Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 6 (2013), 279–284
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point theorems for cyclic weak contractions in compact metric spaces

Jackie Harjani, Belén López, Kishin Sadarangani*

Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain.

Abstract

The purpose of this paper is to present a fixed point theorem for cyclic weak contractions in compact metric spaces.

Keywords: Fixed point, weak contraction, cyclic representation.
2010 MSC: 47H10.

1. Introduction and Preliminaries

Alber and Guerre-Delabriere in [1] define weakly contractive mappings and they prove some fixed point theorems in the context of Hilbert spaces. In [5] Rhoades extends some results of [1] to complete metric spaces.

Recently, E. Karapinar in [3] proves a fixed point theorem for an operator T on a complete metric space X when X has a cyclic representation with respect to T.

Firstly, we present some definitions.

Definition 1.1. Let X be a nonempty set, m a positive integer and $T : X \longrightarrow X$ an operator.
$X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of X with respect to T if

(i) A_i, $i = 1, 2, \ldots, m$ are nonempty subsets of X.

(ii) $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset A_m, \quad T(A_m) \subset A_1$.

*Corresponding author
Email addresses: `jharjani@dma.ulpgc.es` (Jackie Harjani), `blopez@dma.ulpgc.es` (Belén López), `ksadaran@dma.ulpgc.es` (Kishin Sadarangani)

Received 2012-3-5

In [3] the author uses the class of functions $\mathfrak{J}$ given by

$$\mathfrak{J} = \{\phi : [0,\infty) \longrightarrow [0,\infty) : \quad \text{continuous, nondecreasing} \quad \phi(t) > 0 \ \text{ for } \ t \in (0,\infty), \phi(0) = 0\}.$$

Examples of functions in $\mathfrak{J}$ are $\phi(t) = \lambda t$ with $\lambda > 0$; $\phi(t) = \ln(1+t)$; $\phi(t) = \arctan x$.

We use in this paper the class of functions $\mathfrak{F}$ given by

$$\mathfrak{F} = \{\varphi : [0,\infty) \longrightarrow [0,\infty) : \quad \text{nondecreasing}, \quad \varphi(t) > 0 \ \text{ for } \ t \in (0,\infty) \quad \varphi(0) = 0\}.$$

Obviously, $\mathfrak{J} \subset \mathfrak{F}$.

The function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ given by

$$\varphi(t) = \begin{cases} t & \text{for } \ t \in [0,1] \\ 2t & \text{for } \ t \in (1,\infty) \end{cases}$$

belongs to $\mathfrak{F}$ but it is not an element of $\mathfrak{J}$.

The following definition appears in [3] (Definition 2).

Definition 1.2. Let (X,d) be a metric space, m a positive integer, $A_1, A_2, \ldots, A_m$ closed non-empty subsets of X and $X = \bigcup_{i=1}^m A_i$. An operator $T : X \longrightarrow X$ is called a cyclic weak contraction if

(*i*) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T.

(*ii*) $d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$ for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1,2,\ldots,m$, where $A_{m+1} = A_1$ and $\phi \in \mathfrak{J}$.

The main result in [3] is the following.

Theorem 1.3. *(Theorem 6 of [3]). Let (X,d) be a complete metric space, m a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $X = \bigcup_{i=1}^m A_i$. Let $T : X \longrightarrow X$ be an operator such that*

(*a*) *$X = \bigcup_{i}^m A_i$ is a cyclic representation of X with respect to T.*

(*b*) *T is a cyclic weak contraction for certain $\phi \in \mathfrak{J}$.*

Then T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Remark 1.4. If we look at the proof of Theorem 1 in [3], the author starts with a point $x_0 \in X$ and considers the Picard iteration $x_{n+1} = Tx_n$. He proves that (x_n) is a Cauchy sequence and, therefore, $\lim_{n\to\infty} x_n = x$ for certain $x \in X$.

Using (a), it is proved that the sequence (x_n) has an infinite number of terms in each A_i $(i = 1,2,\ldots,m)$ and in this point, the author uses that the sets A_i are closed and proves that $x \in \bigcap_{i=1}^m A_i$.

Finally, as $\bigcap_{i=1}^m A_i$ is closed (here, it is also used the fact that the sets A_i $(i = 1,2,\ldots,m)$ are closed) and so complete, the author reduces the problem to an operator of the complete metric space $\bigcap_{i=1}^m A_i$ into itself and he applies a result of [5].

The purpose of this paper is to give a version of Theorem 1 when X is a compact metric space.

2. Main results

Theorem 2.1. *Let (X,d) be a compact metric space and $T : X \longrightarrow X$ a continuous operator.*

Suppose that m is a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of X, $X = \bigcup_{i=1}^m A_i$ satisfying

(*i*) *$X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T.*

(*ii*) *$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \mathfrak{F}$.*

Then T has a unique fixed point.

Proof. Firstly, we will prove that $\inf\{d(x,Tx) : x \in X\} = 0$.

