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Common coupled fixed point results for probabilistic φ-contractions in Menger PGM-spaces

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Abstract

We consider several hybrid probabilistic contractions with a gauge function φ. Without any continuity or monotonicity conditions for φ, we obtain some new common coupled fixed point theorems in $Menger$ PGM-spaces. Finally, an example is given to illustrate our main results.

Keywords: $Menger$ PGM-space, hybrid probabilistic contraction, gauge function, coupled fixed point.
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1. Introduction and Preliminaries

The concept of a probabilistic metric space was introduced and studied by Menger [9, 14]. Since then, many authors have studied the fixed point property for mappings defined on probabilistic metric spaces (see [4, 5, 16, 17, 18, 20, 21]). Jachymski [6] has proved some fixed point theorems for probabilistic nonlinear contractions with a gauge function φ and discussed the relations between several assumptions concerning φ. Mustafa and Sims [10] defined the concept of a G-metric space and many fixed point theorems for contractive mappings in G-metric spaces have been studied [1, 2, 11, 15]. Zhou *et al.* [19] defined the notion of a generalized probabilistic metric space (or a PGM-space), which was a generalization of a PM-space and a G-metric space. Since then, some results in $Menger$ PGM-spaces have been studied [22].

Coupled fixed points and their applications for binary mappings have been studied by Bhaskar and Lakshmikantham [3]. Let X be a non-empty set and $T : X \times X \to X$ be a mapping; then an element $(u, v) \in X \times X$ is called a coupled fixed point of T if $T(u, v) = u$ and $T(v, u) = v$. [7, 12, 13] have presented some results for the existence and uniqueness of coupled fixed points for the cases of partially ordered metric spaces, cone metric spaces and fuzzy metric spaces.

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In this paper, we introduce and investigate several hybrid probabilistic contractions with a gauge function φ. Our main results prove some common coupled fixed point theorems in *Menger PGM*-spaces without any continuity or monotonicity conditions for φ.

Let $\mathbb{R}$ denote the set of reals, $\mathbb{R}^+$ the nonnegative reals and $\mathbb{Z}^+$ be the set of all positive integers. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. We will denote by $\mathcal{D}$ the set of all distribution functions, while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied: $\Delta(a, 1) = a; \Delta(a, b) = \Delta(b, a); a \geq b, c \geq d \Rightarrow \Delta(a, c) \geq \Delta(b, d); \Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 1.1. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$\Delta^1(t) = \Delta(t, t), \qquad \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)), \qquad \text{for } m = 2, 3, ..., t \in [0, 1].$

Two examples of t-norm are $\Delta_m(a, b) = \min\{a, b\}$ and $\Delta_p(a, b) = ab$.

Definition 1.2 ([10]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(G-1) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$;
(G-2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ for all $x, y, z \in X$;
(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a generalized metric or a G-metric on X and the pair (X, G) is a G-metric space.

Definition 1.3 ([19]). A *Menger* probabilistic G-metric space (shortly, a *PGM*-space) is a triple (X, G^*, Δ), where X is a nonempty set, Δ is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x, y, z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = ...$(symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t + s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

Lemma 1.4. *Let (X, G) be a G-metric space. Define a mapping $G^* : X \times X \times X \to \mathcal{D}$ by*

$$G^*(x, y, z)(t) = G^*_{x,y,z}(t) = H(t - G(x, y, z)), \tag{1.1}$$

for $x, y, z \in X$ and $t > 0$. Then (X, G^*, Δ) is a Menger PGM-space called the induced Menger PGM-space by (X, G).

Definition 1.5 ([19]). Let (X, G^*, Δ) be a *Menger PGM*-space and x_0 be any point in X. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, and (ϵ, δ)-neighborhood of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\epsilon) > 1-\delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta\},$$

which means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1 - \delta$.

Definition 1.6 ([19]). Let (X, G^*, Δ) be a PGM-space, $\{x_n\}$ is a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (write $x_n \to x$), if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_{x_0}(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$;

(2) $\{x_n\}$ is called a *Cauchy* sequence, if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $n, m, l > M_{\epsilon,\delta}$;

(3) (X, G^*, Δ) is said to be complete if every *Cauchy* sequence in X converges to a point in X.

Lemma 1.7 ([22]). *Let (X, G^*, Δ) be a Menger PGM-space. For each $\lambda \in (0, 1]$, define a function G^*_λ by*

$$G^*_\lambda(x, y, z) = \inf_t\{t \geq 0 : G^*_{x,y,z}(t) > 1 - \lambda\}, \tag{1.2}$$

for any $x, y, z \in X$, then

(1) $G^*_\lambda(x, y, z) < t$ if and only if $G^*_{x,y,z}(t) > 1 - \lambda$;

(2) $G^*_\lambda(x, y, z) = 0$ for all $\lambda \in (0, 1]$ if and only if $x = y = z$;

(3) $G^*_\lambda(x, y, z) = G^*_\lambda(y, x, z) = G^*_\lambda(y, z, x) = ...$;

(4) If $\Delta = \Delta_m$, then for every $\lambda \in (0, 1]$, $G^*_\lambda(x, y, z) \leq G^*_\lambda(x, a, a) + G^*_\lambda(a, y, z)$.

