq k(gx_{n-1}, gy_{n-1})(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&\preceq k(gx_{n-2}, gy_{n-2})(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \\
&\vdots
\end{aligned}$$

$$\preceq \quad k(gx_0, gy_0)(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n))$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$ and $k = k(gx_0, gy_0)$. Then, we have

$$\begin{aligned} & q_n \\ = \quad & q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\ \preceq \quad & kq_{n-1} \\ \preceq \quad & \dots \\ \preceq \quad & k^n q_0 \end{aligned}$$

Let $m > n \geq 1$. It follows that

$$q(gx_n, gx_m) \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m) \text{ and}$$
$$q(gy_n, gy_m) \preceq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \ldots + q(gy_{m-1}, gy_m).$$

Then we have

$$\begin{aligned} q(gx_n, gx_m) + q(gy_n, gy_m) &\preceq q_n + q_{n+1} + \ldots + q_{m-1} \\ &\preceq k^n q_0 + k^{n+1} q_0 + \ldots + k^{m-1} q_0 \preceq \frac{k^n}{1-k} q_0 \end{aligned} \tag{3.2}$$

From (3.2) we have

$$q(gx_n, gx_m) \preceq \frac{k^n}{1-k} q_0 \tag{3.3}$$

and also

$$q(gy_n, gy_m) \preceq \frac{k^n}{1-k} q_0 \tag{3.4}$$

Thus, Lemma 2.23(3) shows that gx_n and gy_n are Cauchy sequences in X. Since X is complete, there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get $\lim_{n\to\infty} ggx_n = gx^*$ and $\lim_{n\to\infty} ggy_n = gy^*$. Commutativity of F and g now implies that $ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1})$ for all $n \in \mathbb{N}$ and $ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1})$ for all $n \in N$. Since F is continuous, therefore,

$$gx^* = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) = F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) = F(x^*, y^*),$$

$$gy^* = \lim_{n\to\infty} ggy_n = \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) = F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) = F(y^*, x^*)$$

Thus (x^*, y^*) is a coupled coincidence point of F and g. □

Corollary 3.2. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Let $F : X \times X \to X$ be a continuous and $k : X \times X \to [0, 1)$ be such that*

(i) *$k(F(x, y), F(y, x)) \leq k(x, y)$ for all $x, y \in X$ and*
(ii) *$q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(x, y)(q(x, u) + q(y, v))$ for all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$.*

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0, F(y_0, x_0), x_0, y_0)) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$, the identity function on X in Theorem 3.1. □

Theorem 3.3. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let $k : X \times X \to [0, 1)$ be any given function such that*

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ *for all* $x, y \in X$
(ii) $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ *for all* $x, y, u, v \in X$ *with* $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$ *and*
(iii) *Whenever* $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ *or* $(x_n, y_n, x_{n+1}, y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$, *then* $(x_n, y_n, x, y) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ inM, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Consider Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ as in the proof of Theorem 3.1. Since $(g(X), d)$ is complete, there exists $x^*, y^* \in X$ such that $gx_n \to gx^*$ and $gy_n \to gy^*$. By q3, (3.3) and (3.4) we have for all $n \geq 0$,

$$q(gx_n, gx^*) \preceq \frac{k^n}{1-k} q_0, \tag{3.5}$$

$$q(gy_n, gy^*) \preceq \frac{k^n}{1-k} q_0 \tag{3.6}$$

Adding (3.5) and (3.6) we get $q(gx_n, gx^*) + q(gy_n, gy^*) \preceq \frac{2k^n}{1-k} q_0$ for all $n \geq 0$. Since $(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M$ for all $n \geq 0$ and $gx_n \to gx$, $gy_n \to gy$, we have $(gx_{n+1}, gy_{n+1}, gx, gy) \in M$ for all $n \geq 0$. Thus for all $n \in \mathbb{N}$

$$\begin{aligned}
& q(gx_n, F(x^*, y^*)) + q(gy_n, F(y^*, x^*)) \\
& = q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) + q(F(y_{n-1}, x_{n-1}), F(y^*, x^*)) \\
& \preceq k(gx_{n-1}, gy_{n-1})[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
& = k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
& \preceq k(gx_{n-2}, gy_{n-2})[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
& \vdots \\
& \preceq k(gx_0, gy_0)[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
& = k[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\
& \preceq k \frac{2k^{n-1}}{1-k} q_0 = \frac{2k^n}{1-k} q_0
\end{aligned}$$

