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A new multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces

Birol Gunduz

Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.

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Abstract

Sufficient conditions for the convergence of a new multistep iteration to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in the framework of convex metric spaces are obtained. As an application, related results for a new three step iteration are derived. Our convergence results generalize and refine many known results.

Keywords: Iteration process, convex metric spaces, asymptotically quasi-nonexpansive mappings, common fixed point, strong convergence.
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1. Introduction and basic definitions

In the sequel, we designate the set $\{1, 2, ..., r\}$ by I and the set of natural numbers by $\mathbb{N}$. Denote by $F(T)$ the set of fixed points of T and by $F := (\bigcap_{i=1}^{r} F(T_i))$ the set of common fixed points of a finite family of mappings $\{T_i : i \in I\}$.

For the sake of convenience, we recall some definitions and notations.

Definition 1.1. Let X be a metric space and $T : X \to X$ a mapping.

1. The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \le d(x, y) \text{ for all } x, y \in X.$$

Email address: `birolgndz@gmail.com` (Birol Gunduz)

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2. The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d\left(Tx,p\right) \leq d\left(x,p\right) \text{ for all } x \in X \text{ and } p \in F(T).$$

3. The mapping T is said to be asymptotically nonexpansive [2] if there exists a $u_n \in [1,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that

$$d\left(T^n x, T^n y\right) \leq u_n d\left(x,y\right) \text{ for } \forall x,y \in X \text{ and } n \in \mathbb{N}.$$

4. The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a $u_n \in [1,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that

$$d\left(T^n x, p\right) \leq u_n d\left(x,p\right) \forall x \in X,\ \forall p \in F(T) \text{ and } n \in \mathbb{N}.$$

Remark 1.2. From the above definition, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. However, the converses of these statements are not true, in general (see, for example, [2, 6, 14, 21]).

In 1970, Takahashi [18] introduced the concept of convexity in metric spaces and studied the properties of such spaces.

Definition 1.3 ([18])**.** A convex structure in a metric space (X,d) is a mapping $W : X \times X \times [0,1] \to X$ satisfying for all $x,y,u \in X$ and all $\lambda \in [0,1]$,

$$d\left(W\left(x,y,\lambda\right),u\right) \leq \lambda d\left(x,u\right) + \left(1-\lambda\right) d\left(y,u\right).$$

A metric space together with a convex structure is called a convex metric space.

A nonempty subset C of X is said to be convex if $W\left(x,y,\lambda\right) \in C$ for all $(x,y,\lambda) \in C \times C \times [0,1]$. The following holds in convex metric spaces [18] for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(W(x,y,\lambda),x) = (1-\lambda)d(x,y) \text{ and } d(W(x,y,\lambda),y) = \lambda d(x,y).$$

Every normed space (and Banach space) is a special convex metric space with a convex structure $W\left(x,y,\lambda\right) = \lambda x + \left(1-\lambda\right) y$ for all $x,y \in X$ and $\lambda \in [0,1]$. In fact,

$$\begin{aligned} d\left(W\left(x,y,\lambda\right),u\right) &= \left\|\left(\lambda x + \left(1-\lambda\right) y\right) - u\right\| \\ &\leq \lambda \left\|x-u\right\| + \left(1-\lambda\right)\left\|y-u\right\| \\ &= \lambda d\left(x,u\right) + \left(1-\lambda\right) d\left(y,u\right) \quad \forall u \in X. \end{aligned}$$

But there are many examples of convex metric spaces which cannot be embedded in any normed space [3].

In 2008, Khan et al. [9] introduced the following iterative process for a finite family of mappings. Let $\{T_i : i \in I\}$ be a family of self-mappings of C, where C is a convex subset of a Banach space X. Suppose that $\alpha_{in} \in [0,1]$ for all $n = 1,2,\ldots$ and $i = 1,2,\ldots,r$. For $x_1 \in C$, let the sequence $\{x_n\}$ be defined by the following algorithm:

