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Some inequalities of Hermite–Hadamard type for functions whose second derivatives are (α, m)-convex

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Abstract

In the paper, the authors establish a new integral identity and by this identity with the Hölder integral inequality, discover some new Hermite–Hadamard type integral inequalities for functions whose second derivatives are (α, m)-convex.

Keywords: Hermite–Hadamard type inequality, (α, m)-convex function, second derivative, Hölder integral inequality.
2010 MSC: 26D15, 26A51, 41A55.

1. Introduction

It is common knowledge in mathematical analysis that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval I such that $a, b \in I$ and $a < b$. Then the well-known Hermite–Hadamard integral inequality reads that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \,\mathrm{d}\, x \le \frac{f(a)+f(b)}{2}.$$

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Definition 1.1 ([9]). For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an m-convex function on $[0, b]$.

Definition 1.2 ([5]). For $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m\bigl(1-\lambda^\alpha\bigr)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

For the above convex functions, we recite the following theorems.

Theorem 1.3 ([2, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)}{8}\bigl(|f'(a)| + |f'(b)|\bigr).$$

Theorem 1.4 ([1, Theorem 2.2]). *Let $I \supseteq \mathbb{R}_0 = [0, \infty)$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a, b]$ for some $m \in (0, 1]$ and $q \ge 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right)^{1/q}, \left(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right)^{1/q}\right\}.$$

Theorem 1.5 ([1, Theorem 3.1]). *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $|f'|^q$ is (α, m)-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \times \min\Biggl\{&\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q},\\ &\left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\Biggr\},\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \textit{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right).$$

In recent years, some other kinds of Hermite–Hadamard type inequalities were generated in, for example, [4, 7, 8, 10, 11, 12]. For more systematic information, please refer to monographs [3, 6] and related references therein.

In this paper, we will establish a new integral identity and discover some new Hermite–Hadamard type integral inequalities for functions whose second derivatives are (α, m)-convex.

2. A lemma

For establishing some new integral inequalities of Hermite–Hadamard type for functions whose second derivatives are (α, m)-convex, we need the following lemma.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $f'' \in L([a,b])$, then*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \\
&\qquad = \frac{(b-a)^2}{54}\biggl[\int_0^1 (2-t-t^2) f''\left(ta + (1-t)\frac{2a+b}{3}\right)\mathrm{d}\,t \\
&\qquad\quad + \int_0^1 (2+t-t^2) f''\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\mathrm{d}\,t \\
&\qquad\quad + \int_0^1 (3t-t^2) f''\left(t\frac{a+2b}{3} + (1-t)b\right)\mathrm{d}\,t\biggr].
\end{aligned}$$

Proof. Integrating by part and changing the variable of definite integral yield

$$\begin{aligned}
&\int_0^1 (2-t-t^2) f''\left(ta + (1-t)\frac{2a+b}{3}\right)\mathrm{d}\,t \\
&\quad = -\frac{6}{a-b} f'\left(\frac{2a+b}{3}\right) + \frac{3}{a-b}\int_0^1 (1+2t) f'\left(ta + (1-t)\frac{2a+b}{3}\right)\mathrm{d}\,t \\
&\quad = -\frac{6}{a-b} f'\left(\frac{2a+b}{3}\right) + \frac{27}{(a-b)^2} f(a) - \frac{9}{(a-b)^2} f\left(\frac{2a+b}{3}\right) \\
&\qquad - \frac{18}{(a-b)^2}\int_0^1 f\left(ta + (1-t)\frac{2a+b}{3}\right)\mathrm{d}\,t \int_0^1 (2+t-t^2) f''\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\mathrm{d}\,t \\
&\quad = \frac{6}{a-b} f'\left(\frac{2a+b}{3}\right) - \frac{6}{a-b} f'\left(\frac{a+2b}{3}\right) \\
&\qquad - \frac{3}{a-b}\int_0^1 (1-2t) f'\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\mathrm{d}\,t \\
&\quad = \frac{6}{a-b}\left[f'\left(\frac{2a+b}{3}\right) - f'\left(\frac{a+2b}{3}\right)\right] + \frac{9}{(a-b)^2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] \\
&\qquad - \frac{18}{(a-b)^2}\int_0^1 f\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\mathrm{d}\,t
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 (3t-t^2) f''\left(t\frac{a+2b}{3} + (1-t)b\right)\mathrm{d}\,t \\
&\qquad = \frac{6}{a-b} f'\left(\frac{a+2b}{3}\right) - \frac{3}{a-b}\int_0^1 (3-2t) f'\left(t\frac{a+2b}{3} + (1-t)b\right)\mathrm{d}\,t \\
&\qquad = \frac{6}{a-b} f'\left(\frac{a+2b}{3}\right) - \frac{9}{(a-b)^2} f\left(\frac{a+2b}{3}\right) + \frac{27}{(a-b)^2} f(b) \\
&\qquad\quad - \frac{18}{(a-b)^2}\int_0^1 f\left(t\frac{a+2b}{3} + (1-t)b\right)\mathrm{d}\,t.
\end{aligned}$$

