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Uniqueness and global exponential stability of almost periodic solution for Hematopoiesis model on time scales

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Abstract

This paper deals with almost periodic Hematopoiesis dynamic equation on time scales. By applying a novel method based on the fixed point theorem of decreasing operator, we establish sufficient conditions for the existence of unique almost periodic positive solution. Particularly, we give iterative sequence which converges to the almost periodic positive solution. Moreover, we investigate global exponential stability of the almost periodic positive solution by means of Gronwall inequality.

Keywords: Hematopoiesis model on time scales, almost periodic solution, global exponential stability, fixed point theorem of decreasing operator, exponential dichotomy.
2010 MSC: 34K14, 34K20.

1. Introduction

In 1977, Mackey and Glass [15] investigated the Hematopoiesis model

$$x'(t) = -ax(t) + \frac{\beta}{1 + x^N(t-\tau)},$$

which described the production of blood cells. Gyori and Ladas [6] have investigated the global attractively of positive equilibrium for this model. Moreover, the above model and some generalized models have been investigated by many authors, see [4, 7, 11, 12, 16, 17, 18]. Due to the various seasonal effects of the environmental factors in real life situation (e.g., seasonal effects of weather, food supplies, mating habits,

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harvesting, etc.), it is rational and practical to study the biological system with periodic coefficients or almost periodic coefficients. Some authors [17, 18] have studied nonautonomous differential equations with periodic coefficients of the above model.

As we know, in the real world, some processes vary continuously while others vary discretely. These processes can be modeled by differential equations and difference equations, respectively. However, there are also many processes that vary both continuously and discretely. The theory of time scale calculus and dynamic equations on time scales provides us with a powerful tool for solving such mixed processes. The calculus on time scales (see [1, 2] and references cited therein) was initiated by Stefan Hilger in his 1988 Ph.D. dissertation [9] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received great attention. The two main features of the calculus on time scales are unification and extension.

The existence and stability of periodic solution or almost periodic solution for differential equations and difference equations are very basic and important problems. It is natural to ask whether we can explore such existence and stability problems in a unified way and offer more general conclusions. The study of dynamic equations on time scales can unify and extend the fields of differential and difference equations.

Motivated by the above facts, in this paper, we investigate the following nonautonomous almost periodic Hematopoiesis dynamic equation on time scales

$$x^{\Delta}(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + x^{N_i}(t - \tau_i(t))}. \tag{1.1}$$

Almost periodicity is closer to the reality in biological systems [5, 10]. However, to our knowledge, no papers deal with the existence and global exponential stability of unique almost periodic positive solution for the above model (1.1) on time scales.

In this paper, we aim to establish sufficient conditions that guarantee the existence of unique almost periodic positive solution of model (1.1) on time scales. The technique used in this paper is different from the usual methods employed to solve almost periodic cases such as the contraction mapping principle and Liapunov functional. Our method is based on the fixed point theorem of decreasing operator. Moreover, we also investigate global exponential stability of almost periodic positive solution by means of Gronwall inequality. The results of this paper are new and more valuable in applications, which complement and extend the previously obtained results in [4, 6, 7, 11, 12, 16, 17, 18]. Our study reveals that it is unnecessary to prove results for differential equations and separately again for difference equations. We can unify such existence and stability problems in the framework of dynamic equations on time scales.

2. Preliminaries

In this section, we present some basic definitions and preliminary results from the calculus on time scales and almost periodic functions. For more details, see [1, 2, 13, 14].

The symbol $\mathbb{T}$ denotes a time scale, which is a nonempty closed subset of $\mathbb{R}$. Some examples of such time scales are

$$\mathbb{R}, \quad \mathbb{Z}, \quad \bigcup_{k\in\mathbb{Z}}[2k, 2k+1], \quad \bigcup_{k\in\mathbb{Z}}\bigcup_{n\in\mathbb{N}}\{k+\frac{1}{n}\}.$$

Definition 2.1. The forward and backward jump operators σ, ρ: $\mathbb{T} \to \mathbb{T}$ and the graininess μ: $\mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$.

