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Proper CQ^*-ternary algebras

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Abstract

In this paper, modifying the construction of a C^*-ternary algebra from a given C^*-algebra, we define a proper CQ^*-ternary algebra from a given proper CQ^*-algebra.

We investigate homomorphisms in proper CQ^*-ternary algebras and derivations on proper CQ^*-ternary algebras associated with the Cauchy functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|.$$

We moreover prove the Hyers-Ulam stability of homomorphisms in proper CQ^*-ternary algebras and of derivations on proper CQ^*-ternary algebras associated with the Cauchy functional equation

$$f(x+y+z)=f(x)+f(y)+f(z).$$

Keywords: proper CQ^*-ternary homomorphism, proper CQ^*-ternary derivation, Cauchy functional equation, Hyers-Ulam stability.
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1. Introduction and preliminaries

Let A be a linear space and A_0 is a $*$-algebra contained in A as a subspace. We say that A is a quasi $*$-algebra over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are defined and linear;

(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

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A quasi $*$-algebra (A, A_0) is said to be a locally convex quasi $*$-algebra if in A a locally convex topology τ is defined such that

(i) the involution is continuous and the multiplications are separately continuous;

(ii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra $(A[\tau], A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer to [3].

Many authors have considered a special class of quasi $*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;

(iii) $\|y\|_0 = \sup_{a\in A, \|a\|\le 1} \|ay\|$ for all $y \in A_0$.

Several mathematician have contributed works on these subjects (see [4]–[10], [43], [44]).

Following the terminology of [2], a non-empty set G with a ternary operation $[\cdot,\cdot,\cdot] : G\times G\times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\cdot,\cdot,\cdot])$. The ternary groupoid $(G, [\cdot,\cdot,\cdot])$ is called *commutative* if $[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations σ of $\{1, 2, 3\}$.

If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G, [\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is *associative*, i.e., if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see [2]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [2, 48]).

If a C^*-ternary algebra $(A, [\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [2]).

We define a proper CQ^*-ternary algebra and investigate the properties of proper CQ^*-ternary algebras.

Definition 1.1. A proper CQ^*-algebra (A, A_0), endowed with the triple product

$$[\cdot,\cdot,\cdot] : A_0 \times A \times A_0 \to A,$$

which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable and satisfies that $[w_0, w, w_1] \in A_0$ for all $w_0, w, w_1 \in A_0$, is called a *proper CQ^*-ternary algebra*, and denoted by $(A, A_0, [\cdot,\cdot,\cdot])$.

Example 1.2. *(1) Let (A, A_0) be a proper CQ^*-algebra. Let*

$$[z, x, w] := zx^*w$$

for all $x \in A$ and all $z, w \in A_0$. Then $(A, A_0, [\cdot,\cdot,\cdot])$ is a proper CQ^*-ternary algebra.

(2) A proper JCQ^*-triple $(A, A_0, \{\cdot,\cdot,\cdot\})$ is a proper CQ^*-algebra (A, A_0), endowed with the Jordan triple product

$$\{z, x, w\} := \frac{1}{2}(zx^*w + wx^*z)$$

for all $x \in A$ and all $z, w \in A_0$ (see [29]). It is obvious that every proper JCQ^*-triple is a proper CQ^*-ternary algebra if we let $[\cdot,\cdot,\cdot] := \{\cdot,\cdot,\cdot\}$.

Definition 1.3. Let $(A, A_0, [\cdot,\cdot,\cdot])$ and $(B, B_0, [\cdot,\cdot,\cdot])$ be proper CQ^*-ternary algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper CQ^*-ternary homomorphism* if $H(z), H(w) \in B_0$ and

$$H([z, x, w]) = [H(z), H(x), H(w)]$$

for all $z, w \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *proper CQ^*-ternary derivation* if

$$\delta([w_0, w_1, w_2]) = [w_2, \delta(w_0)^*, w_1^*] + [w_0, \delta(w_1), w_2] + [w_1^*, \delta(w_2)^*, w_0]$$

for all $w_0, w_1, w_2 \in A_0$.