Then

$$q(gx_n, F(x^*, y^*)) \preceq \frac{2k^n}{1-k} q_0 \tag{3.7}$$

and

$$q(gy_n, F(y^*, x^*)) \preceq \frac{2k^n}{1-k} q_0 \tag{3.8}$$

By Lemma 2.24, (3.5) and (3.7), we have $F(x^*, y^*) = gx^*$. Similarly, by Lemma 2.24, (3.6) and (3.8) we have $F(y^*, x^*) = gy^*$. Thus (x^*, y^*) is a coupled coincidence point of F and g. □

Corollary 3.4. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Let $k : X \times X \to [0, 1)$ be any given function such that*

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ *for all* $x, y \in X$

(ii) $q(F(x, y), F(u, v))+q(F(y, x), F(v, u)) \preceq k(x, y)(q(x, u)+q(y, v))$ *for all* $x, y, u, v \in X$ *with* $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$ *and*

(iii) *Whenever* $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ *or* $(x_n, y_n, x_{n+1}, y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$*, then* $(x_n, y_n, x, y) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ inM, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$, the identity map on X in Theorem 3.3. □

Theorem 3.5. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let $k,\ l : X \times X \to [0, 1)$ be any given functions such that*

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ *and* $l(F(x, y), F(y, x)) \leq l(gx, gy)$ *for all* $x, y \in X$,

(ii) $k(x, y) = k(y, x)$ *and* $l(x, y) = l(y, x)$ *for all* $x, y \in X$,

(iii) $(k + l)(x, y) < 1$ *for all* $x, y \in X$ *and*

(iv) $q(F(x, y), F(u, v)) \preceq k(gx, gy)q(gx, gu)+l(gx, gy)q(gy, gv)$ *for all* $x,\ y,\ u,\ v \in X$ *with* $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Given $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Then we have

$$\begin{aligned} q(F(x, y), F(u, v)) &\preceq k(gx, gy)q(gx, gu) + l(gx, gy)q(gy, gv)), \\ q(F(y, x), F(v, u)) &\preceq k(gy, gx)q(gy, gv) + l(gy, gx)q(gx, gu) \\ &= k(gx, gy)q(gy, gv) + l(gx, gy)q(gx, gu) \end{aligned}$$

Thus $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u))$
$\preceq (k + l)(gx, gy)(q(gx, gu) + q(gy, gv))$ where $(k + l) : X \times X \to [0, 1)$ satisfies

$$\begin{aligned} (k + l)(F(x, y), F(y, x)) &= k(F(x, y), F(y, x)) + l(F(x, y), F(y, x)) \\ &\leq k(gx, gy) + l(gx, gy) = (k + l)(gx, gy) \end{aligned}$$

for all $x, y \in X$. Result follows by Theorem 3.1. □

Corollary 3.6. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Let $k,\ l : X \times X \to [0, 1)$ be any given functions such that*

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ *and* $l(F(x, y), F(y, x)) \leq l(x, y)$ *for all* $x, y \in X$,

(ii) $k(x, y) = k(y, x)$ *and* $l(x, y) = l(y, x)$ *for all* $x, y \in X$,

(iii) $(k + l)(x, y) < 1$ *for all* $x, y \in X$ *and*

(iv) $q(F(x, y), F(u, v)) \preceq k(x, y)q(x, u) + l(x, y)q(y, v)$ *for all* $x, y, u, v \in X$ $x, y, u, v \in X$ *with* $(x, y, u, v) \in M$ *or* $(u, v, x, y) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. *Take $g = I_X$ the identity function on X in Theorem 3.5.*

Corollary 3.7. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let $k : X \times X \to [0, \frac{1}{2})$ be any given function such that*