Lemma 2.1 is thus proved. □

3. Hermite–Hadamard type inequalities for (α, m)-convex functions

Now we start out to establish some new Hermite–Hadamard type integral inequalities for functions whose second derivatives are (α, m)-convex.

Theorem 3.1. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be twice differentiable and $f''\in L([a,b])$ for $0\le a<b$. If $|f''|^q$ is an (α,m)-convex function on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]^2$ and $q\ge1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}\right. & \left.-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le & \frac{(b-a)^2}{324}\left(\frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{1/q} \\ & \times\left\{7^{1-1/q}\left[6(3\alpha+7)\left|f''(a)\right|^q+m\alpha\big(7\alpha^2+42\alpha+59\big)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q}\right. \\ & +13^{1-1/q}\left[6\big(2\alpha^2+11\alpha+13\big)\left|f''\left(\frac{2a+b}{3}\right)\right|^q+m\alpha\big(13\alpha^2+66\alpha+77\big)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q} \\ & \left.+7^{1-1/q}\left[6\big(2\alpha^2+9\alpha+7\big)\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m\alpha\big(7\alpha^2+30\alpha+23\big)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Proof. Because $|f''|^q$ is (α,m)-convex on $\left[0,\frac{b}{m}\right]$, by Lemma 2.1 and the Hölder integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}\right. & \left.-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le & \frac{(b-a)^2}{54}\left[\int_0^1(2-t-t^2)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|\mathrm{d}\,t\right. \\ & +\int_0^1(2+t-t^2)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|\mathrm{d}\,t \\ & \left.+\int_0^1(3t-t^2)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|\mathrm{d}\,t\right] \\ \le & \frac{(b-a)^2}{54}\left\{\left(\int_0^1(2-t-t^2)\,\mathrm{d}\,t\right)^{1-1/q}\right. \\ & \times\left[\int_0^1(2-t-t^2)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q} \\ & +\left(\int_0^1(2+t-t^2)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1(2+t-t^2)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q} \\ & \left.+\left(\int_0^1(3t-t^2)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1(3t-t^2)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\} \\ \le & \frac{(b-a)^2}{54}\left\{\left(\frac{7}{6}\right)^{1-1/q}\left[\int_0^1(2-t-t^2)\left(t^\alpha\left|f''(a)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\ & +\left(\frac{13}{6}\right)^{1-1/q}\left[\int_0^1(2+t-t^2)\left(t^\alpha\left|f''\left(\frac{2a+b}{3}\right)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q} \\ & \left.+\left(\frac{7}{6}\right)^{1-1/q}\left[\int_0^1(3t-t^2)\left(t^\alpha\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right\} \\ = & \frac{(b-a)^2}{324}\left(\frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{1/q} \\ & \times\left\{7^{1-1/q}\left[6(3\alpha+7)\left|f''(a)\right|^q+m\alpha\big(7\alpha^2+42\alpha+59\big)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q}\right.\end{aligned}$$

$$+ 13^{1-1/q}\left[6\big(2\alpha^2+11\alpha+13\big)\left|f''\left(\frac{2a+b}{3}\right)\right|^q + m\alpha\big(13\alpha^2+66\alpha+77\big)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q}$$
$$+ 7^{1-1/q}\left[6\big(2\alpha^2+9\alpha+7\big)\left|f''\left(\frac{a+2b}{3}\right)\right|^q + m\alpha\big(7\alpha^2+30\alpha+23\big)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\}.$$