The stability problem of functional equations originated from a question of Ulam [45] concerning the stability of group homomorphisms. Hyers [18] gave a first affirmative answer to the question of Ulam for Banach spaces. Th.M. Rassias [33] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.4. *[33] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Th.M. Rassias [34] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Gajda [16] following the same approach as in Th.M. Rassias [33], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Th.M. Rassias and Šemrl [38] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [16], as well as of Th.M. Rassias and Šemrl [38] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [17], Jung [22], who among others studied the Hyers-Ulam stability of functional equations. The inequality (1) that was introduced for the first time by Th.M. Rassias [33] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept (cf. the books of Czerwik [13, 14], Hyers, Isac and Th.M. Rassias [19]).

J.M. Rassias [30] following the spirit of the innovative approach of Th.M. Rassias [33] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \ne 1$ (see also [31] for a number of other new results).

Găvruta [17] provided a further generalization of Th.M. Rassias' Theorem. Isac and Th.M. Rassias [21] applied the Hyers-Ulam stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [20], Hyers, Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (see [1, 11, 12, 15], [24]–[28], [32], [35]–[42], [46, 47]).

This paper is organized as follows: In Sections 2 and 3, we investigate homomorphisms and derivations in proper CQ^*-ternary algebras associated with the Cauchy functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|.$$

In Sections 4 and 5, we prove the Hyers-Ulam stability of homomorphisms in proper CQ^*-ternary algebras and of derivations on proper CQ^*-ternary algebras associated with the Cauchy functional equation

$$f(x+y+z) = f(x)+f(y)+f(z).$$

Throughout this paper, assume that $(A, A_0, [\cdot,\cdot,\cdot])$ is a proper CQ^*-ternary algebra with C^*-norm $\|\cdot\|_{A_0}$ and norm $\|\cdot\|_A$, and that $(B, B_0, [\cdot,\cdot,\cdot])$ is a proper CQ^*-ternary algebra with C^*-norm $\|\cdot\|_{B_0}$ and norm $\|\cdot\|_B$.

2. Homomorphisms in proper CQ^*-ternary algebras

In this section, we investigate homomorphisms in proper CQ^*-ternary algebras.

Theorem 2.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\begin{aligned} \|\mu f(x)+f(y)+f(z)\|_B &\le \|f(\mu x+y+z)\|_B, &(2)\\ \|f([w_0,x,w_1]) - [f(w_0),f(x),f(w_1)]\|_B &\le \theta(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}) &(3)\end{aligned}$$

for all $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $w_0, w_1 \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \to B$ is a proper CQ^*-ternary homomorphism.

Proof. Let $\mu = 1$ in (2). By [29, Proposition 2.1], the mapping $f : A \to B$ is Cauchy additive.

Letting $z = 0$ and $y = -\mu x$ in (2), we get

$$\mu f(x) - f(\mu x) = \mu f(x) + f(-\mu x) = 0$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. So $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}$ and all $x \in A$. By the same reasoning as in the proof of [23, Theorem 2.1], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (3),

$$\begin{aligned} \|f([w_0,x,w_1]) &- [f(w_0),f(x),f(w_1)]\|_B \\ &= \lim_{n\to\infty} \frac{1}{8^n}\|f(8^n[w_0,x,w_1]) - [f(2^n w_0), f(2^n x), f(2^n w_1)]\|_B \\ &\le \lim_{n\to\infty} \frac{8^{nr}}{8^n}\theta(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}) = 0\end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$f([w_0,x,w_1]) = [f(w_0),f(x),f(w_1)]$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w_0,x,w_1]) = [f(w_0),f(x),f(w_1)]$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Since $f(w) \in B_0$ for all $w \in A_0$, the mapping $f : A \to B$ is a proper CQ^*-ternary homomorphism, as desired.