If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.
If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

Definition 2.3. For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.
If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_r^t f(s)\Delta s = F(t) - F(r) \quad for \quad t, r \in \mathbb{T}.$$

Definition 2.4. A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\Re = \Re(\mathbb{T}, \mathbb{R})$. We define the set $\Re^+ = \Re^+(\mathbb{T}, \mathbb{R}) = \{p \in \Re : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 2.5. If p is a regressive function, then the generalized exponential function e_p is defined as the unique solution of the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$, where $s \in \mathbb{T}$.
An explicit formula for $e_p(t, s)$ is given by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}\left(p(\tau)\right)\Delta\tau\right\} \quad \text{for all} \quad s, t \in \mathbb{T}$$

with

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & if \;\; h \neq 0, \\ z, & if \;\; h = 0. \end{cases}$$

Definition 2.6. Let $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Lemma 2.7. *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $e_0(t, s) \equiv 1$, $e_p(t, t) \equiv 1$ *;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))\, e_p(t, s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$*;*

(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$, $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$*;*

(v) $(e_p(t, s))^{\Delta} = pe_p(t, s)$*;*

(vi) If $a, b, c \in \mathbb{T}$*, then* $\int_a^b p(s)e_p(c, \sigma(s))\Delta s = e_p(c, a) - e_p(c, b)$.

Definition 2.8. ([13]) Let Γ be a collection of sets which is constructed by subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called an almost periodic time scale with respect to Γ, if

$$\Gamma^* = \left\{\pm\tau \in \bigcap_{\Lambda \in \Gamma} \Lambda : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\right\} \neq \emptyset$$

and Γ^* is called the smallest almost periodic set of $\mathbb{T}$.

Definition 2.9. ([13]) Let $\mathbb{T}$ be an almost periodic time scale with respect to Γ. A function $f(t) \in C(\mathbb{T}, \mathbb{R}^n)$ is called almost periodic if for any given $\varepsilon > 0$, the set $E(f, \varepsilon) = \{\tau \in \Gamma^* : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$ is relatively dense in $\mathbb{T}$; that is, for any given $\varepsilon > 0$, there exists a real number $l = l(\varepsilon) > 0$ such that each interval of length l contains at least one $\tau = \tau(\varepsilon) \in E(f, \varepsilon)$ satisfying $|f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}$.
The set $E(f, \varepsilon)$ is called ε-translation set of $f(t)$, τ is called ε-translation number of $f(t)$, and $l(\varepsilon)$ is called contain interval length of $E(f, \varepsilon)$.

Remark 2.10. If $\Gamma = \{\mathbb{R}\}$ and $\mathbb{T} = \mathbb{R}$, then $\Gamma^* = \mathbb{R}$, in this case, Definition 2.9 is equivalent to the definition of almost periodic function in [5]. If $\Gamma = \{\mathbb{Z}\}$ and $\mathbb{T} = \mathbb{Z}$, then $\Gamma^* = \mathbb{Z}$, in this case, Definition 2.9 is equivalent to the definition of almost periodic sequence in [3].

Lemma 2.11. ([13]) *Let $f \in C(\mathbb{T}, \mathbb{R}^n)$ be an almost periodic function, then $f(t)$ is bounded on $\mathbb{T}$.*

Lemma 2.12. ([13]) *If $f, g \in C(\mathbb{T}, \mathbb{R}^n)$ are almost periodic, then $f + g$ and fg are almost periodic.*

Lemma 2.13. ([13]) *If $f(t)$ is almost periodic and $G(\cdot)$ is uniformly continuous defined on the value field of $f(t)$, then $G \circ f$ is almost periodic.*

