Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 246–254
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Local convergence of deformed Halley method in Banach space under Holder continuity conditions

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Communicated by Yeol Je Cho

Abstract

We present a local convergence analysis for deformed Halley method in order to approximate a solution of a nonlinear equation in a Banach space setting. Our methods include the Halley and other high order methods under hypotheses up to the first Fréchet-derivative in contrast to earlier studies using hypotheses up to the second or third Fréchet-derivative. The convergence ball and error estimates are given for these methods. Numerical examples are also provided in this study.

Keywords: Chebyshev method, Banach space, convergence ball, local convergence.
2010 MSC: 65D10, 65D99, 65G99, 47H17, 49M15

1. Introduction

Many problems in computational sciences and other disciplines can be brought in the form of

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y using mathematical modeling [2, 3, 4, 5, 11, 14, 15].

In this study we are concerned with approximating a solution x^* of the equation (1.1). In general the solutions of (1.1) can not be found in closed form, so one has to consider some iterative methods for solving (1.1). Usually the convergence analysis of iterative methods are two types: semi-local and local convergence analysis. The semi-local convergence analysis is, based on the information around an initial

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Received 2012-3-5

point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of equation (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev-Halley-type methods [1, 3, 5, 7]–[16] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive [10]–[13] or for quadratic equations the second Fréchet-derivative is constant [4, 12]. Moreover, in some applications involving stiff systems [2], [5], [9], high order methods are usefull. That is why we study the local convergence of deformed Halley method DHM defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n + \alpha F'(x_n)^{-1}F(x_n), \\ H_n &= \frac{1}{\lambda}F'(x_n)^{-1}[F'(x_n + \lambda(z_n - x_n)) - F'(x_n)] \\ x_{n+1} &= y_n + \frac{1}{2}H_n(I - \frac{1}{2}H_n)^{-1}(y_n - x_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\lambda \in (0, 1]$ and $\alpha \in \mathbb{R}$ are given parameters. Deformed methods have been introduced to improve on the convergence order of Newton's method or Newton-like methods [2, 3, 10, 11, 14, 15, 16]. In particular, DHM was proposed in [17] as an alternative to the famous Halley method defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ L_n &= F'(x_n)^{-1}F''(x_n)F'^{-1}(x_n)F(x_n) \\ x_{n+1} &= y_n + \frac{1}{2}L_n(I - \frac{1}{2}L_n)^{-1}(y_n - x_n). \end{aligned} \tag{1.3}$$

Notice that the computation of the expensive in general second Fréchet derivative $F''(x_n)$ is required in method (1.3) but not in DHM.

The semilocal convergence analysis of DHM was given in [17] under Lipschitz continuity conditions on up to the second Fréchet-derivative in the special case when $\alpha = 1$ and $\lambda > 0$.

The usual conditions for the semi-local convergence of these methods are $(\mathcal{C})$: There exist constants β, η, β_1, β_2 such that

$(\mathcal{C}_1)$ There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \le \beta$;

$(\mathcal{C}_2)$
$$\|\Gamma_0 F(x_0)\| \le \eta;$$

$(\mathcal{C}_3)$
$$\|F''(x)\| \le \beta_1 \quad \text{for each } \ x \in D;$$

$(\mathcal{C}_4)$
$$\|F''(x) - F''(y)\| \le \beta_2\|x - y\|^p \quad \text{for each } \ x, y \in D \text{ and some } p \in (0, 1].$$

The local convergence conditions are similar but x_0 is x^* in $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. There is a plethora of local and semi-local convergence results under the $(\mathcal{C})$ conditions [1]–[17]. The conditions $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ restrict the applicability of these methods.

As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Notice that $f'''(x)$ is unbounded on D. That is condition $(\mathcal{C}_4)$ is not satisfied. Hence, the results depending on $(\mathcal{C}_4)$ cannot apply in this case. However, using (2.8)-(2.11) that follow we have $f'(x^*) = 3$ and $f(x^*) = 0$. That is, conditions (2.8)-(2.8) are satisfied for $p = 1$, $L_0 = L = 146.6629073$, $M = 101.5578008$. Hence, the results of our Theorem 2.1 that follows can apply to solve equation $f(x) = 0$ using DHM. Hence, the applicability of DHM is expanded under our new conditions.