$$\begin{aligned} x_{n+1} &= \left(1-\alpha_{rn}\right) x_n + \alpha_{rn} T_r^n y_{(r-1)n}, \\ y_{(r-1)n} &= \left(1-\alpha_{(r-1)n}\right) x_n + \alpha_{(r-1)n} T_{r-1}^n y_{(r-2)n}, \\ y_{(r-2)n} &= \left(1-\alpha_{(r-2)n}\right) x_n + \alpha_{(r-2)n} T_{r-2}^n y_{(r-3)n}, \\ &\vdots \\ y_{2n} &= \left(1-\alpha_{2n}\right) x_n + \alpha_{2n} T_2^n y_{1n}, \\ y_{1n} &= \left(1-\alpha_{1n}\right) x_n + \alpha_{1n} T_1^n y_{0n}, \end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all n. The iterative process (1.1) is a generalized form of the modified Mann (one-step) iterative process of Schu [15], the modified Ishikawa (two-step) iterative process of Tan and Xu [20], and the three-step iterative process of Xu and Noor [22].

In [7], Khan and Ahmed transformed process (1.1) for convex metric spaces as follows.

Let C be a convex subset of a convex metric space X and $\{T_i : i = 1, 2, \ldots, r\}$ a finite family of self-mappings of C. Suppose that $\alpha_{in} \in [0, 1]$ for all $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, r$. Given $x_0 \in X$, let the sequence $\{x_n\}$ be defined by the iteration process

$$\begin{aligned} x_{n+1} &= W\left(T_r^n y_{(r-1)n}, x_n, \alpha_{rn}\right), \\ y_{(r-1)n} &= W\left(T_{r-1}^n y_{(r-2)n}, x_n, \alpha_{(r-1)n}\right), \\ y_{(r-2)n} &= W\left(T_{r-2}^n y_{(r-3)n}, x_n, \alpha_{(r-2)n}\right), \\ &\vdots \\ y_{2n} &= W\left(T_2^n y_{1n}, x_n, \alpha_{2n}\right), \\ y_{1n} &= W\left(T_1^n y_{0n}, x_n, \alpha_{1n}\right), \end{aligned} \tag{1.2}$$

where $y_{0n} = x_n$ for all n.

Inspired and motivated by these definitions, we introduce a new more general multistep iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces as follows.

Let (X, d) be a convex metric space with convex structure W and $T_i : X \to X$ $(i \in I)$ a finite family of asymptotically quasi-nonexpansive mappings. Then for a given $x_1 \in X$ and $n \geq 1$, compute iteratively the sequences $\{x_n\}$, $\{y_n\}$,$\ldots$,$\{y_{n+r-2}\}$ by

$$\begin{aligned} y_n &= W\left(T_r^n x_n, x_n, \alpha_{nr}\right), \\ y_{n+1} &= W\left(T_{r-1}^n x_n, W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), \alpha_{n(r-1)}\right), \\ y_{n+2} &= W\left(T_{r-2}^n y_n, W\left(T_{r-2}^n y_{n+1}, x_n, \frac{\beta_{n(r-2)}}{1-\alpha_{n(r-2)}}\right), \alpha_{n(r-2)}\right), \\ &\vdots \\ y_{n+r-2} &= W\left(T_2^n y_{n+r-4}, W\left(T_2^n y_{n+r-3}, x_n, \frac{\beta_{n2}}{1-\alpha_{n2}}\right), \alpha_{n2}\right), \\ x_{n+1} &= W\left(T_1^n y_{n+r-3}, W\left(T_1^n y_{n+r-2}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right), \end{aligned} \tag{1.3}$$

where $\{\alpha_{ni}\}, \{\beta_{ni}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_{ni} + \beta_{ni} < 1$ for all $i \in I$. The iteration (1.3) is called the modified multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. For $r = 3$, (1.3) reduces to the modified three-step iteration:

$$\begin{aligned} y_n &= W\left(T_3^n x_n, x_n, \alpha_{n3}\right), \\ y_{n+1} &= W\left(T_2^n x_n, W\left(T_2^n y_n, x_n, \frac{\beta_{n2}}{1-\alpha_{n2}}\right), \alpha_{n2}\right), \\ x_{n+1} &= W\left(T_1^n y_n, W\left(T_1^n y_{n+1}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right), \end{aligned} \tag{1.4}$$