The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Under the conditions of Theorem 3.1, if $\alpha = m = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{7(b-a)^2}{324}\left\{\left[\frac{5|f''(a)|^q + 9\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{14}\right]^{1/q} + \frac{13}{7}\left[\frac{\left|f''\left(\frac{2a+b}{3}\right)\right|^q + \left|f''\left(\frac{a+2b}{3}\right)\right|^q}{2}\right]^{1/q} + \left[\frac{9\left|f''\left(\frac{a+2b}{3}\right)\right|^q + 5\left|f''(b)\right|^q}{14}\right]^{1/q}\right\}.$$

Corollary 3.3. *Under the conditions of Theorem 3.1, if $\alpha = m = q = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^2}{12}\left[\frac{5|f''(a)| + 22\left|f''\left(\frac{2a+b}{3}\right)\right| + 22\left|f''\left(\frac{a+2b}{3}\right)\right| + 5|f''(b)|}{54}\right].$$

Theorem 3.4. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be twice differentiable and $f'' \in L([a,b])$ for $0 \le a < b$. If $|f''|^q$ is (α, m)-convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1]^2$ and $q > 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{(b-a)^2}{54}\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\left(\frac{1}{2(\alpha+1)(\alpha+2)}\right)^{1/q}\\ &\times\Bigg\{\big(3^{p+2}-2^{p+1}(p+4)\big)^{1/p}\left[2\left|f''(a)\right|^q + m\alpha(\alpha+3)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q}\\ &+ \big(2^{p+1}(p+4)-2p-5\big)^{1/p}\left[2(2\alpha+3)\left|f''\left(\frac{2a+b}{3}\right)\right|^q + m\alpha(3\alpha+5)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q}\\ &+ \big(3^{p+2}-2^{p+1}(p+4)\big)^{1/p}\left[2(\alpha+1)\left|f''\left(\frac{a+2b}{3}\right)\right|^q + m\alpha(\alpha+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\},\end{aligned} \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemma 2.1 and the Hölder integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{(b-a)^2}{54}\left[\int_0^1 (1-t)\,(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|\mathrm{d}\,t\right.\\ &+ \int_0^1 (1+t)(2-t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|\mathrm{d}\,t\\ &\left.+ \int_0^1 t(3-t)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|\mathrm{d}\,t\right]\end{aligned}$$

$$
\begin{aligned}
&\le \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (1-t)(2+t)^p \,\mathrm{d}\, t\right)^{1/p}\left[\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q \mathrm{d}\, t\right]^{1/q}\right.\\
&\quad+\left(\int_0^1 (1+t)(2-t)^p \,\mathrm{d}\, t\right)^{1/p}\left[\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q \mathrm{d}\, t\right]^{1/q}\\
&\quad\left.+\left(\int_0^1 t(3-t)^p \,\mathrm{d}\, t\right)^{1/p}\left[\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q \mathrm{d}\, t\right]^{1/q}\right\},
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{gathered}
\int_0^1 (1-t)(2+t)^p \,\mathrm{d}\, t=\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}, \quad \int_0^1 (1+t)(2-t)^p \,\mathrm{d}\, t=\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)},\\
\int_0^1 t(3-t)^p \,\mathrm{d}\, t=\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}.
\end{gathered}
\tag{3.3}
$$