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

The rest of the paper is organized as follows: In Section 2 we present the local convergence of these methods. The numerical examples are given in the concluding Section 3.

2. Local convergence

In this section we present the local convergence analysis of DHM. Let $L_0 > 0, L > 0, M > 0, \alpha \in \mathbb{R}, \lambda \in (0, 1]$ and $p \in [0, 1]$ be given parameters. It is convenient for the local convergence analysis that follows to introduce some functions and parameters.

Define functions on the interval $[0, (\frac{1}{L_0})^p)$ by

$$\begin{aligned} g_1(t) &= \frac{Lt^p}{(1+p)(1-L_0t^p)}, \\ g_2(t) &= g_1(t) + \frac{|1+\alpha|M}{1-L_0^p}, \\ g_3(t) &= \frac{L||\alpha|^p\lambda|^{p-1}M^pt^p}{2(1-L_0t^p)^{1+p}}, \\ \bar{g}_3(t) &= L||\alpha|^p\lambda|^{p-1}M^pt^p - 2(1-L_0t^p)^{1+p}, \\ g_4(t) &= g_1(t) + \frac{g_3(t)M}{(1-g_3(t))(1-L_0t^p)}, \\ \bar{g}_4(t) &= g_4(t) - 1 \end{aligned} \tag{2.1}$$

and parameters

$$r_1 = (\frac{1+p}{(1+p)L_0 + L})^{\frac{1}{p}} < (\frac{1}{L_0})^{\frac{1}{p}}$$

and

$$r_2 = (\frac{(1+p)(1-M|1+\alpha|)}{(1+p)L_0 + L})^{\frac{1}{p}}.$$

Suppose that

$$M|1+\alpha| < 1. \tag{2.2}$$

Then, r_2 is well defined and

$$0 < r_2 < r_1.$$

We also have that

$$0 \le g_1(t) < 1,$$

and

$$0 \le g_2(t) < 1 \text{ for each } t \in [0, r_2].$$

Using the definition of function $\bar{g}_3$ we get that $\bar{g}_3(0) = -2 < 0$ and $\bar{g}_3((\frac{1}{L_0})^{\frac{1}{p}}) = \frac{L|\lambda|^{p-1}|\alpha|^p M^p}{L_0} > 0$. It then follows from the Intermediate Value Theorem that function $\bar{g}_3$ has zeros in $(0, (\frac{1}{L_0})^{\frac{1}{p}})$. Denote by r_3 the smallest such zero. Then, we have that

$$0 \leq g_3(t) < 1 \text{ for each } t \in [0, r_3). \tag{2.3}$$

Similarly using the definition of function $\bar{g}_4$ we have that $\bar{g}_4(0) = -1 < 0$ and $\bar{g}_4(t) \to \infty$ as $t \to ((\frac{1}{L_0})^{\frac{1}{p}})^-$. Hence, function $\bar{g}_4$ has zeros in $(0, (\frac{1}{L_0})^{\frac{1}{p}})$. Denote by r_4 the smallest such zero. Define

$$r = \min\{r_2, r_3, r_4\} \tag{2.4}$$

Then, we have that

$$0 \leq g_1(t) < 1, \tag{2.5}$$

$$0 \leq g_2(t) < 1, \tag{2.6}$$

$$0 \leq g_3(t) < 1 \tag{2.7}$$

and

$$0 \leq g_4(t) < 1 \text{ for each } t \in [0, r). \tag{2.8}$$

Next using the preceding notation, we present the local convergence result for DHM.

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M > 0$, $\alpha \in \mathbb{R}|$, $\lambda \in (0, 1]$ and $p \in (0, 1]$ such that for each $x, y \in D$*

$$M|1 + \alpha| < 1,$$

$$F(x^*) = 0, \;\; F'(x^*)^{-1} \in L(Y, X), \tag{2.9}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|^p, \tag{2.10}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|^p, \tag{2.11}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{2.12}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.13}$$

where the radius r is given by (2.4). Then, sequence $\{x_n\}$ generated by DHM for $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$.