Theorem 2.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \to B$ a mapping satisfying (2) and $f(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f([w_0, x, w_1]) - [f(w_0), f(x), f(w_1)]\|_B \leq \theta \cdot \|w_0\|_A^r \cdot \|x\|_A^r \cdot \|w_1\|_A^r \tag{4}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. Then the mapping $f : A \to B$ is a proper CQ^*-ternary homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (4),

$$\begin{aligned} \|f([w_0, x, w_1]) &- [f(w_0), f(x), f(w_1)]\|_B \\ &= \lim_{n\to\infty} \frac{1}{8^n} \|f(8^n[w_0, x, w_1]) - [f(2^n w_0), f(2^n x), f(2^n w_1)]\|_B \\ &\leq \lim_{n\to\infty} \frac{8^{nr}}{8^n} \theta \cdot \|w_0\|_A^r \cdot \|x\|_A^r \cdot \|w_1\|_A^r = 0 \end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$f([w_0, x, w_1]) = [f(w_0), f(x), f(w_1)]$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \to B$ satisfies

$$f([w_0, x, w_1]) = [f(w_0), f(x), f(w_1)]$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \to B$ is a proper CQ^*-ternary homomorphism.

3. Derivations on proper CQ^*-ternary algebras

In this section, we investigate derivations on proper CQ^*-ternary algebras.

Theorem 3.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A_0 \to A$ a mapping such that*

$$\|\mu f(x) + f(y) + f(z)\|_A \leq \|f(\mu x + y + z)\|_A, \tag{5}$$

$$\begin{aligned} \|f([w_0, w_1, w_2]) - [w_2, f(w_0)^*, w_1^*] &- [w_0, f(w_1), w_2] \\ -[w_1^*, f(w_2)^*, w_0]\|_A &\leq \theta(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}) \end{aligned} \tag{6}$$

for all $\mu \in \mathbb{T}$ and all $w_0, w_1, w_2, x, y, z \in A_0$. Then the mapping $f : A_0 \to A$ is a proper CQ^*-ternary derivation.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A_0 \to A$ is $\mathbb{C}$-linear.

(i) Assume that $r < 1$. By (6),

$$\begin{aligned} \|f([w_0, w_1, w_2]) &- [w_2, f(w_0)^*, w_1^*] - [w_0, f(w_1), w_2] - [w_1^*, f(w_2)^*, w_0]\|_A \\ &= \lim_{n\to\infty} \frac{1}{8^n} \|f(8^n[w_0, w_1, w_2]) - [2^n w_2, f(2^n w_0)^*, 2^n w_1^*] \\ &- [2^n w_0, f(2^n w_1), 2^n w_2] - [2^n w_1^*, f(2^n w_2)^*, 2^n w_0]\|_A \\ &\leq \lim_{n\to\infty} \frac{8^{nr}}{8^n} \theta(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}) = 0 \end{aligned}$$

for all $w_0, w_1, w_2 \in A_0$. So

$$f([w_0, w_1, w_2]) = [w_2, f(w_0)^*, w_1^*] + [w_0, f(w_1), w_2] + [w_1^*, f(w_2)^*, w_0]$$

for all $w_0, w_1, w_2 \in A_0$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A_0 \to A$ satisfies

$$f([w_0, w_1, w_2]) = [w_2, f(w_0)^*, w_1^*] + [w_0, f(w_1), w_2] + [w_1^*, f(w_2)^*, w_0]$$

for all $w_0, w_1, w_2 \in A_0$.

Therefore, the mapping $f : A_0 \to A$ is a proper CQ^*-ternary derivation.

Theorem 3.2. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A_0 \to A$ a mapping satisfying* (5) *such that*

$$\begin{aligned} \|f([w_0, w_1, w_2]) &- [w_2, f(w_0)^*, w_1^*] - [w_0, f(w_1), w_2] \\ &- [w_1^*, f(w_2)^*, w_0]\|_A \leq \theta \cdot \|w_0\|_A^r \cdot \|w_1\|_A^r \cdot \|w_2\|_A^r \end{aligned} \tag{7}$$

for all $w_0, w_1, w_2 \in A_0$. Then the mapping $f : A_0 \to A$ is a proper CQ^*-ternary derivation.

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.1.