Using the (α, m)-convexity of $|f''|^q$ on $\left[0, \frac{b}{m}\right]$, we obtain

$$
\begin{aligned}
\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q \mathrm{d}\, t &\le \int_0^1 (1-t)\left(t^\alpha\left|f''(a)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right)\mathrm{d}\, t\\
&=\frac{1}{2(\alpha+1)(\alpha+2)}\left[2\left|f''(a)\right|^q+m\alpha(\alpha+3)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right],
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q \mathrm{d}\, t &\le \int_0^1 t\left(t^\alpha\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\, t\\
&=\frac{1}{2(\alpha+1)(\alpha+2)}\left[2(\alpha+1)\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m\alpha(\alpha+1)\left|f''\left(\frac{b}{m}\right)\right|^q\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q \mathrm{d}\, t\\
&\qquad\le \int_0^1 (1+t)\left(t^\alpha\left|f''\left(\frac{2a+b}{3}\right)\right|^q+m(1-t^\alpha)\times\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right)\mathrm{d}\, t\\
&\qquad=\frac{1}{2(\alpha+1)(\alpha+2)}\left[2(2\alpha+3)\left|f''\left(\frac{2a+b}{3}\right)\right|^q+m\alpha(3\alpha+5)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right];
\end{aligned}
$$

Substituting the equalities in (3.3) into the above inequalities and the inequality (3.2) yields the inequality (3.1). The proof of Theorem 3.4 is complete. □

Corollary 3.5. *Under the conditions of Theorem 3.4, if $\alpha=m=1$, we have*

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\, x\right| &\le \frac{(b-a)^2}{54}\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\left(\frac{1}{6}\right)^{1/q}\\
&\quad\times\left\{\left(3^{p+2}-2^{p+1}(p+4)\right)^{1/p}\left[\left|f''(a)\right|^q+2\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]^{1/q}\right.\\
&\quad+\left(2^{p+1}(p+4)-2p-5\right)^{1/p}\left[5\left|f''\left(\frac{2a+b}{3}\right)\right|^q+4\left|f''\left(\frac{a+2b}{3}\right)\right|^q\right]^{1/q}\\
&\quad\left.+\left(3^{p+2}-2^{p+1}(p+4)\right)^{1/p}\left[2\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}
$$

Theorem 3.6. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be twice differentiable and $f''\in L([a,b])$ for $0\le a<b$. If $|f''|^q$ is an (α,m)-convex function on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]^2$, $q>1$, and $0<r\le q$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\quad\le\frac{(b-a)^2}{54}\Bigg\{\left(\frac{(q-1)(7q-2r-5)}{(2q-r-1)(3q-r-2)}\right)^{1-1/q}\times\left[\frac{3\alpha+2r+5}{\alpha+r+2}B(\alpha+1,r+1)\left|f''(a)\right|^q\right.\\
&\quad\quad+m\left(\frac{2r+5}{(r+1)(r+2)}-\frac{3\alpha+2r+5}{\alpha+r+2}B(\alpha+1,r+1)\right)\\
&\quad\quad\left.\times\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q}+\left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)^{1-1/q}\left[\frac{\alpha+3}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right.\\
&\quad\quad\left.+m\frac{\alpha(3\alpha+7)}{2(\alpha+1)(\alpha+2)}\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q}+\left(\frac{(q-1)(7q-2r-5)}{(2q-r-1)(3q-r-2)}\right)^{1-1/q}\\
&\quad\quad\times\left[\frac{2\alpha+2r+5}{(\alpha+r+1)(\alpha+r+2)}\left|f''\left(\frac{a+2b}{3}\right)\right|^q\right.\\
&\quad\quad\left.+\ m\left(\frac{2r+5}{(r+1)(r+2)}-\frac{2\alpha+2r+5}{(\alpha+r+1)(\alpha+r+2)}\right)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\Bigg\},
\end{aligned}\tag{3.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $B(\alpha,\beta)$ denotes the well-known Beta function which may be defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t,\quad \alpha,\beta>0.$$