Lemma 1.8 ([22]). *Let (X, G^*, Δ) be a Menger PGM-space and let $\{G^*_\lambda\}$, $\lambda \in (0, 1]$ be a family of functions on X defined by (1.2). If Δ is a t-norm of H-type, then for each $\lambda \in (0, 1]$, there exists $\mu \in [0, \lambda]$, such that for each $m \in \mathbb{Z}^+$,*

$$G^*_\lambda(x_0, x_m, x_m) \leq \sum_{i=0}^{m-1} G^*_\mu(x_i, x_{i+1}, x_{i+1}),$$

$$G^*_\lambda(x_0, x_0, x_m) \leq \sum_{i=0}^{m-1} G^*_\mu(x_i, x_i, x_{i+1}),$$

for all $x_0, x_1, ..., x_m \in X$.

Lemma 1.9 ([6]). *Suppose that $F \in \mathcal{D}$. For each $n \in \mathbb{Z}^+$, let $F_n : \mathbb{R} \to [0, 1]$ be nondecreasing and $g_n : (0, +\infty) \to (0, +\infty)$ satisfy $\lim_{n\to\infty} g_n(t) = 0$ for any $t > 0$. If*

$$F_n(g_n(t)) \geq F(t) \quad for\ any\ t > 0,$$

then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.

Definition 1.10 ([13]). Let X be a non-empty set. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings. A is said to be commutative with T if $AT(x, y) = T(Ax, Ay)$ for all $x, y \in X$. A point $u \in X$ is called a common coupled fixed point of T and A if $u = Au = T(u, u)$.

Lemma 1.11 ([17]). *Let X be a non-empty set. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings. If $T(X \times X) \subset A(X)$, then there exist two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$.*

2. Main results

Theorem 2.1. *Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\Sigma_{n=1}^\infty \varphi^n(t) < +\infty$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that*

$$G^*_{T(x,y),T(p,q),T(h,l)}(\varphi(t)) \geq [\Delta(G^*_{Ax,Ap,Ah}(t), G^*_{Ay,Aq,Al}(t))]^{\frac{1}{2}}, \tag{2.1}$$

for all $x, y, p, q, h, l \in X$, where $T(X \times X) \subset A(X)$, A is continuous and commutative with T. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. By Lemma 1.11, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$. Suppose that $t > 0$. From (2.1), we have

$$\begin{aligned} G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi(t)) &= G^*_{T(x_{n-1},y_{n-1}),T(x_n,y_n),T(x_{n+1},y_{n+1})}(\varphi(t)) \\ &\geq [\Delta(G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t), G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t))]^{\frac{1}{2}}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(\varphi(t)) &= G^*_{T(y_{n-1},x_{n-1}),T(y_n,x_n),T(y_{n+1},x_{n+1})}(\varphi(t)) \\ &\geq [\Delta(G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t), G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t))]^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Suppose that $G_n(t) = [\Delta(G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t), G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t))]^{\frac{1}{2}}$. Then, operating by t-norm Δ on (2.2) and (2.3), from $\Delta \geq \Delta_p$ we obtain

$$G_{n+1}(\varphi(t)) \geq [\Delta(G_n(t), G_n(t))]^{\frac{1}{2}} = G_n(t). \tag{2.4}$$

Thus, it follows from (2.2), (2.3), and (2.4) that

$$G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi^n(t)) \geq G_n(\varphi^{n-1}(t)) \geq \cdots \geq G_1(t), \tag{2.5}$$

$$G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(\varphi^n(t)) \geq G_n(\varphi^{n-1}(t)) \geq \cdots \geq G_1(t). \tag{2.6}$$

Next, we show that $\{Ax_n\}$ is a *Cauchy* sequence. For each $\lambda \in (0, 1]$, suppose that $D_\lambda = \inf\{t > 0 : G_1(t) > 1 - \lambda\}$. Then, $G_1(D_\lambda + 1) > 1 - \lambda$. From (2.5) we see that $G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi^n(D_\lambda + 1)) > 1 - \lambda$. By Lemma 1.7, we have

$$G^*_\lambda(Ax_n, Ax_{n+1}, Ax_{n+2}) < \varphi^n(D_\lambda + 1), \qquad \lambda \in (0, 1]. \tag{2.7}$$