Lemma 2.14. ([13]) *If $f(t) \in C(\mathbb{T}, \mathbb{R}^n)$ is almost periodic, then $F(t)$ is almost periodic if and only if $F(t)$ is bounded on $\mathbb{T}$, where $F(t) = \int_0^t f(s)\Delta s$.*

Definition 2.15. ([13, 19]) Let $Q(t)$ be $n \times n$ rd-continuous matrix function on $\mathbb{T}$.
The linear system

$$x^{\Delta}(t) = Q(t)x(t), \qquad t \in \mathbb{T} \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\left\|X(t)PX^{-1}(\sigma(s))\right\| \le ke_{\ominus\alpha}(t, \sigma(s)) \quad \text{for } \ t \ge \sigma(s), \quad s, t \in \mathbb{T},$$

$$\left\|X(t)(I-P)X^{-1}(\sigma(s))\right\| \le ke_{\ominus\alpha}(\sigma(s), t) \quad \text{for } \ t \le \sigma(s), \quad s, t \in \mathbb{T}.$$

Consider almost periodic system

$$x^{\Delta}(t) = Q(t)x(t) + g(t), \qquad t \in \mathbb{T}, \tag{2.2}$$

where $Q(t)$ is an almost periodic matrix function, $g(t)$ is an almost periodic vector function.

Lemma 2.16. ([13, 14]) *If the linear system* (2.1) *admits an exponential dichotomy, then the almost periodic system* (2.2) *has a unique almost periodic solution $x(t)$ as follows*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))g(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))g(s)\Delta s.$$

Lemma 2.17. ([1]) *Let $Q(t)$ be a regressive $n \times n$ matrix-valued function on $\mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$, then the initial value problem*

$$x^{\Delta}(t) = Q(t)x(t), \qquad x(t_0) = x_0$$

has a unique solution $x(t)$ as follows

$$x(t) = e_Q(t, t_0)x_0.$$

Lemma 2.18. ([13]) *Let $c_i(t)$ be almost periodic function on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \Re^+$, $\forall t \in \mathbb{T}$ and* $\min\limits_{1 \le i \le n} \left\{ \inf\limits_{t \in \mathbb{T}} c_i(t) \right\} > 0.$
Then the linear system

$$x^{\Delta}(t) = diag\left(-c_1(t), -c_2(t), \cdots, -c_n(t)\right)x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

By Lemma 2.17, we can get

Lemma 2.19. *Let* $-C = diag\,(-c_1(t), -c_2(t), \cdots, -c_n(t))$*, then* $X(t) = e_{-C}(t, t_0)$ *is a fundamental solution matrix of the linear system* $x^{\Delta}(t) = diag\,(-c_1(t), -c_2(t), \cdots, -c_n(t))\, x(t)$.

Definition 2.20. Let X be a Banach space and P be a closed,nonempty subset of X, P is called a cone if (i)$x \in P, \lambda \geq 0$ implies $\lambda x \in P$; (ii)$x \in P, -x \in P$ implies $x = \theta$. (θ is zero element).
Every cone $P \subset X$ induces an ordering in X, we define '$\leq$' with respect to P by $x \leq y$ if and only if $y - x \in P$.

Definition 2.21. A cone P of X is called normal cone if there exists a positive constant σ, such that $||x + y|| \geq \sigma$ for any $x, y \in P, ||x|| = ||y|| = 1$.

Definition 2.22. Let P be a cone of X and $A : P \to P$ an operator. A is called decreasing if $\theta \leq x \leq y$ implies $Ax \geq Ay$.

The following fixed point theorem of decreasing operator (see [8]) is an important tool in our proofs.