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.14}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.15}$$

$$\|\frac{1}{2}H_n\| \leq g_3(\|x_n - x^*\|) < 1 \tag{2.16}$$

and

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.17}$$

where the “g“ functions are given by (2.1).

Proof. By hypothesis $x_0 \in U(x^*, r)$. Using the definition of radius r and (2.9), we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x - x^*\|^p < L_0 r^p < 1. \tag{2.18}$$

It follows from (2.18) and the Banach Lemma on invertible operators [14] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x^*)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x - x^*\|^p} < \frac{1}{1 - L_0 r^p}. \tag{2.19}$$

Moreover y_0, z_0 are well defined by first and second substep of DHM for $n = 0$. Using the first substep of DHM for $n = 0$, we get that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= [F'(x_0)^{-1}F'(x^*)][\int_0^1 F'(x^*)^{-1} \\ &\quad \times [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta]. \end{aligned} \tag{2.20}$$

Then, by the definition of function g_1, (2.4), (2.10), (2.19) and (2.20), we obtain that

$$\begin{aligned} \|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1} \\ &\quad \times [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta\|\|x_0 - x^*\| \\ &\le \frac{L\|x_0 - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_0 - x^*\|)} \\ &\le g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &< \|x_k - x^*\| < r, \end{aligned}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*, r)$. Similarly, using the second substep of DHM for $n = 0$, we get that

$$z_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 + \alpha)F'(x_0)^{-1}F(x_0). \tag{2.21}$$

Then, by (2.5), (2.12), (2.19), (2.21) the definition of function g_2 and (2.14) (for $n = 0$), we obtain for $F(x_0) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta$

$$\begin{aligned} \|z_0 - x^*\| &\le \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\ &\quad + \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta\| \\ &\quad \times \|x_0 - x^*\| \\ &\le g_2(\|x_0 - x^*\|) + \frac{|1 + \alpha|M}{1 - L_0\|x_0 - x^*\|}]\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned}$$

which shows (2.15) for $n = 0$ and $z_0 \in U(x^*, r)$. We have by the definition of λ and (2.14), (2.15)(for $n = 0$) that

$$x_0 - x^* + \lambda(z_0 - x_0)\| \le |1 - \lambda|\|x_0 - x^*\| + |\lambda|\|z_0 - x^*\| < (|1 - \lambda| + |\lambda|)r \le r,$$

which shows that $x_0 + \lambda(z_0 - x_0) \in U(x^*, r)$ and H_0 is well defined. We need an estimate on $\|H_0\|$. Using the definition of H_0, g_3, (2.19) and (2.11) we get in turn that

$$\|\frac{1}{2}H_0\| \le \frac{1}{2|\lambda|}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F'(x^* + \lambda(z_0 - x_0) - F'(x_0)]\|$$

$$\le \frac{1}{2|\lambda|}\frac{L|\lambda|^p\|z_0 - x_0\|^p}{1 - L_0\|x_0 - x^*\|^p}$$

$$
\begin{aligned}
&\leq \frac{|\alpha|^p L|\lambda|^{p-1}(\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\|)^p}{2(1-L_0\|x_0-x^*\|^p)} \\
&\leq \frac{|\alpha|^p L|\lambda|^{p-1}M^p\|x_0-x^*\|^p}{2(1-L_0\|x_0-x^*\|^p)^{1+p}} \\
&= g_3(\|x_0-x^*\|) < 1,
\end{aligned}
$$

which shows (2.16) for $n=0$. Hence, we have

$$\|(I-\frac{1}{2}H_0)^{-1}\| \leq \frac{1}{1-g_3(\|x_0-x^*\|)}.$$

Then, using the last substep of DHM for $n=0$, we get

$$
\begin{aligned}
\|x_1-x^*\| &\leq \|y_0-x^*\| + \|\frac{1}{2}H_0\|\|(I-\frac{1}{2}H_0)^{-1}\|\|y_0-x_0\| \\
&\leq g_1(\|x_0-x^*\|)\|x_0-x^*\| + \frac{g_3(\|x_0-x^*\|)}{1-g_3(\|x_0-x^*\|)} \\
&\quad \times\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq g_1(\|x_0-x^*\|)\|x_0-x^*\| + \frac{g_3(\|x_0-x^*\|)}{1-g_3(\|x_0-x^*\|)} \\
&\quad \times\frac{M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|} \\
&= g_4(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r,
\end{aligned}
$$

which shows (2.17) for $n=0$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.14)-(2.17). Finally using the estimate $\|x_{k+1}-x^*\| < \|x_k-x^*\| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$.