4. Stability of homomorphisms in proper CQ^*-ternary algebras

We prove the Hyers-Ulam stability of homomorphisms in proper CQ^*-ternary algebras.

Theorem 4.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (3) *such that $f(w) \in B_0$ for all $w \in A_0$ and*

$$\begin{aligned} \|f(\mu x + \mu y + \mu z) &- \mu f(x) - \mu f(y) - \mu f(z)\|_B \\ &\leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \end{aligned} \tag{8}$$

$$\begin{aligned} \|f(w_0 + w_1 + w_2) &- f(w_0) - f(w_1) - f(w_2)\|_{B_0} \\ &\leq \theta(\|w_0\|_{A_0}^r + \|w_1\|_{A_0}^r + \|w_2\|_{A_0}^r) \end{aligned} \tag{9}$$

for all $\mu \in \mathbb{T}$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper CQ^*-ternary homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{3\theta}{3^r - 3}\|x\|_A^r \tag{10}$$

for all $x \in A$.

Proof. Let us assume $\mu = 1$ and $x = y = z$ in (8). Then we get

$$\|f(3x) - 3f(x)\|_B \leq 3\theta\|x\|_A^r \tag{11}$$

for all $x \in A$. So

$$\|f(x) - 3f(\frac{x}{3})\|_B \leq \frac{3\theta}{3^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m})\|_B \leq \sum_{j=l}^{m-1} \|3^j f(\frac{x}{3^j}) - 3^{j+1} f(\frac{x}{3^{j+1}})\|_B \leq \frac{3\theta}{3^r} \sum_{j=l}^{m-1} \frac{3^j}{3^{rj}}\|x\|_A^r \tag{12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\{3^n f(\frac{x}{3^n})\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (12), we get (10).

It follows from (8) that

$$\begin{aligned}
&\|H(\mu x + \mu y + \mu z) - \mu H(x) - \mu H(y) - \mu H(z)\|_B \\
&= \lim_{n\to\infty} 3^n \|f(\frac{\mu x + \mu y + \mu z}{3^n}) - \mu f(\frac{x}{3^n}) - \mu f(\frac{y}{3^n}) - \mu f(\frac{z}{3^n})\|_B \\
&\le \lim_{n\to\infty} \frac{3^n\theta}{3^{nr}}(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. So

$$H(\mu x + \mu y + \mu z) = \mu H(x) + \mu H(y) + \mu H(z)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. By the same reasoning as in the proof of [23, Theorem 2.1], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Now, let $T : A \to B$ be another additive mapping satisfying (10). Then we have

$$\begin{aligned}
\|H(x) - T(x)\|_B &= 3^n\|H(\frac{x}{3^n}) - T(\frac{x}{3^n})\|_B \\
&\le 3^n(\|H(\frac{x}{3^n}) - f(\frac{x}{3^n})\|_B + \|T(\frac{x}{3^n}) - f(\frac{x}{3^n})\|_B) \\
&\le \frac{6\cdot 3^n\theta}{3^{nr}(3^r-3)}\|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H.

It follows from (9) that $H(w) = \lim_{n\to\infty} 3^n f(\frac{w}{3^n}) \in B_0$ for all $w \in A_0$. So it follows from (3) that

$$\begin{aligned}
\|H([w_0, x, w_1]) &- [H(w_0), H(x), H(w_1)]\|_B \\
&= \lim_{n\to\infty} 3^{3n}\|f(\frac{[w_0, x, w_1]}{3^{3n}}) - [f(\frac{w_0}{3^n}), f(\frac{x}{3^n}), f(\frac{w_1}{3^n})]\|_B \\
&\le \lim_{n\to\infty} \frac{3^{3n}}{3^{3nr}}\theta(\|w_0\|_A^{3r} + \|x\|_A^{3r} + \|w_1\|_A^{3r}) = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$ and all $x \in A$. So

$$H([w_0, x, w_1]) = [H(w_0), H(x), H(w_1)]$$

for all $w_0, w_1 \in A_0$ and all $x \in A$.

Thus the mapping $H : A \to B$ is a unique proper CQ^*-ternary homomorphism satisfying (10), as desired.