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ *for all* $x, y \in X$,
(ii) $k(x, y) = k(y, x)$ *for all* $x, y \in X$ *and*
(iii) $q(F(x, y), F(u, v)) \preceq k(gx, gy)(q(gx, gu) + q(gy, gv))$ *for all* $x, y, u, v \in X$ *with* $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ inM, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Take $k(x, y) = l(x, y)$ in Theorem 3.5. □

Corollary 3.8. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4.Let $k : X \times X \to [0, \frac{1}{2})$ be any given functions such that*

(i) $k(F(x, y), F(y, x)) \leq k(x, y)$ *for all* $x, y \in X$,
(ii) $k(x, y) = k(y, x)$ *for all* $x, y \in X$ *and*
(iii) $q(F(x, y), F(u, v)) \preceq k(x, y)(q(x, u) + q(y, v))$ *for all* $x, y, u, v \in X$ *with* $x, y, u, v \in X$ *with* $(x, y, u, v) \in M$ *or* $(u, v, x, y) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ inM, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $k(x, y) = l(x, y)$ and $g = I_X$ in Theorem 3.5. □

Theorem 3.9. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let $k,\ l : X \times X \to [0, 1)$ be any given functions such that*

(i) $k(F(x, y), F(y, x)) \leq k(gx, gy)$ *and* $l(F(x, y), F(y, x)) \leq l(gx, gy)$ *for all* $x, y \in X$,
(ii) $k(x, y) = k(y, x)$ *and* $l(x, y) = l(y, x)$ *for all* $x, y \in X$,
(iii) $(k + l)(x, y) < 1$ *for all* $x, y \in X$
(iv) $q(F(x, y), F(u, v)) \preceq k(gx, gy)q(gx, gu) + l(gx, gy)q(gy, gv)$ *for all* $x, y, u, v \in X$ *with* $x, y, u, v \in X$ *with* $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$ *and*
(v) *Whenever* $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ *or* $(x_n, y_n, x_{n+1}, y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$, *then* $(x_n, y_n, x, y) \in M$. $(gx, gy, gu, gv) \in M$ *or* $(gu, gv, gx, gy) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. It follows form Theorem 3.3 by the similar argument to those given in the proof of Theorem 3.5. □

Corollary 3.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on Xand M be an (F, g) invariant subset of X^4. Let $k,\ l : X \times X \to [0, 1)$ be any given functions such that*

(i) $k(F(x,y),F(y,x)) \leq k(x,y)$ *and* $l(F(x,y),F(y,x)) \leq l(x,y)$ *for all* $x,y \in X$,
(ii) $k(x,y) = k(y,x)$ *and* $l(x,y) = l(y,x)$ *for all* $x,y \in X$,
(iii) $(k+l)(x,y) < 1$ *for all* $x,y \in X$
(iv) $q(F(x,y),F(u,v)) \preceq k(x,y)q(x,u) + l(x,y)q(y,v)$ *for all* $x,y,u,v \in X$ *with* $x,y,u,v \in X$ *with* $(gx,gy,gu,gv) \in M$ *or* $(gu,gv,gx,gy) \in M$ *and*
(v) *Whenever* $(x_{n+1},y_{n+1},x_n,y_n) \in M$ *or* $(x_n,y_n,x_{n+1},y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$, *then* $(x_n,y_n,x,y) \in M$.

If there exist $x_0,y_0 \in X$ *satisfying* $(F(x_0,y_0),F(y_0,x_0),x_0,y_0)$ *in* M, *then there exist* $x^*,y^* \in X$ *such that* $F(x^*,y^*) = x^*$ *and* $F(y^*,x^*) = y^*$, *that is,* F *has a coupled fixed point* (x^*,y^*).