□

Remark 2.2. (a) Condition (2.10) can be dropped, since this condition follows from $(\mathcal{A}_3)$. Notice, however that

$$L_0 \leq L \tag{2.22}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2]–[6].

(b) In view of condition (2.10) and the estimate

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}[F'(x)-F'(x^*)] + I\| \\
&\leq 1 + \|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\
&\leq 1 + L_0\|x-x^*\|^p,
\end{aligned}
$$

condition (2.12) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t^p. \tag{2.23}$$

(c) The convergence ball of radius r_1 was given by us in [2, 3, 5] for Newton's method under conditions (2.10) and (2.11). Estimate $r_2 < r_1$ shows that the convergence ball of higher than two DHM methods are smaller than the convergence ball DHM. The convergence ball given by Rheinboldt [15] for Newton's method is

$$r_R = \frac{2}{3L} < r_1 \ \text{(for } p = 1) \tag{2.24}$$

if $L_0 < L$ and $\frac{r_R}{r_1} \to \frac{1}{3}$ as $\frac{L_0}{L} \to 0$. Hence, we do not expect r to be larger than r_1 no matter how we choose L_0, L, M and α.

(d) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2]– [5], [14, 15].

(e) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2]– [5], [14, 15]:

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, $F''(x^*) = F'(x^*)T'(F(x^*)) = T(0)T'(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

(f) We can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right),$$

since the bounds given in Theorem 2.1 may be very pessimistic.

(g) The restriction $\lambda \in (0, 1]$ can be dropped, if (2.13) is replaced by

$$U_1 = \bar{U}(x^*, (|\lambda| + |1 - \lambda|)r) \subseteq D \tag{2.25}$$

for $\lambda \in \mathbb{R}$. Indeed, we will then have

$$\begin{aligned} \|x_n + \lambda(y_n - x_n) - x^*\| &\le |\lambda|\|x_n - x^*\| + |1 - \lambda|\|y_n - x^*\| \\ &\le (|\lambda| + |1 - \lambda|)r \\ &\Rightarrow x_n + \lambda(y_n - x_n) \in U_1. \end{aligned}$$

3. Numerical Examples

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1. Let $X = Y = \mathbb{R}$. Define function F on $D = [1, 3]$ by

$$F(x) = \frac{2}{3}x^{\frac{2}{3}} - x. \tag{3.1}$$

Then, $x^* = \frac{9}{4}$, $F'(x^*)^{-1} = 2$, $L_0 = 1 < L = 2$, $p = 0.5$, $\alpha = -0.6585$, $\lambda = 1$ and $M = 2(\sqrt{3} - 1)$, $r_1 = 0.6547$, $r_2 = 0.4629$, $r_3 = 0.1882$, $r_4 = 0.0215$ and $r = 0.0215$.

Example 3.2. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define F on D for $v = (x,y,z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T. \tag{3.2}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = M = e, p = 1, \alpha = -0.8161$, $\lambda = 0.5$. The values of $r_1 = 0.3249$, $r_2 = 0.1625$, $r_3 = 0.1679$, $r_4 = 0.0819$ and $r = 0.0819$.

Example 3.3. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.3}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $p = 1$, $x^* = 0$, $L_0 = 7.5$, $L = 15, \alpha = -0.9412$, $\lambda = 0.5$ and $M = M(t) = 1 + 7.5t$. The values of $r_1 = 0.0667$, $r_2 = 0.0333$, $r_3 = 0.0135$, $r_4 = 0.0065$ and $r = 0.0065$.

Example 3.4. Returning back to the motivational example at the introduction of this study, we have $p = 1$, $L_0 == L = 146.6629073$, $M = 101.5578008$, $\alpha = -0.9951$, $\lambda = 0.5$. The values of $r_1 = 0.0045$, $r_2 = 0.0023$, $r_3 = 0.0001$, $r_4 = 0.00001$ and $r = 0.00001$.

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