Proof. From Lemma 2.1 and the Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\le\frac{(b-a)^2}{54}\left[\int_0^1(1-t)\,(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|\mathrm{d}\,t\right.\\
&\quad+\int_0^1(1+t)(2-t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|\mathrm{d}\,t\\
&\quad\left.+\int_0^1 t(3-t)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|\mathrm{d}\,t\right]\\
&\le\frac{(b-a)^2}{54}\left\{\left[\int_0^1(1-t)^{\frac{q-r}{q-1}}(2+t)\,\mathrm{d}\,t\right]^{1/p}\left[\int_0^1(1-t)^r(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.\\
&\quad+\left[\int_0^1(1+t)^p(2-t)\,\mathrm{d}\,t\right]^{1/p}\left[\int_0^1(2-t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\\
&\quad\left.+\left[\int_0^1 t^{\frac{q-r}{q-1}}(3-t)\,\mathrm{d}\,t\right]^{1/p}\left[\int_0^1 t^r(3-t)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}\tag{3.5}$$

A straightforward computation gives

$$\begin{aligned}
&\int_0^1(1-t)^{\frac{q-r}{q-1}}(2+t)\,\mathrm{d}\,t=\int_0^1 t^{\frac{q-r}{q-1}}(3-t)\,\mathrm{d}\,t=\frac{(q-1)(7q-2r-5)}{(2q-r-1)(3q-r-2)},\\
&\int_0^1(1+t)^p(2-t)\,\mathrm{d}\,t=\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}.
\end{aligned}\tag{3.6}$$

By the (α, m)-convexity of $|f''|^q$ on $\left[0, \frac{b}{m}\right]$, we have

$$\begin{aligned}
&\int_0^1 (1-t)^r(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q \mathrm{d}\,t \\
&\quad \le \int_0^1 (1-t)^r(2+t)\left(t^\alpha\left|f''(a)\right|^q + m(1-t^\alpha)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right)\mathrm{d}\,t \\
&\quad = \frac{3\alpha+2r+5}{\alpha+r+2}B(\alpha+1,r+1)\left|f''(a)\right|^q \\
&\qquad + m\left(\frac{2r+5}{(r+1)(r+2)} - \frac{3\alpha+2r+5}{\alpha+r+2}B(\alpha+1,r+1)\right)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q;
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 (2-t)\left|f''\left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3}\right)\right|^q \mathrm{d}\,t \\
&\quad \le \int_0^1 (2-t)\left(t^\alpha\left|f''\left(\frac{2a+b}{3}\right)\right|^q + m(1-t^\alpha)\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right)\mathrm{d}\,t \\
&\quad = \frac{\alpha+3}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{2a+b}{3}\right)\right|^q + m\frac{\alpha(3\alpha+7)}{2(\alpha+1)(\alpha+2)}\left|f''\left(\frac{a+2b}{3m}\right)\right|^q;
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 t^r(3-t)\left|f''\left(t\frac{a+2b}{3} + (1-t)b\right)\right|^q \mathrm{d}\,t \\
&\quad \le \int_0^1 t^r(3-t)\left(t^\alpha\left|f''\left(\frac{a+2b}{3}\right)\right|^q + m(1-t^\alpha)\left|f''\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\,t \\
&\quad = \frac{2\alpha+2r+5}{(\alpha+r+1)(\alpha+r+2)}\left|f''\left(\frac{a+2b}{3}\right)\right|^q \\
&\qquad + m\left(\frac{2r+5}{(r+1)(r+2)} - \frac{2\alpha+2r+5}{(\alpha+r+1)(\alpha+r+2)}\right)\left|f''\left(\frac{b}{m}\right)\right|^q.
\end{aligned}$$

Substituting the equalities in (3.6) to the above inequalities and the inequality (3.5) yields the inequality (3.4). The proof of Theorem 3.6 is complete. $\square$

Corollary 3.7. *Under the conditions of Theorem 3.6, if $\alpha = m = 1$, then*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le \frac{(b-a)^2}{54}\left\{\left(\frac{(q-1)(7q-2r-5)}{(2q-r-1)(3q-r-2)}\right)^{1/p}\right. \\
&\quad \times\left[\frac{2(r+4)}{(r+1)(r+2)(r+3)}\left|f''(a)\right|^q + \frac{2r+7}{(r+2)(r+3)}\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]^{1/q} \\
&\quad + \left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)^{1/p}\left[\frac{2}{3}\left|f''\left(\frac{2a+b}{3}\right)\right|^q + \frac{5}{6}\left|f''\left(\frac{a+2b}{3}\right)\right|^q\right]^{1/q} \\
&\quad + \left(\frac{(q-1)(7q-2r-5)}{(2q-r-1)(3q-r-2)}\right)^{1/p}\left[\frac{2r+7}{(r+2)(r+3)}\left|f''\left(\frac{a+2b}{3}\right)\right|^q\right. \\
&\quad \left.\left. + \frac{2(r+4)}{(r+1)(r+2)(r+3)}\left|f''(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