where $\{\alpha_{ni}\}, \{\beta_{ni}\}$ are sequences in $[0, 1]$ such that $\alpha_{ni} + \beta_{ni} < 1$ for each $i \in \{1, 2, 3\}$ and $n \geq 1$. The modified multistep iteration (1.3) coincides with the iterative scheme of Yang [23] in Banach space setting, when $W(x, y, \alpha) = \alpha x + (1 - \alpha) y$. The iteration scheme (1.4) coincides with the modified Noor iterative scheme defined by Suantai [17] in Banach space, when $W(x, y, \alpha) = \alpha x + (1 - \alpha) y$ and $T_i = T$ for all $i \in \{1, 2, 3\}$. Using "$W(x, y, 0) = y$ for any $x, y \in X$ ([19], Proposition 1.2(a))", the modified multistep iteration (1.3) reduces in convex metric spaces to Khan and Ahmed [7] iteration (1.4) with $\alpha_{n1} = \alpha_{n2} = \cdots = \alpha_{n(r-1)} = 0$. By the same argument, iterations (1.3) and (1.4) reduce in Banach spaces to Xu–Noor algorithm [22], Ishikawa algorithm, Mann algorithm, and their convex metric spaces versions.

The purpose of this paper is to discuss strong convergence theorems of the modified multistep iteration (1.3) for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. And to give strong convergence of the modified three-step iteration (1.4) for three asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result is an extension and improvement of corresponding results in [3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 16, 17, 23, 24].

In order to prove our main result, we need the following useful lemma and proposition.

Lemma 1.4 ([12]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty,\ a_{n+1} = (1+b_n)\,a_n,\ n \geq 0.$$

Then

i) $\lim_{n\to\infty} a_n$ *exists,*

ii) if either $\liminf_{n\to\infty} a_n = 0$ *or* $\limsup_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Proposition 1.5 ([9]). *Let X be a convex metric space and $T_i : X \to X$ a finite family of asymptotically quasi-nonexpansive mappings with $F := \left(\bigcap_{i=1}^{r} F(T_i)\right) \neq \emptyset$. Then, there exist a point $p \in F$ and a sequence $\{u_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} u_n = 1$ such that*

$$d\left(T_i^n x, p\right) \leq u_n d\left(x, p\right)$$

for all $x \in X$, for each $i \in I$.

2. Strong Convergence Theorems of Modified Multistep Iteration

The following lemma is crucial for proving the main theorems.

Lemma 2.1. *Let (X,d) be a convex metric space and $T_i : X \to X$ $(i \in I)$ a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty}(u_n - 1) < \infty$ and that $\{x_n\}$ is obtained by modified multistep iteration (1.3). If $\lim_{n\to\infty} d\left(x_n, F\right) = 0$ where $d\left(x, F\right) = \inf\left\{d\left(x,p\right) : p \in F\right\}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For all $p \in F$, we have from Proposition 1.5 and (1.3) that

$$\begin{aligned} d\left(y_n, p\right) &= d\left(W\left(T_r^n x_n, x_n, \alpha_{nr}\right), p\right) \\ &\leq \alpha_{nr} d\left(T_r^n x_n, p\right) + \left(1-\alpha_{nr}\right) d\left(x_n, p\right) \\ &\leq \alpha_{nr} u_n d\left(x_n, p\right) + \left(1-\alpha_{nr}\right) d\left(x_n, p\right) \\ &\leq u_n d\left(x_n, p\right). \end{aligned} \tag{2.1}$$

Using a similar method, together with (2.1), we get

$$\begin{aligned} d\left(y_{n+1}, p\right) &= d\left(W\left(T_{r-1}^n x_n, W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), \alpha_{n(r-1)}\right), p\right) \\ &\leq \alpha_{n(r-1)} d\left(T_{r-1}^n x_n, p\right) + (1-\alpha_{n(r-1)}) d\left(W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), p\right) \\ &\leq \alpha_{n(r-1)} u_n d\left(x_n, p\right) \\ &\quad +(1-\alpha_{n(r-1)})\left(\frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}} d(T_{r-1}^n y_n, p) + \left(1-\frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right) d\left(x_n, p\right)\right) \\ &\leq \alpha_{n(r-1)} u_n d\left(x_n, p\right) + \beta_{n(r-1)} u_n d\left(y_n, p\right) + (1-\alpha_{n(r-1)} - \beta_{n(r-1)}) d\left(x_n, p\right) \\ &\leq \alpha_{n(r-1)} u_n d\left(x_n, p\right) + \beta_{n(r-1)} u_n^2 d\left(x_n, p\right) + (1-\alpha_{n(r-1)} - \beta_{n(r-1)}) d\left(x_n, p\right) \\ &\leq u_n^2 d\left(x_n, p\right). \end{aligned}$$