By Lemma 1.8, for each $\lambda \in (0, 1]$ there exists $\mu \in (0, 1]$ such that

$$\begin{aligned} G^*_\lambda(Ax_n, Ax_m, Ax_l) &< G^*_\lambda(Ax_n, Ax_m, Ax_m) + G^*_\lambda(Ax_m, Ax_m, Ax_l) \\ &\leq \sum_{i=n}^{m-1} G^*_\mu(x_i, x_{i+1}, x_{i+1}) + \sum_{j=m}^{l-1} G^*_\mu(x_j, x_j, x_{j+1}). \end{aligned} \tag{2.8}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0, 1]$ are given. Since $\Sigma_{n=1}^{\infty}\varphi^n(D_\lambda + 1) < \infty$, there exist $N_1, N_2 \in \mathbb{Z}^+$ such that $\Sigma_{i=n}^{m-1}\varphi^n(D_\lambda + 1) < \frac{\epsilon}{2}$ for all $m > n > N_1$ and $\Sigma_{j=m}^{l-1}\varphi^n(D_\lambda + 1) < \frac{\epsilon}{2}$ for all $l > m > N_2$. Then by (2.7) and (2.8), we have $G^*_\lambda(Ax_n, Ax_m, Ax_l) < \epsilon$, for all $l > m > n > N = \max\{N_1, N_2\}$. From Lemma 1.7, we obtain $G^*_{Ax_n,Ax_m,Ax_l}(\epsilon) > 1 - \lambda$, for all $l > m > n > N = \max\{N_1, N_2\}$. *i.e.*, $\{Ax_n\}$ is a *Cauchy* sequence. Similarly, we can also obtain $\{Ay_n\}$ is a *Cauchy* sequence. Since X is complete, there exist $u, v \in X$ such that $\lim_{n\to\infty} Ax_n = u$ and $\lim_{n\to\infty} Ay_n = v$. From the continuity of A, we have

$$\lim_{n\to\infty} AAx_n = Au \quad \textit{and} \quad \lim_{n\to\infty} AAy_n = Av. \tag{2.9}$$

The commutativity of A with T implies that $AAx_{n+1} = AT(x_n, y_n) = T(Ax_n, Ay_n)$. Since $\Sigma_{n=1}^{\infty}\varphi^n(t) < +\infty$, we have $\lim_{n\to\infty}\varphi^n(t) = 0$, so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (2.1) we have

$$\begin{aligned} G^*_{AAx_{n+1},AAx_{n+2},T(u,v)}(t) &\geq G^*_{AAx_{n+1},AAx_{n+2},T(u,v)}(\varphi^{n_0}(t)) \\ &= G^*_{T(Ax_n,Ay_n),T(Ax_{n+1},Ay_{n+1}),T(u,v)}(\varphi^{n_0}(t)) \\ &\geq [\Delta(G^*_{AAx_n,AAx_{n+1},Au}(\varphi^{n_0-1}(t)), G^*_{AAy_n,AAy_{n+1},Av}(\varphi^{n_0-1}(t)))]^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

Letting $n \to \infty$ in (2.10), we have $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} AAx_{n+1} = T(u, v)$. By (2.9), $T(u, v) = Au$. Similarly, we can also obtain $T(v, u) = Av$. Following, we show that $Au = v$ and $Av = u$. From (2.1) we have

$$\begin{aligned} G^*_{Au,Ay_n,Ay_{n+1}}(\varphi(t)) &= G^*_{T(u,v),T(y_{n-1},x_{n-1}),T(y_n,x_n)}(\varphi(t)) \\ &\geq [\Delta(G^*_{Au,Ay_{n-1},Ay_n}(t), G^*_{Av,Ax_{n-1},Ax_n}(t))]^{\frac{1}{2}} \\ &\geq [G^*_{Au,Ay_{n-1},Ay_n}(t) G^*_{Av,Ax_{n-1},Ax_n}(t)]^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

Similarly, we can have

$$G^*_{Av,Ax_n,Ax_{n+1}}(\varphi(t)) \geq [G^*_{Av,Ax_{n-1},Ax_n}(t) G^*_{Au,Ay_{n-1},Ay_n}(t)]^{\frac{1}{2}}. \tag{2.12}$$

Suppose that $Q_n(t) = G^*_{Au,Ay_n,Ay_{n+1}}(t) G^*_{Av,Ax_n,Ax_{n+1}}(t)$. By (2.11) and (2.12), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$, and

$$Q_n(\varphi^n(t)) \geq Q_{n-1}(\varphi^{n-1}(t)) \geq \cdots \geq Q_0(t). \tag{2.13}$$

Furthermore, from (2.11), (2.12), and (2.13), it follows that

$$G^*_{Au,Ay_n,Ay_{n+1}}(\varphi^n(t)) \geq [Q_0(t)]^{\frac{1}{2}}, \qquad G^*_{Av,Ax_n,Ax_{n+1}}(\varphi^n(t)) \geq [Q_0(t)]^{\frac{1}{2}}. \tag{2.14}$$