Theorem 3.8. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be twice differentiable and $f'' \in L([a,b])$ for $0 \le a < b$. If $|f''|^q$ is an (α, m)-convex function on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1]^2$ and $q > 1$, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\quad\le\frac{(b-a)^2}{54}\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\bigg\{\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p} \\
&\quad\times\left[B(\alpha+1,q+1)\left|f''(a)\right|^q+m\left(\frac{1}{q+1}-B(\alpha+1,q+1)\right)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q} \\
&\quad+\left(2^{p+1}(p+4)-2p-5\right)^{1/p}\left[\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right. \\
&\quad\left.+m\frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)}\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q}+\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p} \\
&\quad\times\left[\frac{1}{\alpha+q+1}\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m\frac{\alpha}{(\alpha+q+1)(q+1)}\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\bigg\},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $B(\alpha,\beta)$ *is the Beta function.*

Proof. Since $|f''|^q$ is (α,m)-convex on $\left[0,\frac{b}{m}\right]$, by Lemma 2.1 and the Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le\frac{(b-a)^2}{54}\left\{\left(\int_0^1(2+t)^p\,\mathrm{d}\,t\right)^{1/p}\times\left[\int_0^1(1-t)^q\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right. \\
&+\left(\int_0^1(1+t)(2-t)^p\,\mathrm{d}\,t\right)^{1/p}\left[\int_0^1(1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q\mathrm{d}\,t\right]^{1/q} \\
&\left.+\left(\int_0^1(3-t)^p\,\mathrm{d}\,t\right)^{1/p}\left[\int_0^1 t^q\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le\frac{(b-a)^2}{54}\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\bigg\{\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p} \\
&\times\left[B(\alpha+1,q+1)\left|f''(a)\right|^q+m\left(\frac{1}{q+1}-B(\alpha+1,q+1)\right)\left|f''\left(\frac{2a+b}{3m}\right)\right|^q\right]^{1/q} \\
&+\left(2^{p+1}(p+4)-2p-5\right)^{1/p}\left[\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\left|f''\left(\frac{2a+b}{3}\right)\right|^q+m\frac{3\alpha^2+5\alpha}{2(\alpha+1)(\alpha+2)}\left|f''\left(\frac{a+2b}{3m}\right)\right|^q\right]^{1/q} \\
&+\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p}\left[\frac{1}{\alpha+q+1}\left|f''\left(\frac{a+2b}{3}\right)\right|^q+m\frac{\alpha}{(\alpha+q+1)(q+1)}\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\bigg\}.
\end{aligned}
$$

The proof of Theorem 3.8 is complete. □

Corollary 3.9. *Under the conditions of Theorem 3.8, if* $\alpha=m=1$, *then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\quad\le\frac{(b-a)^2}{54}\left(\frac{1}{(p+1)(p+2)}\right)^{1/p}\times\left\{\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p}\right. \\
&\quad\times\left[\frac{1}{(q+1)(q+2)}|f''(a)|^q+\frac{1}{q+2}\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]^{1/q}
\end{aligned}
$$

$$+\left(2^{p+1}(p+4)-2p-5\right)^{1/p}\left[\frac{5}{6}\left|f''\left(\frac{2a+b}{3}\right)\right|^q+\frac{2}{3}\left|f''\left(\frac{a+2b}{3}\right)\right|^q\right]^{1/q}$$
$$+\left(\left(3^{p+1}-2^{p+1}\right)(p+2)\right)^{1/p}\left[\frac{1}{q+2}\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\frac{1}{(q+1)(q+2)}|f''(b)|^q\right]^{1/q}\Bigg\}.$$

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