Lemma 2.23. ([8]) *Suppose that*
(i) P *is normal cone of Banach space* X*, operator* $A : P \to P$ *is decreasing;*
(ii) $A\theta > \theta, A^2\theta \geq \varepsilon_0 A\theta$*, where* $\varepsilon_0 > 0$*;*
(iii) For $\forall 0 < c < d < 1$*, there exists* $\eta = \eta(c, d) > 0$ *such that*
$$A(\lambda x) \leq [\lambda(1 + \eta)]^{-1} Ax \qquad \text{for } \forall c \leq \lambda \leq d \text{ and } \theta < x \leq A\theta.$$
Then, the operator A *has a unique positive fixed point* $x^* > \theta$*. Moreover,* $\|x_k - x^*\| \to 0$*,* $(k \to \infty)$ *, where* $x_k = Ax_{k-1}$ $(k = 1, 2, \cdots)$ *for any initial* $x_0 \in P$.

Remark 2.24. In Lemma 2.23, the operator A does not need continuity and compactness.

3. Existence of the unique almost periodic positive solution

In this paper, we use notations: for any bounded function $f(t)$, we denote $\bar{f} = \sup_{t \in \mathbb{T}} f(t)$, $\underline{f} = \inf_{t \in \mathbb{T}} f(t)$.

Throughout this paper, we assume that the bounded almost periodic functions $a(t), \beta_i(t)$, $\tau_i(t)$ satisfy $0 < \underline{a} \leq a(t) \leq \bar{a}$, $0 < \underline{\beta_i} \leq \beta_i(t) \leq \overline{\beta_i}$, $0 < \underline{\tau_i} \leq \tau_i(t) \leq \overline{\tau_i}$, $-a(t) \in \Re^+$ and $N_i > 0$ $(i = 1, 2, \cdots, m)$.

Due to biological significance, we restrict our attention to positive solutions of equation (1.1). The initial condition associated with equation (1.1) is given by
$$x(t; \phi) = \phi(t) > 0 \quad \text{for} \quad t \in [-\tau^*, 0]_{\mathbb{T}},\ \tau^* = \max_{1 \leq i \leq m} \{\overline{\tau_i}\}.$$
Let $X = \{w(t) | w \in C(\mathbb{T}, \mathbb{R}), w(t)$ is almost periodic function$\}$ with the norm $\|w\| = \sup_{t \in \mathbb{T}} |w(t)|$, then X is Banach space.
For $w(t) \in X$, we consider equation

$$x^{\Delta}(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + w^{N_i}(t - \tau_i(t))}. \tag{3.1}$$

Since $\inf_{t \in \mathbb{T}} a(t) = \underline{a} > 0$, then from Lemma 2.18 we know that the linear equation $x^{\Delta}(t) = -a(t)x(t)$ admits exponential dichotomy on $\mathbb{T}$.
Hence, by Lemma 2.16, we know that equation (3.1) has exactly one almost periodic solution:

$$x_w(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + w^{N_i}(s - \tau_i(s))} \Delta s.$$

We define operator $A : X \to X$,

$$(Aw)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + w^{N_i}(s - \tau_i(s))} \Delta s, \qquad w \in X.$$

Obviously, $w(t)$ is the almost periodic solution of equation (1.1) if and only if w is the fixed point of the operator A.
Define a cone $\Omega = \{w|w \in X, w(t) \geq 0, t \in \mathbb{T}\}$.
Let $M = \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{\beta_i}$.
We make assumptions:
(C_1) $0 < N_i \leq 1$;
(C_2) $N_i > 1$, $(N_i - 1)M^{N_i} \leq 1$.

Theorem 3.1. *Assume that for any $i \in \{1, 2, \cdots, m\}$, (C_1) or (C_2) holds, then equation (1.1) has a unique almost periodic positive solution $w^*(t)$. Moreover, $\|w_k - w^*\| \to 0$, $(k \to \infty)$, $w_k = Aw_{k-1}$ $(k = 1, 2, \cdots)$ for any initial $w_0 \in \Omega$.*

Proof. Firstly, we prove that $A\Omega \subset \Omega$.
For $\forall w \in \Omega$, then

$$(Aw)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + w^{N_i}(s - \tau_i(s))} \Delta s > 0. \tag{3.2}$$