It is obvious that $Q_0(t) \in \mathcal{D}^+$. Since $\lim_{n\to\infty} \varphi^n(t) = 0$ from (2.14) and Lemma 1.9 we have

$$\lim_{n\to\infty} Ax_n = Av, \quad \lim_{n\to\infty} Ay_n = Au.$$

This shows that $u = Av = T(v, u)$ and $v = Au = T(u, v)$. Now, we prove that $u = v$. By (2.1) we have

$$\begin{aligned} G^*_{u,v,v}(\varphi(t)) &= G^*_{T(v,u),T(u,v),T(u,v)}(\varphi(t)) \\ &\geq [G^*_{Av,Au,Au}(t) G^*_{Au,Av,Av}(t)]^{\frac{1}{2}} = [G^*_{u,v,v}(t) G^*_{v,u,u}(t)]^{\frac{1}{2}}, \end{aligned} \tag{2.15}$$

$$G^*_{u,u,v}(\varphi(t)) \geq [G^*_{u,v,v}(t) G^*_{v,u,u}(t)]^{\frac{1}{2}}. \tag{2.16}$$

Suppose $F(t) = G^*_{u,v,v}(t) G^*_{v,u,u}(t)$, then $F(\varphi^n(t)) \geq F(t)$. Using Lemma 1.9, we have $F(t) = 1$, i.e. $u = v$. So, the proof is finished.

□

Theorem 2.2. *Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that*

$$G^*_{T(x,y),T(p,q),T(h,l)}(\varphi(t)) \geq [G^*_{Ax,Ap,Ah}(t) G^*_{Ay,Aq,Al}(t)]^{\frac{1}{2}}, \tag{2.17}$$

for all $x, y, p, q, h, l \in X$, where $T(X \times X) \subset A(X)$, A is continuous and commutative with T. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. The process of the proof is similar to Theorem 2.1, except the proof of $\{Ax_n\}$ and $\{Ay_n\}$ are *Cauchy* sequences. So, we just show the the difference in the following. By Lemma 1.11, we can construct two sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$. Suppose that $t > 0$. From (2.17), we have

$$\begin{aligned} G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi(t)) &= G^*_{T(x_{n-1},y_{n-1}),T(x_n,y_n),T(x_{n+1},y_{n+1})}(\varphi(t)) \\ &\geq [G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t) G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t)]^{\frac{1}{2}}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(\varphi(t)) &= G^*_{T(y_{n-1},x_{n-1}),T(y_n,x_n),T(y_{n+1},x_{n+1})}(\varphi(t)) \\ &\geq [G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t)G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t)]^{\frac{1}{2}}. \end{aligned} \tag{2.19}$$

Suppose that $P_n(t) = [G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t)G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t)]^{\frac{1}{2}}$. Then, from (2.18) and (2.19),we obtain $P_{n+1}(\varphi(t)) \geq P_n(t)$, which implies that

$$G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi^n(t)) \geq P_n(\varphi^{n-1}(t)) \geq \cdots \geq P_1(t), \tag{2.20}$$

$$G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(\varphi^n(t)) \geq P_n(\varphi^{n-1}(t)) \geq \cdots \geq P_1(t). \tag{2.21}$$

Since $P_1(t) \in \mathcal{D}^+$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, by Lemma 1.9 we have

$$\lim_{n\to\infty} G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) = 1, \qquad \lim_{n\to\infty} G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(t) = 1. \tag{2.22}$$

Thus, by (2.22), we have

$$\lim_{n\to\infty} P_n(t) = 1 \quad for\ all\ t > 0. \tag{2.23}$$

We claim that, for any $k \in \mathbb{Z}^+$,

$$G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(t) \geq \Delta^k(P_n(t - \varphi(t))), \qquad G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(t) \geq \Delta^k(P_n(t - \varphi(t))). \tag{2.24}$$

In fact, this is obvious for $k = 1$ by (2.18) and (2.19). Assume that (2.24) holds for some k. Since $\varphi(t) < t$, by (2.18), we have $G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) \geq G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi(t)) \geq P_n(t)$. By (2.17) and (2.24) we have

$$G^*_{Ax_{n+1},Ax_{n+k+1},Ax_{n+k+2}}(t) \geq [G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(t)G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(t)]^{\frac{1}{2}} \geq \Delta^k(P_n(t - \varphi(t))).$$

Then, we can obtain

$$\begin{aligned} G^*_{Ax_n,Ax_{n+k+1},Ax_{n+k+2}}(t) &= G^*_{Ax_n,Ax_{n+k+1},Ax_{n+k+2}}(t - \varphi(t) + \varphi(t)) \\ &\geq \Delta(G^*_{Ax_n,Ax_{n+1},Ax_{n+1}}(t - \varphi(t)), G^*_{Ax_{n+1},Ax_{n+k+1},Ax_{n+k+2}}(t)) \\ &\geq \Delta(G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t - \varphi(t)), \Delta^k(P_n(t - \varphi(t)))) \\ &\geq \Delta(P_n(t - \varphi(t)), \Delta^k(P_n(t - \varphi(t)))) \\ &= \Delta^{k+1}(P_n(t - \varphi(t))). \end{aligned}$$