In fact, we take $x_0 \in X$ and consider the Picard iteration given by $x_{n+1} = Tx_n$.

If there exists $n_0 \in \mathbb{N}$ with $x_{n_0+1} = x_{n_0}$ then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and, thus, the existence of the fixed point is proved.

Suppose that $x_{n+1} \neq x_n$ for all $n = 0, 1, 2 \dots$

Then, by (i), for any $n > 0$ there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n}$ and using (ii) we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \varphi(d(x_{n-1}, x_n)) \leq d(x_{n-1}, x_n). \tag{2.1}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is a nondecreasing sequence of nonnegative real numbers. This fact implies the existence of $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

Now, taking $n \to \infty$ in (2.1), we obtain

$$r \leq r - \lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) \leq r$$

and, thus

$$\lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) = 0. \tag{2.2}$$

Suppose that $r > 0$.

Since that $r = \inf\{d(x_n, x_{n+1}) : n \in \mathbb{N}\}$,

$$0 < r \leq d(x_n, x_{n+1}) \quad \text{for} \quad n = 0, 1, 2 \dots$$

and, since φ is nondecreasing and $\varphi(t) > 0$ for $t \in (0, \infty)$ we have

$$0 < \varphi(r) \leq \varphi(d(x_n, x_{n+1})).$$

Letting $n \to \infty$ in the last inequality

$$0 < \varphi(r) \leq \lim_{n\to\infty} \varphi(d(x_n, x_{n+1}))$$

and this contradicts to (2.2).

Therefore, $r = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

This fact and, since $x_{n+1} = Tx_n$, gives us that

$$\inf\{d(x, Tx) : x \in X\} = 0. \tag{2.3}$$

Now, we consider the mapping

$$X \longrightarrow \mathbb{R}^+$$
$$x \mapsto d(x, Tx).$$

This mapping is, obviously, continuous and, as X is compact, we find $z \in X$ such that

$$d(z, Tz) = \inf\{d(x, Tx) : x \in X\}.$$

By (2.3), $d(z, Tz) = 0$ and, consequently, $z = Tz$.

This proves the existence of a fixed point of T.

For the uniqueness, suppose that z and y are two fixed points of T.

As $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T, we have that $z, y \in \bigcap_{i=1}^m A_i$.

By (ii)

$$d(z, y) = d(Tz, Ty) \leq d(z, y) - \varphi(d(z, y)) \leq d(z, y).$$

Therefore, $\varphi(d(z, y)) = 0$.

Since $\varphi \in \mathfrak{F}$, $d(z, y) = 0$ and, thus, $z = y$.

This finishes the proof. $\square$

Remark 2.2. Under assumption that X is compact, Theorem 1 is true under weaker assumptions. More precisely, the sets A_i $(i = 1, 2, \ldots, m)$ are not necessarily closed and the function φ is not necessarily continuous.

Theorem 2.3. *Under assumptions of Theorem 2, the fixed point problem for T is well posed, that is, if there exists a sequence $\{y_n\}$ in X with $d(y_n, Ty_n) \to 0$ as $n \to \infty$, then $y_n \to z$ as $n \to \infty$, where z is the unique fixed point of T (whose existence is guaranteed by Theorem 2).*

Proof. As z is a fixed point of T, by (i) of Theorem 2, $z \in \bigcap_{i=1}^m A_i$.

Now, we take $\{y_n\}$ in X with $d(y_n, Ty_n) \to 0$ as $n \to \infty$.

Using the triangular inequality, (ii) of Theorem 2 and the fact that $z \in \bigcap_{i=1}^m A_i$ we get

$$d(y_n, z) \leq d(y_n, Ty_n) + d(Ty_n, Tz) \leq d(y_n, Ty_n) + d(y_n, z) - \varphi(d(y_n, z)).$$

From the last inequality we have

$$\varphi(d(y_n, z)) \leq d(y_n, Ty_n)$$

and letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \varphi(d(y_n, z)) = 0. \tag{2.4}$$

In order to prove that $\lim_{n\to\infty} d(y_n, z) = 0$, suppose, that this is false. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $p_n \geq n$ with $d(y_{p_n}, z) \geq \varepsilon$.

Since ϕ is nondecreasing and $\phi(t) > 0$ for $t \in (0, \infty)$,

$$0 < \phi(\varepsilon) \leq \phi(d(y_{p_n}, z)).$$

Letting $n \to \infty$, we get

$$0 < \phi(\varepsilon) \leq \lim_{n\to\infty} \phi(d(y_{p_n}, z))$$

and this contradicts to (2.4).

Therefore, $\lim_{n\to\infty} d(y_n, z) = 0$.

This finishes the proof. □

Remark 2.4. In [3], the proof that $\lim_{n\to\infty} d(y_n, z) = 0$ in Theorem 3 is easily deduced from (2.4) because the author uses the continuity of φ.