Proof. Take $g = I_X$ the identity function on X in Theorem 3.9. □

Corollary 3.11. *Let* $(X,\sqsubseteq)$ *be a partially ordered set and suppose that* (X,d) *is a cone metric space. Let* q *be a c-distance on* X *and* M *be an* (F,g) *invariant subset of* X^4. *Suppose* $F : X\times X \to X$ *and* $g : X \to X$ *be two functions such that* $F(X\times X) \subseteq g(X)$ *and* $(g(X),d)$ *is complete subspace of* X. *Let* $k : X\times X \to [0,\frac{1}{2})$ *be any given functions such that*

(i) $k(F(x,y),F(y,x)) \leq k(gx,gy)$ *for all* $x,y \in X$,
(ii) $k(x,y) = k(y,x)$ *for all* $x,y \in X$ *and*
(iii) $q(F(x,y),F(u,v)) \preceq k(gx,gy)(q(gx,gu) + q(gy,gv))$ *for all* $x,y,u,v \in X$ *with* $(gx,gy,gu,gv) \in M$ *or* $(gu,gv,gx,gy) \in M$ *and*
(iv) *Whenever* $(x_{n+1},y_{n+1},x_n,y_n) \in M$ *or* $(x_n,y_n,x_{n+1},y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$, *then* $(x_n,y_n,x,y) \in M$.

If there exist $x_0,y_0 \in X$ *satisfying* $(F(x_0,y_0),F(y_0,x_0),x_0,y_0)$ *in* M, *then there exist* $x^*,y^* \in X$ *such that* $F(x^*,y^*) = gx^*$ *and* $F(y^*,x^*) = gy^*$, *that is,* F *and* g *have a coupled coincidence point* (x^*,y^*).

Proof. Take $k(x,y) = l(x,y)$ in Theorem 3.9. □

Corollary 3.12. *Let* $(X,\sqsubseteq)$ *be a partially ordered set and suppose that* (X,d) *is a complete cone metric space. Let* q *be a c-distance on* X *and* M *be an* (F,g) *invariant subset of* X^4. *Let* $k : X\times X \to [0,\frac{1}{2})$ *be any given functions such that*

(i) $k(F(x,y),F(y,x)) \leq k(x,y)$ *for all* $x,y \in X$,
(ii) $k(x,y) = k(y,x)$ *for all* $x,y \in X$ *and*
(iii) $q(F(x,y),F(u,v)) \preceq k(x,y)(q(x,u) + q(y,v))$ *for all* $x,y,u,v \in X$ *with* $(gx,gy,gu,gv) \in M$ *or* $(gu,gv,gx,gy) \in M$ *and*
(iv) *Whenever* $(x_{n+1},y_{n+1},x_n,y_n) \in M$ *or* $(x_n,y_n,x_{n+1},y_{n+1}) \in M$ *and* $\{x_n\} \to x, \{y_n\} \to y$, *then* $(x_n,y_n,x,y) \in M$.

If there exist $x_0,y_0 \in X$ *satisfying* $(F(x_0,y_0),F(y_0,x_0),x_0,y_0)$ *in* M, *then there exist* $x^*,y^* \in X$ *such that* $F(x^*,y^*) = x^*$ *and* $F(y^*,x^*) = y^*$, *that is,* F *has a coupled fixed point* (x^*,y^*).

Proof. Take $k(x,y) = l(x,y)$ and $g = I_X$ in Theorem 3.9. □

Theorem 3.13. *Let* $(X,\sqsubseteq)$ *be a partially ordered set and suppose that* (X,d) *is a complete cone metric space. Let* q *be a c-distance on* X *and* M *be an* (F,g) *invariant subset of* X^4. *Suppose* $F : X\times X \to X$ *and* $g : X \to X$ *be two continuous and commuting functions with* $F(X\times X) \subseteq g(X)$. *Let* $k,l : X\times X \to [0,1)$ *be any given functions such that*

(i) $k(F(x,y),F(y,x)) \leq k(gx,gy)$ *and* $l(F(x,y),F(y,x)) \leq l(gx,gy)$ *for all* $x,y \in X$,
(ii) $(k+l)(x,y) < 1$ *for all* $x,y \in X$ *and*
(iii) $q(F(x,y),F(u,v)) \preceq k(gx,gy)q(gx,F(x,y)) + l(gx,gy)q(gu,F(u,v))$ *for all* $x,y,u,v \in X$ *with* $(gx,gy,gu,gv) \in M$ *or* $(gu,gv,gx,gy) \in M$.