By the same process, we can obtain $G^*_{Ay_n,Ay_{n+k+1},Ay_{n+k+2}}(t) \geq \Delta^{k+1}(P_n(t-\varphi(t)))$. Therefore, by induction, (2.24) holds for all $k \in \mathbb{Z}^+$. Suppose that $\epsilon > 0$ and $\lambda \in (0, 1]$ are given. By the hypothesis, Δ is a t-norm of H-type, there exists $\delta > 0$ such that

$$\Delta^k(s) > 1 - \lambda, \quad s \in (1 - \delta, 1],\ k \in \mathbb{Z}^+. \tag{2.25}$$

By (2.23), there exists $N \in \mathbb{Z}^+$ such that $P_n(\epsilon - \varphi(\epsilon)) > 1 - \delta$ for all $n > N$. Hence, from (2.24) and (2.25) we get $G^*_{Ax_n,Ax_{n+k+1},Ax_{n+k+2}}(\epsilon) > 1 - \lambda$ and $G^*_{Ay_n,Ay_{n+k+1},Ay_{n+k+2}}(\epsilon) > 1 - \lambda$, for all $n \geq N$ and $k \in \mathbb{Z}^+$. Therefore, $\{Ax_n\}$ and $\{Ay_n\}$ are *Cauchy* sequences.

The next proof is similar to Theorem 2.1. □

Theorem 2.3. *Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$ and $\Sigma_{n=1}^{\infty}\varphi^n(t) = +\infty$ for any $t > 0$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that*

$$G^*_{T(x,y),T(p,q),T(h,l)}(t) \geq \min\{(G^*_{Ax,Ap,Ah}(\varphi(t)), G^*_{Ay,Aq,Al}(\varphi(t)))\}, \tag{2.26}$$

for all $x, y, p, q, h, l \in X$, where $T(X \times X) \subset A(X)$, A is continuous and commutative with T. Then there exists a unique $u \in X$ such that $u = Au = T(u, u)$.

Proof. By Lemma 1.11, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$. Suppose that $t > 0$. From (2.26), we have

$$\begin{aligned} G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) &= G^*_{T(x_{n-1},y_{n-1}),T(x_n,y_n),T(x_{n+1},y_{n+1})}(t) \\ &\geq \min\{G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(\varphi(t)), G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(\varphi(t))\}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(t) &= G^*_{T(y_{n-1},x_{n-1}),T(y_n,x_n),T(y_{n+1},x_{n+1})}(t) \\ &\geq \min\{G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(\varphi(t)), G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(\varphi(t))\}. \end{aligned} \tag{2.28}$$

Suppose that $E_n(t) = \min\{G^*_{Ax_{n-1},Ax_n,Ax_{n+1}}(t), G^*_{Ay_{n-1},Ay_n,Ay_{n+1}}(t)\}$. Then, from (2.27) and (2.28), we obtain $E_{n+1}(t) \geq E_n(\varphi(t))$, which implies that

$$E_{n+1}(t) \geq E_n(\varphi(t)) \geq E_{n-1}(\varphi^2(t)) \geq \cdots \geq E_1(\varphi^n(t)). \tag{2.29}$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for each $t > 0$, we have $\lim_{n\to\infty} E_1(\varphi^n(t)) = 1$. Moreover, by (2.27), (2.28), (2.29), we have $G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) \geq E_1(\varphi^n(t))$ and $G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(t) \geq E_1(\varphi^n(t))$. Hence, $\lim_{n\to\infty} G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) = 1$ and $\lim_{n\to\infty} G^*_{Ay_n,Ay_{n+1},Ay_{n+2}}(t) = 1$. This implies that

$$\lim_{n\to\infty} E_n(t) = 1, \quad t > 0. \tag{2.30}$$

In the next step we show that, for any $k \in \mathbb{Z}^+$,

$$G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(\varphi(t)) \geq \Delta^k(E_n(\varphi(t) - t)),, \qquad G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(t) \geq \Delta^k(E_n(\varphi(t) - t)). \tag{2.31}$$

In fact, this is obvious for $k = 1$ by (2.27) and (2.28). Assume that (2.31) holds for some k. Since $\varphi(t) > t$, by (2.27), we have $G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(t) \geq E_n(\varphi(t)) \geq E_n(t)$. By (2.26) and (2.31) we have

$$G^*_{Ax_{n+1},Ax_{n+k+1},Ax_{n+k+2}}(t) \geq \min\{G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(\varphi(t)), G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(\varphi(t))\} \geq \Delta^k(E_n(\varphi(t) - t)).$$