In addition, for $\forall w \in \Omega$, we know that equation (3.1) has exactly one almost periodic solution

$$x_w(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + w^{N_i}(s - \tau_i(s))} \Delta s.$$

Since $x_w(t)$ is almost periodic, then $(Aw)(t)$ is almost periodic.
This, together with (3.2), implies $Aw \in \Omega$. So we have $A\Omega \subset \Omega$.
It is clear that Ω is normal cone, $A : \Omega \to \Omega$ is decreasing operator.
Now, we will show that condition (ii) of Lemma 2.23 is satisfied.
Note that

$$(A\theta)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \beta_i(s) \Delta s \leq \int_{-\infty}^{t} e_{-\underline{a}}(t, \sigma(s)) \sum_{i=1}^{m} \overline{\beta_i} \Delta s$$

$$= \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{\beta_i} = M,$$

and

$$(A\theta)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \beta_i(s) \Delta s$$

$$\geq \int_{-\infty}^{t} e_{-\bar{a}}(t, \sigma(s)) \sum_{i=1}^{m} \underline{\beta_i} \Delta s = \frac{1}{\bar{a}} \sum_{i=1}^{m} \underline{\beta_i} > 0,$$

which implies $A\theta > \theta$.

Again, we have

$$\begin{aligned}(A^2\theta)(t) &= \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + (A\theta)^{N_i}(s - \tau_i(s))} \Delta s \\ &\geq \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1 + M^{N_i}} \Delta s \\ &\geq \frac{1}{1 + B} \int_{-\infty}^{t} e_{-a}(t, \sigma(s)) \sum_{i=1}^{m} \beta_i(s) \Delta s = \varepsilon_0 (A\theta)(t),\end{aligned}$$

this implies $A^2\theta \geq \varepsilon_0 A\theta$, here $\varepsilon_0 = \frac{1}{1+B}$, $B = \max\limits_{1\leq i\leq m} \left\{M^{N_i}\right\}$.

Finally, we show that condition (iii) of Lemma 2.23 is satisfied.
Let $\forall 0 < c < d < 1$, for $\forall c \leq \lambda \leq d$ and $\theta < x \leq A\theta$, we have $0 < ||x|| \leq ||A\theta|| \leq M$.

$$A(\lambda x)(t) = \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+\lambda^{N_i}x^{N_i}\left(s-\tau_i(s)\right)}\Delta s$$

$$= \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)} \frac{1+x^{N_i}\left(s-\tau_i(s)\right)}{1+\lambda^{N_i}x^{N_i}\left(s-\tau_i(s)\right)}\Delta s.$$

Note that

$$\frac{1+x^{N_i}\left(s-\tau_i(s)\right)}{1+\lambda^{N_i}x^{N_i}\left(s-\tau_i(s)\right)} = \lambda^{-N_i}\left(1+\frac{\lambda^{N_i}-1}{1+\lambda^{N_i}x^{N_i}\left(s-\tau_i(s)\right)}\right)$$

$$\leq \lambda^{-N_i}\left(1+\frac{\lambda^{N_i}-1}{1+\lambda^{N_i}M^{N_i}}\right) = \frac{1+M^{N_i}}{1+\lambda^{N_i}M^{N_i}}.$$

So we obtain

$$A(\lambda x)(t) \leq \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)} \frac{1+M^{N_i}}{1+\lambda^{N_i}M^{N_i}}\Delta s$$

$$= \frac{1}{\lambda}\int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)} \frac{(1+M^{N_i})\lambda}{1+\lambda^{N_i}M^{N_i}}\Delta s.$$

Let $f_i(t) = \dfrac{(1+M^{N_i})t}{1+t^{N_i}M^{N_i}}$, we have

$$f_i^{'}(t) = \frac{\left(1+M^{N_i}\right)\left[1+(1-N_i)t^{N_i}M^{N_i}\right]}{\left(1+t^{N_i}M^{N_i}\right)^2}.$$

