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Uniform exponential stability for evolution families on the half-line

Petre Preda, Raluca Mureşan*

Department of Mathematics, West University of Timişoara, 4, Blvd. Vasile Parvan, Timişoara, Romania.

Dedicated to the memory of Professor Viorel Radu

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Abstract

In this paper we give a characterization for the uniform exponential stability of evolution families $\{\Phi(t,t_0)\}_{t\geq t_0}$ on $\mathbb{R}_+$ that do not have an exponential growth, using the hypothesis that the pairs of function spaces $(L^1(X), L^\infty(X))$ and $(L^p(X), L^q(X))$, $(p,q) \neq (1,\infty)$, are admissible to the evolution families.

Keywords: Evolution family, admissibility, uniform exponential stability.
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1. Introduction

The situation where the evolution families considered have an uniform exponential growth, i.e. there exist $M, \omega > 0$ such that

$$||\Phi(t,t_0)|| \leq Me^{\omega(t-t_0)}, \text{ for all } t \geq t_0 \geq 0,$$

and the pairs of spaces considered are $(L^1(X), L^\infty(X))$ and $(L^p(X), L^p(X))$ is well known and is discussed by N. van Minh and T. Huy in [11] (2001).

The novelty in our paper is that we replace the hypothesis of uniform exponential growth with the admissibility of the pair $(L^1(X), L^\infty(X))$ to the evolution families. In addition, we consider the pair $(L^p(X), L^q(X))$, where $p \neq q$, $(p,q) \neq (1,\infty)$.

In this way we generalize some results referring to differential systems found in [9] (J. L. Massera, J. J. Schäffer, 1966). Therefore our results concerning the asymptotic properties of evolution families are discussed in the more general frame of differential equations and asymptotic properties of their solutions.

*Corresponding author
Email addresses: `preda@math.uvt.ro, rmuresan@math.uvt.ro` (Petre Preda), `rmuresan@math.uvt.ro` (Raluca Mureşan)

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As a starting point for a vast literature concerning this subject, we have to mention the pioneering work of O. Perron [13] (1930), who was the first to establish the connection between the asymptotic behavior of the solution of the differential equation

$$(A) \qquad \dot{x}(t) = A(t)x(t)$$

and the associated non-homogeneous equation

$$(A, f) \qquad \dot{x}(t) = A(t)x(t) + f(t)$$

in finite dimensional spaces, where A is a $n \times n$ dimensional, continuous and bounded matrix and f is a continuous and bounded function on $\mathbb{R}_+$.

This idea was later developed by W. A. Coppel [4] (1978) and P. Hartman [7] (1964) for differential systems in finite dimensional spaces.

Further developments for differential systems in infinite dimensional spaces can be found in the monographs of J. L. Daleckij, M. G. Krein [5] (1974) and J. L. Massera, J. J. Schäffer [9] (1966). The case of dynamical systems described by evolution processes was studied by C. Chicone, Y. Latushkin [3] (1999) and B. M. Levitan, V. V. Zhikov [8] (1982).

Other results concerning uniform and non-uniform exponential stability, exponential dichotomy and admissibility of exponentially bounded evolution families were obtained by L. Barreira and C. Valls in [1], [2], N. van Minh in [10], [11], [12], F. Räbiger and R. Schnaubelt in [12], P. Preda and C. Preda in [14], [15], [16].

The starting point for our paper is [11] (N. van Minh, N. T. Huy 2001). Inspired by this paper, we use the input-output technique, i.e. we choose carefully selected input functions that allow us to prove our main result, and some well known results in functional analysis.

In this paper we firstly prove that the evolution families $\{\Phi(t, t_0)\}_{t \geq t_0}$ on $\mathbb{R}_+$ are uniformly stable, i.e. there exists a positive constant N such that

$$||\Phi(t, t_0)x|| \leq N||x||$$

for all $t \geq t_0 \geq 0$ and x in X, using the admissibility of $(L^1(X), L^\infty(X))$. The admissibility of the other pair $(L^p(X), L^q(X))$, $(p, q) \neq (1, \infty)$, is used to prove the uniform exponential stability of the evolution families.

2. Preliminaries

Let X be a Banach space and $\mathbb{B}(X)$ the space of all linear and bounded operators acting on X. The norms on X and on $\mathbb{B}(X)$ will be denoted by $|| \cdot ||$.