By the monotonicity of Δ, we can obtain

$$\begin{aligned} G^*_{Ax_n,Ax_{n+k+1},Ax_{n+k+2}}(\varphi(t)) &= G^*_{Ax_n,Ax_{n+k+1},Ax_{n+k+2}}(\varphi(t) - t + t) \\ &\geq \Delta(G^*_{Ax_n,Ax_{n+1},Ax_{n+1}}(\varphi(t) - t), G^*_{Ax_{n+1},Ax_{n+k+1},Ax_{n+k+2}}(t)) \\ &\geq \Delta(G^*_{Ax_n,Ax_{n+1},Ax_{n+2}}(\varphi(t) - t), \Delta^k(E_n(\varphi(t) - t))) \\ &\geq \Delta(E_n(\varphi(t) - t), \Delta^k(E_n(\varphi(t) - t))) \\ &= \Delta^{k+1}(E_n(\varphi(t) - t)). \end{aligned}$$

By the same process, we can obtain $G^*_{Ay_n,Ay_{n+k+1},Ay_{n+k+2}}(\varphi(t)) \geq \Delta^{k+1}(E_n(\varphi(t) - t))$. Therefore, by induction, (2.31) holds for all $k \in \mathbb{Z}^+$. Furthermore, by (2.26) and (2.31) we have

$$G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(t) \geq \Delta^k(E_{n-1}(\varphi(t) - t)), \quad G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(t) \geq \Delta^k(E_{n-1}(\varphi(t) - t)). \tag{2.32}$$

Suppose that $\epsilon > 0$ and $\lambda \in (0, 1]$ are given. By the hypothesis, Δ is a t-norm of H-type, there exists $\delta > 0$ such that

$$\Delta^k(s) > 1 - \lambda, \quad s \in (1 - \delta, 1],\ k \in \mathbb{Z}^+. \tag{2.33}$$

By (2.30), there exists $N \in \mathbb{Z}^+$ such that $E_{n-1}(\varphi(\epsilon) - \epsilon) > 1 - \delta$ for all $n \geq N$. Hence, from (2.32) and (2.33) we get $G^*_{Ax_n,Ax_{n+k},Ax_{n+k+1}}(\epsilon) > 1 - \lambda$ and $G^*_{Ay_n,Ay_{n+k},Ay_{n+k+1}}(\epsilon) > 1 - \lambda$, for all $n > N$ and $k \in \mathbb{Z}^+$. Then, $\{Ax_n\}$ and $\{Ay_n\}$ are *Cauchy* sequences. Since X is complete, there exist $u, v \in X$ such that $\lim_{n\to\infty} Ax_n = u$ and $\lim_{n\to\infty} Ay_n = v$. From the continuity of A, we have

$$\lim_{n\to\infty} AAx_n = Au, \qquad \lim_{n\to\infty} AAy_n = Av.$$

From (2.26) and the commutativity of A with T it follows that

$$\begin{aligned} G^*_{AAx_{n+1},AAx_{n+2},T(u,v)}(t) &= G^*_{T(Ax_n,Ay_n),T(Ax_{n+1},Ay_{n+1}),T(u,v)}(t) \\ &\geq \min\{G^*_{AAx_n,AAx_{n+1},Au}(\varphi(t)), G^*_{AAy_n,AAy_{n+1},Av}(\varphi(t))\}. \end{aligned} \tag{2.34}$$

Letting $n \to \infty$ in (2.34), we have $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} AAx_{n+1} = T(u,v)$. Hence, $T(u,v) = Au$. Similarly, we can also obtain $T(v,u) = Av$. Following, we show that $Au = v$ and $Av = u$. From (2.26) we have

$$\begin{aligned} G^*_{Au,Ay_n,Ay_{n+1}}(t) &= G^*_{T(u,v),T(y_{n-1},x_{n-1}),T(y_n,x_n)}(t) \\ &\geq \min\{G^*_{Au,Ay_{n-1},Ay_n}(\varphi(t)), G^*_{Av,Ax_{n-1},Ax_n}(\varphi(t))\}. \end{aligned} \tag{2.35}$$

Similarly, we can have

$$G^*_{Av,Ax_n,Ax_{n+1}}(t) \geq \min\{G^*_{Av,Ax_{n-1},Ax_n}(\varphi(t)), G^*_{Au,Ay_{n-1},Ay_n}(\varphi(t))\}. \tag{2.36}$$

Suppose that $M_n(t) = \min\{G^*_{Au,Ay_{n-1},Ay_n}(\varphi(t)), G^*_{Av,Ax_{n-1},Ax_n}(\varphi(t))\}$. By (2.35) and (2.36), we have $M_n(t) \geq M_{n-1}(\varphi(t)) \geq \cdots \geq M_0(\varphi^n(t))$. Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$M_0(\varphi^n(t)) = \min\{G^*_{Au,Ay_0,Ay_1}(\varphi^n(t)), G^*_{Av,Ax_0,Ax_1}(\varphi^n(t))\} \to 1 \ (n \to \infty).$$