Since (C_1) $0 < N_i \leq 1$ or (C_2) $N_i > 1$, $(N_i-1)M^{N_i} \leq 1$ $(i = 1, 2, \cdots, m)$ holds, then we know $f_i^{'}(t) > 0$ for $0 < t < 1$, so we have $0 = f_i(0) < f_i(c) \leq f_i(\lambda) \leq f_i(d) < f_i(1) = 1$.
Hence we get

$$A(\lambda x)(t) \leq \frac{1}{\lambda}\int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)} f_i(\lambda)\Delta s$$

$$\leq \frac{1}{\lambda}\int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)} f_i(d)\Delta s$$

$$\leq \frac{1}{\lambda}g(d)\int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+x^{N_i}\left(s-\tau_i(s)\right)}\Delta s$$

$$= \frac{1}{\lambda}g(d)(Ax)(t) = \frac{1}{\lambda}\cdot\frac{1}{1+\left(\frac{1}{g(d)}-1\right)}(Ax)(t) = \frac{1}{\lambda}\cdot\frac{1}{1+\eta(d)}(Ax)(t),$$

here $g(d) = \max\limits_{1\leq i\leq m}\left\{f_i(d)\right\}$, $0 < g(d) < 1$, $\eta = \eta(d) = \frac{1}{g(d)} - 1 > 0$.

By Lemma 2.23, we know the operator A has a unique positive fixed point $w^* > \theta$, which means equation (1.1) has a unique almost periodic positive solution $w^*(t)$. Moreover, $\|w_k - w^*\| \to 0, (k \to \infty)$, $w_k = Aw_{k-1}(k = 1, 2, \cdots)$ for any initial $w_0 \in \Omega$. The proof of Theorem 3.1 is completed. $\square$

Remark 3.2. Theorem 3.1 of this paper not only gives sufficient conditions for the existence of unique almost periodic positive solution, but also gives iterative sequence $\{w_k(t)\}$, which converges to the almost periodic positive solution $w^*(t)$.

Remark 3.3. From the above proof, we have

$$\begin{aligned} w^*(t) = (Aw^*)(t) &= \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+w^{*N_i}(s-\tau_i(s))}\Delta s \\ &\le \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \beta_i(s)\Delta s \le \int_{-\infty}^{t} e_{-\underline{a}}(t,\sigma(s)) \sum_{i=1}^{m} \overline{\beta_i}\Delta s \\ &= \frac{1}{\underline{a}} \sum_{i=1}^{m} \overline{\beta_i} = M. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} w^*(t) = (Aw^*)(t) &= \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+w^{*N_i}(s-\tau_i(s))}\Delta s \\ &\ge \int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \frac{\beta_i(s)}{1+M^{N_i}}\Delta s \\ &\ge \frac{1}{1+B}\int_{-\infty}^{t} e_{-a}(t,\sigma(s)) \sum_{i=1}^{m} \beta_i(s)\Delta s \\ &\ge \frac{1}{1+B}\int_{-\infty}^{t} e_{-\bar{a}}(t,\sigma(s)) \sum_{i=1}^{m} \underline{\beta_i}\Delta s = \frac{1}{\bar{a}(1+B)} \sum_{i=1}^{m} \underline{\beta_i}. \end{aligned}$$

So we get

$$\frac{1}{\bar{a}(1+B)} \sum_{i=1}^{m} \underline{\beta_i} \le w^*(t) \le M, \qquad \text{here } \ B = \max_{1\le i\le m}\left\{M^{N_i}\right\}.$$

4. Global exponential stability of almost periodic positive solution

Theorem 4.1. *Assume that* $N_i \ge 1$, $(N_i-1)M^{N_i} \le 1$, $(i = 1,2,\cdots,m)$ *and* $\underline{a} > \sum_{i=1}^{m} \overline{\beta_i} N_i$. *Then equation* (1.1) *has a unique globally exponentially stable almost periodic positive solution.*