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. By the similar argument as in Theorem 3.1 we can find the sequences $\{gx_n\}$ and $\{gy_n\}$ satisfying (3.1). Now for all $n \in \mathbb{N}$

$$\begin{aligned}
& q(gx_n, gx_{n+1}) \\
= \; & q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
\preceq \; & k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, F(x_{n-1}, y_{n-1})) + l(gx_{n-1}, gy_{n-1})q(gx_n, F(x_n, y_n)) \\
= \; & k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_n) + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_n, gx_{n+1}) \\
\preceq \; & k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) + l(gx_{n-2}, gy_{n-2})q(gx_n, gx_{n+1}) \\
\vdots \; & \\
\preceq \; & k(gx_0, gy_0)q(gx_{n-1}, gx_n) + l(gx_0, gy_0)q(gx_n, gx_{n+1})
\end{aligned}$$

Put $q_n = q(gx_n, gx_{n+1})$ and $d = \frac{k(gx_0, gy_0)}{1 - l(gx_0, gy_0)}$. Then $d \in [0, 1)$ and we have

$$q_n = q(gx_n, gx_{n+1}) \preceq dq_{n-1} \preceq \ldots \preceq d^n q_0$$

$$\begin{aligned}
\textit{Also} \quad & q(gy_n, gy_{n+1}) \\
& = q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
& \preceq k(gy_{n-1}, gx_{n-1})q(gy_{n-1}, F(y_{n-1}, x_{n-1})) + l(gy_{n-1}, gx_{n-1})q(gy_n, F(y_n, x_n)) \\
& = k(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gy_{n-1}, gy_n) + l(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gy_n, gy_{n+1}) \\
& \preceq k(gy_{n-2}, gx_{n-2})q(gy_{n-1}, gy_n) + l(gy_{n-2}, gx_{n-2})q(gy_n, gy_{n+1}) \\
& \vdots \\
& \preceq k(gy_0, gx_0)q(gy_{n-1}, gy_n) + l(gy_0, gx_0)q(gy_n, gy_{n+1})
\end{aligned}$$

Put $r_n = q(gy_n, gy_{n+1})$ and $h = \frac{k(gy_0, gx_0)}{1 - l(gy_0, gx_0)}$. Then $h \in [0, 1)$ and we have

$$r_n = q(gy_n, gy_{n+1}) \preceq hr_{n-1} \preceq \ldots \preceq h^n r_0$$

Let $m > n \geq 1$. It follows that

$$\begin{aligned}
q(gx_n, gx_m) & \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m) \\
& = q_n + q_{n+1} + \ldots + q_{m-1} \\
& \preceq d^n q_0 + d^{n+1} q_0 + \ldots + d^{m-1} q_0 \\
& \preceq \frac{d^n}{1 - d} q_0
\end{aligned}$$

$$\begin{aligned}
\textit{Also} \quad q(gy_n, gy_m) & \preceq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \ldots + q(gy_{m-1}, gy_m) \\
& = r_n + r_{n+1} + \ldots + r_{m-1} \\
& \preceq h^n r_0 + h^{n+1} r_0 + \ldots + h^{m-1} r_0 \\
& \preceq \frac{h^n}{1 - h} r_0
\end{aligned}$$

Thus, Lemma 2.23(3) shows that gx_n and gy_n are Cauchy sequences in X. since X is complete, there exists there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get $\lim_{n\to\infty} ggx_n = gx^*$ and $\lim_{n\to\infty} ggy_n = gy^*$. Commutativity of F and g now implies that $ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1})$ for all $n \in \mathbb{N}$ and $ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1})$ for all $n \in \mathbb{N}$. Since F is continuous, therefore,

$$gx^* = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) = F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) = F(x^*, y^*),$$

$$gy^* = \lim_{n\to\infty} ggy_n = \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) = F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) = F(y^*, x^*)$$

Thus (x^*, y^*) is a coupled coincidence point of F and g. □

Corollary 3.14. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let $k, l \in [0, 1)$ be any given numbers such that $k + l < 1$ and*

$$q(F(x, y), F(u, v)) \preceq kq(gx, F(x, y)) + lq(gu, F(u, v))$$

for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Take $k(x, y) = k$ and $l(x, y) = l$ in Theorem 3.13. □

Corollary 3.15. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4.Let $k, l : X \times X \to [0, 1)$ be any given functions such that*