3. Examples and some remarks

In the sequel, we relate our results with the ones appearing in [4].
Previously, we present the main result of [4]

Theorem 3.1. *Let (X, d) be a complete metric space, m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of X, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a (c)-comparison function (this means that φ is increasing and the series $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$) and $T : X \longrightarrow X$ an operator. Assume that*

(i) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T.

(ii) $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}$, $i = 1, 2, \cdot, m$, where $A_{m+1} = A_1$.

Then T has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$ and the Picard iteration $\{x_n\}$ converges to x^* for any starting point $x_0 \in X$.

Since compact metric space is a complete metric space, Theorem 4 can be applied when (X, d) is compact.

In what follows, we present an example which can be treated by Theorem 2 and Theorem 4 cannot be applied.

Example 3.2. Consider $([0,1],d)$ where d is the usual distance given by $d(x,y)=|x-y|$. Let $T:[0,1]\longrightarrow[0,1]$ be the mapping defined by $Tx=\dfrac{x}{1+x}$.

In this case, $m=1$.

Moreover, for $x,y\in[0,1]$

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x}{1+x}-\frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \le \frac{|x-y|}{1+|x-y|} \\ &= T(|x-y|) = d(x,y)-(d(x,y)-T(|x-y|)). \end{aligned}$$

Therefore, condition (ii) of Theorem 2 is satisfied for the function $\varphi:[0,\infty)\longrightarrow[0,\infty)$ given by

$$\varphi(t)=t-\frac{t}{1+t}=\frac{t^2}{1+t}.$$

Moreover, it is easily seen that $\varphi\in\mathfrak{F}$.

By Theorem2, T has a unique fixed point (which is $x=0$).

On the other hand, the function $\Psi:[0,\infty)\longrightarrow[0,\infty)$ given by $\Psi(t)=\dfrac{t}{1+t}$, is not a (c)-comparison function since $\Psi^n(t)=\dfrac{t}{1+nt}$ and, consequently, for $t>0$ the series $\displaystyle\sum_{k=0}^{\infty}\Psi^k(t)$ diverges.

This proves that our example cannot be treated by Theorem 4.

For the following example, we need the following lemma whose proof appears in [2].

Lemma 3.3. *Let $\rho:[0,\infty)\longrightarrow[0,\frac{\pi}{2})$ be the function defined by $\rho(x)=\arctan(x)$. Then*

$$\rho(x)-\rho(y)\le\rho(x-y) \qquad \textit{for} \qquad x\ge y.$$

Now, we consider the function $\Psi:[0,\infty)\longrightarrow[0,\infty)$ given by

$$\Psi(x)=\begin{cases}\arctan x & \textit{if} \quad 0\le x\le 1\\ \alpha & \textit{if} \quad 1<x,\end{cases}$$

where $1-\dfrac{\pi}{4}<\alpha<1$.

Example 3.4. Consider the same metric space $([0,1],d)$ that in Example 1 and the operator $T:[0,1]\longrightarrow[0,1]$ given by

$$Tx=\arctan x.$$

In this case, $m=1$. Moreover, taking into account Lemma 1, for $x,y\in[0,1]$ we can obtain

$$\begin{aligned} d(Tx,Ty) &= |\arctan x-\arctan y| \le \arctan(|x-y|) \\ &= \Psi(|x-y|) = d(x,y)-(d(x,y)-\Psi(d(x,y))) \\ &= d(x,y)-\varphi(d(x,y)), \end{aligned}$$

where $\varphi:[0,\infty)\longrightarrow[0,\infty)$ is defined as $\varphi(x)=x-\Psi(x)$.

Notice that

$$\varphi(x)=\begin{cases}x-\arctan x & \text{if} \quad 0\le x\le 1\\ x-\alpha & \text{if} \quad x>1\end{cases}$$

It is easily seen that $\varphi\in\mathfrak{F}$ and φ is not continuous. Therefore, this example can be studied by Theorem 2 while Theorem 4 cannot be applied.

Acknowledgements:

Research partially supported by "Ministerio de Educación y Ciencia", Project MTM 2007/65706.

References

[1] Ya. I. Alber, S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert space*, in: I. Gohberg, Yu. Lyubich (Eds), New Results in Operator Theory, Advances and Appl., **98**, Birkhauser. Verlag, 1997, 7–22.

[2] J. Caballero, J. Harjani, K. Sadarangani, *Uniqueness of positive solutions for a class of fourth-order boundary value problems*, Abs. Appl. Anal., vol(**2011**), Article ID 543035, 13 pages. 3.2

[3] E. Karapinar, *Fixed point theory for cyclic weak φ-contraction*, Appl. Math. Lett., **24** (2011), 822–825.

[4] M. Pacurar, I.A.Rus, *Fixed point theory for cyclic φ-contractions*, Nonlinear Anal., **72** (2010), 1181–1187. 3

[5] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693.