This shows that $M_n(t) \to 1$ as $n \to \infty$, and so

$$\lim_{n\to\infty} Ax_n = Av, \quad \lim_{n\to\infty} Ay_n = Au.$$

This shows that $u = Av = T(v,u)$ and $v = Au = T(u,v)$. Now, we prove that $u = v$. By (2.26) we have

$$\begin{aligned} G^*_{u,v,v}(t) &= G^*_{T(v,u),T(u,v),T(u,v)}(t) \\ &\geq \min\{G^*_{Av,Au,Au}(\varphi(t)), G^*_{Au,Av,Av}(\varphi(t))\} = \min\{G^*_{u,v,v}(\varphi(t)), G^*_{v,u,u}(\varphi(t))\}, \end{aligned} \tag{2.37}$$

$$G^*_{u,u,v}(t) \geq \min\{G^*_{u,v,v}(\varphi(t)), G^*_{v,u,u}(\varphi(t))\}. \tag{2.38}$$

Suppose $F(t) = \min\{G^*_{u,v,v}(t), G^*_{v,u,u}(t)\}$, since $F(t) \geq F(\varphi(t))$, then $G^*_{u,v,v}(t) \geq F(\varphi(t)) \geq F(\varphi^n(t))$. Letting $n \to \infty$, we have $G^*_{u,v,v}(t) = 1$, *i.e.*, $u = v$. So, the proof is completed. □

For each $x \in X$, if we take the mapping $A : X \to X$ as $Ax = x$, then we can obtain the following consequence from Theorem 2.1.

Corollary 2.4. *Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\Sigma_{n=1}^{\infty}\varphi^n(t) < +\infty$ for any $t > 0$. Let $T : X \times X \to X$ be a mapping such that*

$$G^*_{T(x,y),T(p,q),T(h,l)}(\varphi(t)) \geq [\Delta(G^*_{x,p,h}(t), G^*_{y,q,l}(t))]^{\frac{1}{2}},$$

for all $x, y, p, q, h, l \in X$. Then there exists a unique $u \in X$ such that $u = Au = T(u,u)$.

Since each hybrid contraction with a gauge function φ includes the case of linear contraction as a special case if we take $\varphi(t) = \alpha t$ or $\varphi(t) = \frac{t}{\alpha}$ where $\alpha \in (0,1)$. For example, from Theorem 2.2 we obtain the following consequence.

Corollary 2.5. *Let (X, G^*, Δ) be a complete Menger PGM-space such that Δ is a t-norm of H-type and $\alpha \in (0,1)$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that*

$$G^*_{T(x,y),T(p,q),T(h,l)}(\alpha t) \geq [G^*_{Ax,Ap,Ah}(t) G^*_{Ay,Aq,Al}(t)]^{\frac{1}{2}},$$

for all $x, y, p, q, h, l \in X$, where $T(X \times X) \subset A(X)$, A is continuous and commutative with T. Then there exists a unique $u \in X$ such that $u = Au = T(u,u)$.

3. An application

In this section, we give an example to illustrate the validity of Theorem 2.1.

Example 3.1. Suppose that $\Delta = \Delta_p$. Then Δ_p is a t-norm of H-type. Define a function $G^* : X \times X \times X \to \mathbb{R}^+$ by

$$G^*_{x,y,z}(t) = \begin{cases} e^{-\frac{G(x,y,z)}{t}}, & t > 0, \\ 1, & t \le 0. \end{cases}$$

for all $x, y, z \in X$, where $G(x, y, z) = |x - y| + |y - z| + |z - x|$, then G^* is a G-metric (see [19]). It is easy to see that G^* satisfies (PGM-1)-(PGM-3). Next we show $G^*(x, y, z)(t+s) \ge \Delta\{G^*_{x,a,a}(t), G^*_{a,y,z}(s)\} = G^*_{x,a,a}(t)G^*_{a,y,z}(s)$ for all $x, y, z, a \in X$ and all $s, t > 0$.

Since

$$\begin{aligned} \frac{|x-y|+|y-z|+|z-x|}{t+s} &\le \frac{|x-a|+|a-y|+|y-z|+|z-a|+|a-x|}{t+s} \\ &= \frac{2|x-a|}{t+s} + \frac{|a-y|+|y-z|+|z-a|}{t+s} \\ &< \frac{2|x-a|}{t} + \frac{|a-y|+|y-z|+|z-a|}{s}, \end{aligned}$$

then, $G^*(x, y, z)(t + s) = e^{-\frac{|x-y|+|y-z|+|z-x|}{t+s}} \ge e^{-\{\frac{2|x-a|}{t}+\frac{|a-y|+|y-z|+|z-a|}{s}\}} = G^*_{x,a,a}(t)G^*_{a,y,z}(s)$. Then G^* is a probabilistic G-metric.