(i) *$k(F(x, y), F(y, x)) \leq k(x, y)$ and $l(F(x, y), F(y, x)) \leq l(x, y)$ for all $x, y, \in X$,*
(ii) *$(k + l)(x, y) < 1$ for all $x, y \in X$ and*
(iii) *$q(F(x, y), F(u, v)) \preceq k(x, y)q(x, F(x, y)) + l(x, y)q(u, F(u, v))$ for all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$.*

If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $g = I_X$ in Theorem 3.13. □

Corollary 3.16. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and M be an (F, g) invariant subset of X^4. Let $k, l \in [0, 1)$ be any given numbers such that $k + l < 1$ and*

$$q(F(x, y), F(u, v)) \preceq kq(x, F(x, y)) + lq(u, F(u, v))$$

for all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0)$ in M,then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Proof. Take $k(x, y) = k, l(x, y) = l$ and $g = I_X$ in Theorem 3.13. □

Theorem 3.17. *Under the hypothesis of any one of the theorems from Theorem 3.1, Theorem 3.3, Theorem 3.5, Theorem 3.9 and Theorem 3.13 or any one of the corollaries 3.7, 3.11, and 3.14 we have $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$ where (x^*, y^*) is a coincidence point of F and g.*

Proof. We prove this theorem under the hypothesis of Theorem 3.1. Proofs are similar for other theorems or corollaries and can be obtained by a little adjustment. We have

$$\begin{aligned} q(gx^*, gx^*) + q(gy^*, gy^*) &= q(F(x^*, y^*), F(x^*, y^*) + q(F(y^*, x^*), F(y^*, x^*)) \\ &\preceq k(x^*, y^*)(q(gx^*, gx^*) + q(gy^*, gy^*)) \end{aligned}$$

Since $0 \leq k(x^*, y^*) < 1$, we have $q(gx^*, gx^*) + q(gy^*, gy^*) = \theta$. But $q(gx^*, gx^*) \geq \theta$ and $q(gy^*, gy^*) \geq \theta$, hence $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$. □

Corollary 3.18. *Under the hypothesis of any one of the corollaries 3.2, 3.4, 3.6, 3.8, 3.10, 3.12, 3.15 and 3.16 if $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$ for all $x, y \in X$, then we have $q(x^*, x^*) = \theta$ and $q(y^*, y^*) = \theta$ where (x^*, y^*) is a coupled fixed point of F.*

Proof. Similar to Theorem 3.17 once we work with $g = I_X$. □

Theorem 3.19. *In addition to the hypothesis of any one of the theorems from Theorem 3.1, Theorem 3.3, Theorem 3.5, Theorem 3.9 and Theorem 3.13 or any one of the corollaries 3.7, 3.11, and 3.14 suppose that any two elements x and y of X satisfy $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$ and g is one-one. Then there exists a coupled coincidence point of F and g which is of the form (x^*, x^*) for some $x^* \in X$.*

Proof. Again we prove this theorem under the hypothesis of Theorem 3.1. Proofs are similar for other theorems or corollaries and can be obtained by a little adjustment. Consider coupled coincidence point (x^*, y^*) of F and g. Since $(gx^*, gy^*, gy^*, gx^*) \in M$ or $(gy^*, gx^*, gx^*, gy^*) \in M$, therefore, we have

$$\begin{aligned} q(gx^*, gy^*) + q(gy^*, gx^*) &= q(F(x^*, y^*), F(y^*, x^*) + q(F(y^*, x^*), F(x^*, y^*)) \\ &\preceq k(x^*, y^*)(q(gx^*, gy^*) + q(gy^*, gx^*)) \end{aligned}$$

Since $0 \leq k(x^*, y^*) < 1$,we have $q(gx^*, gy^*) + q(gy^*, gx^*) = \theta$.But $q(gx^*, gy^*) \geq \theta$ and $q(gy^*, gx^*) \geq \theta$, hence $q(gx^*, gy^*) = \theta$ and $q(gy^*, gx^*) = \theta$. Let $u_n = \theta, x_n = gx^*$ for all $n \geq 0$, then we have $q(x_n, gx^*) \preceq u_n$ for all $n \geq 0$ and $q(x_n, gy^*) \preceq u_n$ for all $n \geq 0$. By Lemma 2.23(1) we have $gx^* = gy^*$. Since g is one-one, therefore,$x^* = y^*$. Thus there exists a coupled coincidence point of the form (x^*, x^*) for some $x^* \in X$. This completes the proof. □