By induction, it follows from (1.3) that we have

$$d\left(y_{n+j},p\right)\le u_n^{j+1}d\left(x_n,p\right) \tag{2.2}$$

for some $j=0,1,\ldots,r-2$. Therefore, it follows from (1.3) and (2.2) that

$$\begin{aligned} d\left(x_{n+1},p\right) &= d\left(W\left(T_1^n y_{n+r-3}, W\left(T_1^n y_{n+r-2}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right), p\right)\\ &\le \alpha_{n1} d\left(T_1^n y_{n+r-3}, p\right)\\ &\quad + (1-\alpha_{n1})\left(\frac{\beta_{n1}}{1-\alpha_{n1}} d\left(T_1^n y_{n+r-2}, p\right) + \left(1-\frac{\beta_{n1}}{1-\alpha_{n1}}\right) d\left(x_n,p\right)\right)\\ &\le \alpha_{n1}u_n d\left(y_{n+r-3},p\right) + \beta_{n1}u_n d\left(y_{n+r-2},p\right) + (1-\alpha_{n1}-\beta_{n1}) d\left(x_n,p\right)\\ &\le u_n^r d\left(x_n,p\right)\\ &= \left(1+\left(u_n^r-1\right)\right) d\left(x_n,p\right). \end{aligned} \tag{2.3}$$

Since $0\le t^r-1\le rt^{r-1}(t-1)$ for all $t\ge 1$, the assumption $\sum_{n=1}^{\infty}(u_n-1)<\infty$ implies that $\{u_n\}$ is bounded; then $u_n\in[1,M]$, $\forall n\ge 1$ and for some M. Hence, $u_n^r-1\le rM^{r-1}\left(u_n-1\right)$ holds for all $n\ge 1$. Therefore, $\sum_{n=1}^{\infty}\left(u_n^r-1\right)<\infty$. By Lemma 1.4, we obtain existence of $\lim_{n\to\infty} d\left(x_n,p\right)$.

It is well known that $1+x\le e^x$ for all $x\ge 0$. Using it and (2.3), we get

$$\begin{aligned} d\left(x_{n+m},p\right) &\le \left(1+\left(u_{n+m-1}^r-1\right)\right) d\left(x_{n+m-1},p\right)\\ &\le e^{u_{n+m-1}^r-1} d\left(x_{n+m-1},p\right)\\ &\le e^{u_{n+m-1}^r-1}\left[\left(1+\left(u_{n+m-2}^r-1\right)\right) d\left(x_{n+m-2},p\right)\right]\\ &\le e^{u_{n+m-1}^r-1+u_{n+m-2}^r-1} d\left(x_{n+m-2},p\right)\\ &\ \ \vdots\\ &\le e^{\sum_{j=n}^{n+m-1}\left(u_j^r-1\right)} d\left(x_n,p\right)\\ &\le M d\left(x_n,p\right), \end{aligned}$$

where $M=e^{\sum_{j=n}^{n+m-1}\left(u_j^r-1\right)}<\infty$. That is,

$$d\left(x_{n+m},p\right)\le M d\left(x_n,p\right) \tag{2.4}$$

for all $m,n\in\mathbb{N}$ and $p\in F$.