Theorem 4.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (3), (8) and (9) such that $f(w) \in B_0$ for all $w \in A_0$. Then there exists a unique proper CQ^*-ternary homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{3\theta}{3-3^r}\|x\|_A^r \tag{13}$$

for all $x \in A$.

Proof. It follows from (11) that

$$\|f(x) - \frac{1}{3}f(3x)\|_B \le \theta\|x\|_A^r$$

for all $x \in A$. So

$$\|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}f(3^m x)\|_B \le \sum_{j=l}^{m-1}\|\frac{1}{3^j}f(3^j x) - \frac{1}{3^{j+1}}f(3^{j+1}x)\|_B \le \sum_{j=l}^{m-1}\frac{3^{jr}}{3^j}\theta\|x\|_A^r \tag{14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (14), we get (13).

The rest of the proof is similar to the proof of Theorem 4.1.

5. Stability of derivations on proper CQ^*-ternary algebras

We prove the Hyers-Ulam stability of derivations on proper CQ^*-ternary algebras.

Theorem 5.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A_0 \to A$ be a mapping satisfying (6) such that*

$$\begin{aligned} \|f(\mu w_0 + \mu w_1 + \mu w_2) \; &- \; \mu f(w_0) - \mu f(w_1) - \mu f(w_2)\|_A \\ &\le \; \theta(\|w_0\|_A^r + \|w_1\|_A^r + \|w_2\|_A^r) \end{aligned} \tag{15}$$

for all $\mu \in \mathbb{T}$ and all $w_0, w_1, w_2 \in A_0$. Then there exists a unique proper CQ^*-ternary derivation $\delta : A_0 \to A$ such that

$$\|f(w) - \delta(w)\|_A \le \frac{3\theta}{3^r - 3} \|w\|_A^r \tag{16}$$

for all $w \in A_0$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ satisfying (16). The mapping $\delta : A_0 \to A$ is given by

$$\delta(w) := \lim_{n\to\infty} 3^n f(\frac{w}{3^n})$$

for all $w \in A_0$.

It follows from (6) that

$$\begin{aligned} &\|\delta([w_0, w_1, w_2]) - [w_2, \delta(w_0)^*, w_1^*] - [w_0, \delta(w_1), w_2] - [w_1^*, \delta(w_2)^*, w_0]\|_A \\ &= \lim_{n\to\infty} \|3^{3n} f(\frac{[w_0, w_1, w_2]}{3^{3n}}) - [\frac{3^n w_2}{3^n}, 3^n f(\frac{w_0}{3^n})^*, \frac{3^n w_1^*}{3^n}] - [\frac{3^n w_0}{3^n}, 3^n f(\frac{w_1}{3^n}), \frac{3^n w_2}{3^n}] \\ &-[\frac{3^n w_1^*}{3^n}, 3^n f(\frac{w_2}{3^n})^*, \frac{3^n w_0}{3^n}]\|_A \le \lim_{n\to\infty} \frac{3^{3n}}{3^{3nr}} \theta(\|w_0\|_A^{3r} + \|w_1\|_A^{3r} + \|w_2\|_A^{3r}) = 0 \end{aligned}$$

for all $w_0, w_1, w_2 \in A_0$. So

$$\delta([w_0, w_1, w_2]) = [w_2, \delta(w_0)^*, w_1^*] + [w_0, \delta(w_1), w_2] + [w_1^*, \delta(w_2)^*, w_0]$$

for all $w_0, w_1, w_2 \in A_0$.

Thus the mapping $\delta : A_0 \to A$ is a unique proper CQ^*-ternary derivation satisfying (16), as desired.

Theorem 5.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A_0 \to A$ be a mapping satisfying (6) and (15). Then there exists a unique proper CQ^*-ternary derivation $\delta : A_0 \to A$ such that*

$$\|f(w) - \delta(w)\|_A \le \frac{3\theta}{3 - 3^r} \|w\|_A^r$$

for all $w \in A_0$.

Proof. The proof is similar to the proofs of Theorems 4.2 and 5.1.

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