Proof. Since the condition $N_i \ge 1$, $(N_i-1)M^{N_i} \le 1$, $(i = 1,2,\cdots,m)$ is satisfied, then by Theorem 3.1 we know equation (1.1) has a unique almost periodic positive solution $w^*(t)$, and $\frac{1}{\bar{a}(1+B)} \sum_{i=1}^{m} \underline{\beta_i} \le w^*(t) \le M$. Let $\psi(t)$ be the initial function of $w^*(t)$, $w^*(t;\psi) = \psi(t)$ for $t \in [-\tau^*, 0]_{\mathbb{T}}$. Now we prove $w^*(t)$ is globally exponentially stable.

Suppose $x(t)$ is arbitrary solution of equation (1.1) with initial function $x(t;\phi) = \phi(t) > 0$, $\quad t \in [-\tau^*, 0]_{\mathbb{T}}$.

Let $y(t) = x(t) - w^*(t)$, then we have

$$\begin{aligned} y^{\Delta}(t) &= (x(t) - w^*(t))^{\Delta} \\ &= -a(t)x(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + x^{N_i}(t - \tau_i(t))} - \left(-a(t)w^*(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + w^{*N_i}(t - \tau_i(t))}\right) \\ &= -a(t)(x(t) - w^*(t)) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + x^{N_i}(t - \tau_i(t))} - \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + w^{*N_i}(t - \tau_i(t))}. \end{aligned} \tag{4.1}$$

Let

$$h(t) = \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + x^{N_i}(t - \tau_i(t))} - \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + w^{*N_i}(t - \tau_i(t))},$$

then it follows from (4.1) that

$$y^{\Delta}(t) = -a(t)y(t) + h(t). \tag{4.2}$$

From (4.2), we know that $y(t)$ can be expressed as follows

$$y(t) = e_{-a}(t, t_0)y(t_0) + \int_{t_0}^{t} e_{-a}(t, s)h(s)\Delta s, (t \geq t_0), t_0 \in [-\tau^*, 0]_{\mathbb{T}}. \tag{4.3}$$

Thus, (4.3) implies that

$$y(t) = e_{-a}(t, t_0)(\phi(t_0) - \psi(t_0)) + \int_{t_0}^{t} e_{-a}(t, s)h(s)\Delta s. \tag{4.4}$$

Note that

$$\begin{aligned} |h(t)| &= \left|\sum_{i=1}^{m} \beta_i(t)\left(\frac{1}{1 + x^{N_i}(t - \tau_i(t))} - \frac{1}{1 + w^{*N_i}(t - \tau_i(t))}\right)\right| \\ &\leq \sum_{i=1}^{m} \beta_i(t)\left|\frac{1}{1 + x^{N_i}(t - \tau_i(t))} - \frac{1}{1 + w^{*N_i}(t - \tau_i(t))}\right|. \end{aligned} \tag{4.5}$$

By the mean value theorem, we have

$$\begin{aligned} &\left|\frac{1}{1 + x^{N_i}(t - \tau_i(t))} - \frac{1}{1 + w^{*N_i}(t - \tau_i(t))}\right| \\ &= \left|-\frac{N_i\xi^{N_i-1}}{(1 + \xi^{N_i})^2}\left[x(t - \tau_i(t)) - w^*(t - \tau_i(t))\right]\right| \\ &= \frac{N_i\xi^{N_i-1}}{(1 + \xi^{N_i})^2}\left|x(t - \tau_i(t)) - w^*(t - \tau_i(t))\right|, \end{aligned} \tag{4.6}$$

in which ξ lies between $x(t - \tau_i(t))$ and $w^*(t - \tau_i(t))$.