Suppose that $\varphi(t) = \frac{t}{2}$. For each $x, y \in X$, define $T : X \times X \to X$ as follows: $T(x, y) = x+y$, $A : X \to X$ as: $Ax = 4x$ and $T(X \times X) \subset A(X)$. A is continuous and commutative with T. For each $x, y, p, q, h, l \in X$ and $t > 0$, we have
$\frac{|(x+y)-(p+q)|+|(p+q)-(h+l)|+|(h+l)-(x+y)|}{\frac{t}{2}} \le \frac{4\{|x-p|+|p-h|+|h-x|+|y-q|+|q-l|+|l-y|\}}{t} \times \frac{1}{2}$, and so

$$\begin{aligned} G^*_{T(x,y),T(p,q),T(h,l)}(\frac{t}{2}) &= e^{-\frac{|(x+y)-(p+q)|+|(p+q)-(h+l)|+|(h+l)-(x+y)|}{\frac{t}{2}}} \\ &\ge e^{-\frac{4\{|x-p|+|p-h|+|h-x|+|y-q|+|q-l|+|l-y|\}}{t}\times\frac{1}{2}} \\ &= [e^{-\frac{4\{|x-p|+|p-h|+|h-x|\}}{t}} e^{-\frac{4\{|y-q|+|q-l|+|l-y|\}}{t}}]^{\frac{1}{2}} \\ &= [\Delta_p(G^*_{Ax,Ap,Ah}(t), G^*_{Ay,Aq,Al}(t))]^{\frac{1}{2}} \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied. Therefore, 0 is the unique common coupled fixed point of T and A.

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References

[1] R. P. Agarwal, Z. Kadelburg, S. Radenović, *On coupled fixed point results in asymmetric G-metric spaces*, J. Inequal. Appl., **2013** (2013), 12 pages. 1

[2] R. P. Agarwal, E. Karapınar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed point theory Appl., **2013** (2013), 33 pages. 1

[3] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393. 1

[4] S. Chauhan, B. D. Pant, *Fixed point theorems for compatible and subsequentially continuous mappings in Menger spaces*, J. Nonlinear Sci. Appl., **7** (2014), 78–89. 1

[5] L. Ćirić, *Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces*, Nonlinear Anal., **72** (2010), 2009–2018. 1

[6] J. Jachymski, *On probabilistic φ-contractions on Menger spaces*, Nonlinear Anal., **73** (2010), 2199–2203. 1, 1.9

[7] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl., **59** (2010), 3656–3668. 1

[8] T. Luo, C. X. Zhu, Z. Q. Wu, *Tripled common fixed point theorems under probabilistic φ-contractive conditions in generalized Menger probabilistic metric spaces*, Fixed Point Theory Appl., **2014** (2014), 17 pages.

[9] K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. USA., **28** (1942), 535–537. 1

[10] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1, 1.2

[11] Z. Mustafa, B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed point theory Appl., **2009** (2009), 10 pages. 1

[12] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., **72** (2010), 4508–4517. 1

[13] S. Sedgi, I. Altun, N. Shobec, *Coupled fixed point theorems for contractions in fuzzy metric spaces*, Nonlinear Anal., **72** (2010), 1298–1304. 1, 1.10

[14] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10** (1960), 313–334. 1

[15] N. Tahat, H. Aydi, E. Karapınar, *et al.*, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces*, Fixed Point Theory Appl., **2012** (2012), 9 pages. 1

[16] Z. Q. Wu, C. X. Zhu, J. Li, *Common fixed point theorems for two hybrid pairs of mappings satisfying the common property (E.A) in Menger PM-spaces*, Fixed Point Theory Appl., **2013** (2013), 15 pages. 1

[17] J. Z. Xiao, X. H. Zhu, Y. F. Cao, *Common coupled fixed point results for probabilistic φ-contractions in Menger spaces*, Nonlinear Anal., **74** (2011), 4589–4600. 1, 1.11

[18] J. Z. Xiao, X. H. Zhu, X. Y. Liu, *An alternative characterization of probabilistic Menger spaces with H-type triangular norms*, Fuzzy Sets and Systems, **227** (2013), 107–114. 1

[19] C. Zhou, S. Wang, L. Ćirić, *et al.*, *Generalized probabilistic metric spaces and fixed point theorems*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1, 1.3, 1.5, 1.6

[20] C. X. Zhu, *Several nonlinear operator problems in the Menger PN space*, Nonlinear Anal., **65** (2006), 1281–1284. 1

[21] C. X. Zhu, *Research on some problems for nonlinear operators*, Nonlinear Anal., **71** (2009), 4568–4571. 1

[22] C. X. Zhu, W. Q. Xu, Z. Q. Wu, *Some fixed point theorems in generalized probabilistic metric spaces*, Abst. Appl. Anal., **2014** (2014), 8 pages. 1, 1.7, 1.8