Corollary 3.20. *In addition to hypothesis of any one of the corollaries 3.2, 3.4, 3.6, 3.8, 3.10, 3.12, 3.15 and 3.16, suppose that any two elements of X are comparable. Then there exists a coupled fixed point of F which is of the form (x^*, x^*) for some $x^* \in X$.*

Proof. Similar to Theorem 3.19 once we work with $g = I_X$. □

Example 3.21. Let $E = \mathbb{R}$ and

$$P = \{x \in E : x \geq 0\}.$$

Let $X = [0, 1]$(with usual order) and $d(x, y) = |x - y|$. Then (X, d) is an ordered complete cone metric space. Further, define a subset M of X^4 by

$$M = \{(a, b, c, d) : c \sqsubseteq a, b \sqsubseteq d\}.$$

Then M is (F, g)-invariant. Also let $q : X \times X \to E$ be defined by $q(x, y) = 2d(x, y)$. It is easy to check that q is a c-distance on X. Consider now the function defined by $F(x, y) = x^2/16$ for all $x, y \in X$, $k(x, y) = \frac{1+x+y}{16}$ for all $x, y \in X$ and $gx = x$ for all $x \in X$. Then $F(X \times X) \subseteq g(X)$ and

$$k(F(x, y), F(y, x)) = \frac{1 + \frac{x^2}{16} + \frac{y^2}{16}}{16} \leq \frac{1 + x^2 + y^2}{16} \leq \frac{1 + x + y}{16} = k(gx, gy)$$

for all $x, y \in X$. Further $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) = 2|\frac{x^2}{16} - \frac{u^2}{16}| + 2|\frac{y^2}{16} - \frac{v^2}{16}| = \frac{1}{8}(x+u)|x-u| + \frac{1}{8}(y+v)|y-v| \leq \frac{x+1}{16}.2|x-u| + \frac{y+1}{16}.2|y-v| \leq \frac{1+x+y}{16}.2|x-u| + \frac{1+x+y}{16}.2|y-v| = k(gx, gy)(q(x, u) + q(y, v))$ for all $x, y, u, v \in X$. Further F and g are continuous, commuting and $(F(0, 1), F(1, 0), g(0), g(1)) \in M$. Thus, by Theorem 3.1 , F and g have a coincidence point. Here F and g have a coincidence point at $(0, 0)$.

Example 3.22. Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Let $X = [0,+\infty)$(with usual order), and $d(x,y)(t) = \|x-y\|\, e^t$. Then (X,d) is an ordered cone metric space(see [5] Example 2.9). Further, let $q : X \times X \to E$ be defined by $q(x,y)(t) = y\, e^t$. It is easy to check that q is a c-distance on X. Consider now the function defined by

$$F(x,y) = \begin{cases} \frac{1}{7}(x+y) & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

and $g(x) = \frac{3}{2}x$ for all x. Then $F(X \times X) \subseteq g(X) = X$ and $(g(X),d) = (X,d)$ is complete. Let $k(x,y) = 1/3$ for all $x, y \in X$. Then we have $k(F(x,y),F(y,x)) \leq k(gx,gy)$ for all $x, y \in X$. For $y_1 = 2$ and $y_2 = 3$ we have $gy_1 \sqsubseteq gy_2$ but $F(x,y_1) \sqsubseteq F(x,y_2)$ for all $x > 3$. So F does not satisfy mixed g-monotone property. Hence similar result for mixed g-monotonic function in [3] can not be applied to this example. Also it can be seen easily that $q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \preceq k(x,y)(q(gx,gu)+q(gy,gv))$ for all $(x,y,u,v) \in X^4 = M$. It is easy to see that all other conditions of Theorem 3.3 are satisfied for $M = X^4$. Thus, by Theorem 3.3, F and g have a coincidence point. Here F and g have a unique coincidence point at $(0,0)$.

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