Note that the function $g_i(x) = \dfrac{N_i x^{N_i-1}}{(1 + x^{N_i})^2} < N_i$ for $\forall x \in (0, +\infty)$ and $N_i \geq 1$, $(i = 1, 2, \cdots, m)$. Thus we have

$$\frac{N_i\xi^{N_i-1}}{(1 + \xi^{N_i})^2} < N_i \quad \text{for} \quad N_i \geq 1.$$

From (4.6), we get

$$\left|\frac{1}{1+x^{N_i}\left(t-\tau_i(t)\right)}-\frac{1}{1+w^{*N_i}\left(t-\tau_i(t)\right)}\right|<N_i\left|x\left(t-\tau_i(t)\right)-w^*\left(t-\tau_i(t)\right)\right|. \tag{4.7}$$

Hence, by (4.5) and (4.7), we get

$$|h(t)|<\sum_{i=1}^{m}\beta_i(t)N_i\left|x\left(t-\tau_i(t)\right)-w^*\left(t-\tau_i(t)\right)\right|\le\|x-w^*\|\sum_{i=1}^{m}\overline{\beta_i}N_i.$$

It follows that

$$\|h(t)\|\le\|x-w^*\|\sum_{i=1}^{m}\overline{\beta_i}N_i=\|y\|\sum_{i=1}^{m}\overline{\beta_i}N_i.$$

Take norm at both sides of (4.4), we obtain

$$\begin{aligned}\|y(t)\|&\le e_{-a}(t,t_0)\|\phi-\psi\|+\int_{t_0}^{t}e_{-a}(t,s)\|h(s)\|\Delta s\\&\le e_{-a}(t,t_0)\|\phi-\psi\|+\int_{t_0}^{t}e_{-a}(t,s)\|y\|\sum_{i=1}^{m}\overline{\beta_i}N_i\Delta s.\end{aligned} \tag{4.8}$$

From (4.8), we get

$$\frac{\|y(t)\|}{e_{-a}(t,t_0)}\le\|\phi-\psi\|+\int_{t_0}^{t}\frac{\|y\|}{e_{-a}(s,t_0)}\sum_{i=1}^{m}\overline{\beta_i}N_i\Delta s.$$

By Gronwall inequality (see [1]), we obtain

$$\frac{\|y(t)\|}{e_{-a}(t,t_0)}\le\|\phi-\psi\|e_{\gamma}(t,t_0),\qquad\text{here}\quad\gamma=\sum_{i=1}^{m}\overline{\beta_i}N_i.$$

Hence we get

$$\begin{aligned}\|y(t)\|&\le\|\phi-\psi\|e_{\gamma}(t,t_0)e_{-a}(t,t_0)\\&\le\|\phi-\psi\|e_{\gamma}(t,t_0)e_{-\underline{a}}(t,t_0)\\&=\|\phi-\psi\|e_{-(\underline{a}-\gamma)}(t,t_0).\end{aligned}$$

That is $\|x(t)-w^*(t)\|\le\|\phi-\psi\|e_{-(\underline{a}-\gamma)}(t,t_0)$, here $\underline{a}>\gamma$, which means $w^*(t)$ is globally exponentially stable. The proof of Theorem 4.1 is completed. □

Remark 4.2. As mentioned in the introduction, one of our principal aims is to unify the existence and stability of almost periodic solution for some differential equations and their corresponding discrete analogues.
If $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$, then equation (1.1) reduces to

$$x'(t)=-a(t)x(t)+\sum_{i=1}^{m}\frac{\beta_i(t)}{1+x^{N_i}\left(t-\tau_i(t)\right)},t\in\mathbb{R}$$

and

$$x(k+1)-x(k)=-a(k)x(k)+\sum_{i=1}^{m}\frac{\beta_i(k)}{1+x^{N_i}\left(k-\tau_i(k)\right)},k\in\mathbb{Z},$$

respectively.

Our studies unify differential equations and difference equations. For the existence and stability of almost periodic solution of differential equations and difference equations, it is unnecessary to prove results for differential equations and separately again for their discrete analogues (difference equations). We can unify such problems in the framework of dynamic equations on time scales.

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