Now we use this to prove that $\{x_n\}$ is a Cauchy sequence. From $\lim_{n\to\infty} d\left(x_n,F\right)=0$ for each $\varepsilon>0$ there exists an $n_1\in N$ such that

$$d\left(x_n,F\right)<\frac{\varepsilon}{M+1},\ \forall n\ge n_1.$$

Thus, there exists a $q\in F$ such that

$$d\left(x_n,q\right)<\frac{\varepsilon}{M+1},\ \forall n\ge n_1. \tag{2.5}$$

Using (2.4) and (2.5), we obtain

$$\begin{aligned} d\left(x_{n+m},x_n\right) &\le d\left(x_{n+m},q\right)+d\left(x_n,q\right)\le M d\left(x_n,q\right)+d\left(x_n,q\right)\\ &= (M+1) d\left(x_n,q\right)\\ &< (M+1)\frac{\varepsilon}{M+1}=\varepsilon \end{aligned}$$

for all $n,m\ge n_1$. Therefore $\{x_n\}$ is a Cauchy sequence. □

Now we state and prove our main theorem.

Theorem 2.2. *Let (X,d) be a convex metric space and $T_i : X \to X$ $(i \in I)$ a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty}(u_n - 1) < \infty$ and that $\{x_n\}$ is obtained by modified multistep iteration* (1.3)*. Then*

(i) If $\{x_n\}$ converges to a unique point in F, then $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$.

(ii) If X is complete and either $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, then $\{x_n\}$ converges to a unique point in F.

Proof. (i) Let $p \in F$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists an $n_0 \in N$ such that

$$d(x_n, p) < \varepsilon \ \forall n \geq n_0.$$

Taking infimum over $p \in F$, we obtain

$$d(x_n, F) < \varepsilon \ \forall n \geq n_0.$$

This means $\lim_{n\to\infty} d(x_n, F) = 0$ so that $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$.

(ii) Suppose that X is complete and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Then, we have from Lemma 1.4(ii) $\lim_{n\to\infty} d(x_n, F) = 0$. From the completeness of X and Lemma 2.1, we get that $\lim_{n\to\infty} x_n$ exists and equals $q \in X$, say. Moreover, since the set F of fixed points of asymptotically quasi-nonexpansive mappings is closed, $q \in F$ from $\lim_{n\to\infty} d(x_n, F) = 0$. Hence $\{x_n\}$ converges to a unique point in F. □

Next, we give and prove two corollaries which are consequences of Theorem 2.2.

Corollary 2.3. *Let (X,d) be a complete convex metric space and $T_i : X \to X$ $(i \in I)$ a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty}(u_n - 1) < \infty$ and that $\{x_n\}$ is obtained by modified multistep iteration* (1.3)*. Assume that the following two conditions hold.*

i) $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0;$ (2.6)

ii) for the sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ we have

$$\liminf_{n\to\infty} d(y_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(y_n, F) = 0. \tag{2.7}$$

Then $\{x_n\}$ converges to a unique point in F.

Proof. From (2.6) and (2.7), we have that

$$\liminf_{n\to\infty} d(x_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F) = 0.$$

Therefore, we obtain from Theorem 2.2 (ii) that the sequence $\{x_n\}$ converges to a unique point in F. □

Corollary 2.4. *Let (X,d) be a complete convex metric space and $T_i : X \to X$ $(i \in I)$ a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty}(u_n - 1) < \infty$ and that $\{x_n\}$ is obtained by modified multistep iteration* (1.3)*. Assume that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. If one of the following is true, then the sequence $\{x_n\}$ converges to a unique point in F.*

- *i) There exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0,\infty)$ such that $d(x_n, T_i x_n) \geq g(d(x_n, F))$ for all $n \geq 1$ (See Condition A' of Khan and Fukharuddin* [1]*).*
- *ii) There exists a function $f : [0,\infty) \to [0,\infty)$ which is right continuous at 0, $f(0) = 0$ and $f(d(x_n, T_i x_n)) \geq d(x_n, F)$ for all $n \geq 1$.*

Proof. First suppose that (i) holds. Then

$$\lim_{n\to\infty} g\left(d\left(x_n,F\right)\right) \leq \lim_{n\to\infty} d\left(x_n,T_ix_n\right) = 0.$$

So, $\lim_{n\to\infty} g\left(d\left(x_n,F\right)\right) = 0$ and the properties of g imply $\lim_{n\to\infty} d\left(x_n,F\right) = 0$.

Now all the conditions of Theorem 2.2 are satisfied, therefore by its conclusion, $\{x_n\}$ converges to a point of F.

Next, assume (ii). Then

$$\lim_{n\to\infty} d\left(x_n,F\right) \leq \lim_{n\to\infty} f\left(d\left(x_n,T_ix_n\right)\right) = f\left(\lim_{n\to\infty} d\left(x_n,T_ix_n\right)\right) = f\left(0\right) = 0.$$

Thus, $\liminf_{n\to\infty} d\left(x_n,F\right) = 0$ or $\limsup_{n\to\infty} d\left(x_n,F\right) = 0$. By Theorem 2.2, $\{x_n\}$ converges to a point of F. □

3. Strong Convergence Theorems of Three-Step Iteration

In this section, we give three convergence theorems related to iteration (1.4). We omit their proofs, since they are special cases of theorems given in Section 2.

Theorem 3.1. *Let (X,d) be a convex metric space and $T_1,T_2,T_3 : X \to X$ three asymptotically quasi-nonexpansive mappings with a nonempty common fixed point set F. Suppose that $\sum_{n=1}^{\infty}\left(u_n-1\right) < \infty$ and that $\{x_n\}$ is obtained by modified three-step iteration* (1.4). *Then*

(i) $\liminf_{n\to\infty} d\left(x_n,F\right) = \limsup_{n\to\infty} d\left(x_n,F\right) = 0$ *if $\{x_n\}$ converges to a unique point in F.*

(ii) $\{x_n\}$ converges to a unique point in F if X is complete and either $\liminf_{n\to\infty} d\left(x_n,F\right) = 0$ *or* $\limsup_{n\to\infty} d\left(x_n,F\right) = 0$.

Theorem 3.2. *Let (X,d) be a complete convex metric space and $T_1,T_2,T_3 : X \to X$ three asymptotically quasi-nonexpansive mappings with a nonempty common fixed point set F. Suppose that $\sum_{n=1}^{\infty}\left(u_n-1\right) < \infty$ and that $\{x_n\}$ is obtained by modified three-step iteration* (1.4). *Assume that the following two conditions hold.*

i) $\lim_{n\to\infty} d\left(x_n,x_{n+1}\right) = 0$;

ii) for the sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} d\left(y_n,y_{n+1}\right) = 0$ *we have*

$$\lim inf_{n\to\infty} d\left(y_n,F\right) = 0 \text{ or } \limsup_{n\to\infty} d\left(y_n,F\right) = 0.$$

Then $\{x_n\}$ converges to a unique point in F.

Theorem 3.3. *Let (X,d) be a complete convex metric space and $T_1,T_2,T_3 : X \to X$ three asymptotically quasi-nonexpansive mappings with a nonempty common fixed point set F. Assume that* $\lim_{n\to\infty} d\left(x_n,T_ix_n\right) = 0$ *for $i = 1,2,3$. Suppose that $\sum_{n=1}^{\infty}\left(u_n-1\right) < \infty$ and that $\{x_n\}$ is obtained by modified three-step iteration* (1.4). *If one of the following is true, then the sequence $\{x_n\}$ converges to a unique point in F.*

i) There exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g\left(0\right) = 0$, $g\left(t\right) > 0$ for all $t \in (0,\infty)$ such that $d\left(x_n,T_ix_n\right) \geq g\left(d\left(x_n,F\right)\right)$ for all $n \geq 1$.

ii) There exists a function $f : [0,\infty) \to [0,\infty)$ which is right continuous at 0, $f\left(0\right) = 0$ and $f\left(d\left(x_n,T_ix_n\right)\right) \geq d\left(x_n,F\right)$ for all $n \geq 1$.

Remark 3.4. The results presented in this paper are extensions and improvements of the corresponding results of Yang [23], Khan and Ahmed [7], Khan et al. [9], Suantai [17], Khan and Hussain [10], Gunduz and Akbulut [3, 4], Gunduz et al. [5], Nilsrakoo and Saejung [13], Khan and Kim [11], Petryshyn and Williamson [14], Khan and Yildirim [24], Shahzad and Udomene [16], Khan et al. [8] and Qihou [12].

Remark 3.5. In view of Remark 1.2, our results are valid for asymptotically nonexpansive mappings as well as nonexpansive mappings and quasi-nonexpansive mappings.

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