Definition 2.1. An evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ on $\mathbb{R}_+$ is a family of operators $\Phi(t, t_0) \in \mathbb{B}(X)$, $t \geq t_0 \geq 0$, satisfying:

- (i) $\Phi(t, t) = I$, for all $t \in \mathbb{R}_+$, where I denotes the identity on X;
- (ii) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$ for all $t \geq s \geq t_0 \geq 0$;
- (iii) the map $\Phi(\cdot, t_0)x$ is continuous on $[t_0, \infty)$ for all $x \in X$ and $\Phi(t, \cdot)x$ is continuous on $[0, t]$ for all $x \in X$.

Throughout this paper we use the following function spaces:

$$L^p(X) = \{f : \mathbb{R}_+ \to X : f \text{ is Bochner measurable, } \int_0^\infty ||f(t)||^p dt < \infty\},$$

where $p \geq 1$, and

$$L^\infty(X) = \{f : \mathbb{R}_+ \to X : f \text{ is Bochner measurable, } ess \sup_{t \geq 0} ||f(t)|| < \infty\}.$$

The norm on $L^p(X)$ is

$$||f||_p = (\int_0^\infty ||f(t)||^p dt)^{\frac{1}{p}}$$

and the norm on $L^\infty(X)$ is

$$||f||_\infty = ess\sup_{t\geq 0} ||f(t)||.$$

Definition 2.2. Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. It is uniform exponentially stable if there exist $N, \nu > 0$ such that

$$||\Phi(t,t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||,$$

for all $t \geq t_0 \geq 0$ and $x \in X$.

Definition 2.3. Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. The pair of function spaces $(L^p(X), L^q(X))$ is said to be admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ if for every $f \in L^p(X)$, the function $x_f : \mathbb{R}_+ \to X$,

$$x_f(t) = \int_0^t \Phi(t,\tau)f(\tau)d\tau,$$

is in $L^q(X)$.

3. Main Results

Theorem 3.1. *Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. If the pair of spaces $(L^p(X), L^q(X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then there exists a constant $k > 0$ such that*

$$||x_f||_q \leq k||f||_p,$$

for all f in $L^p(X)$.

Proof. We consider the linear operator $\mathcal{U} : L^p(X) \to L^q(X)$,

$$\mathcal{U}(f) = x_f.$$

We will prove that the operator $\mathcal{U}$ is closed.

Let $f_n \xrightarrow[n\to\infty]{L^p(X)} f$ and $\mathcal{U}f_n \xrightarrow[n\to\infty]{L^q(X)} g$. We show that $\mathcal{U}f = g$ in $L^q(X)$.

We have that

$$||\mathcal{U}f_n(t) - \mathcal{U}f(t)|| = ||\int_0^t \Phi(t,\tau)(f_n(\tau) - f(\tau))d\tau|| \leq$$

$$\leq \int_0^t ||\Phi(t,\tau)(f_n(\tau) - f(\tau))||d\tau,$$

for all $t \geq 0$.

The map $\tau \mapsto \Phi(t,\tau)x : [0,t] \to X$ is continuous, so it is bounded, i.e. there exists a constant $M_{t,x} > 0$ such that

$$||\Phi(t,\tau)x|| \leq M_{t,x},$$

for all $t \geq 0$ and $x \in X$.

By the Uniform Boundedness Principle there exists $M(t) > 0$ such that

$$||\mathcal{U}f_n(t) - \mathcal{U}f(t)|| \leq M(t)\int_0^t ||f_n(\tau) - f(\tau)|| \leq$$

$$\leq M(t)t^{\frac{1}{p'}}||f_n - f||_p \xrightarrow[n\to\infty]{} 0,$$

for all $t \geq 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$. In the above inequality we also used the Hölder inequality.

Therefore

$$\mathcal{U}f_n(t) \xrightarrow[n\to\infty]{} \mathcal{U}f(t),$$

for all $t \geq 0$.

But $\mathcal{U}f_n \xrightarrow{L^q(X)} g$, so

$$\mathcal{U}f = g$$

in $L^q(X)$, which means that $\mathcal{U}$ is a closed linear operator.

By the Closed Graph Theorem, we have that there exists a positive constant k such that

$$||x_f||_q \leq k||f||_p,$$

for all f in $L^p(X)$. □

Theorem 3.2. *Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. If the pairs of spaces $(L^1(X), L^\infty(X))$ and $(L^p(X), L^q(X))$ are admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniform exponentially stable.*

Proof. Firstly we show that the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniformly stable on X, i.e. there exists a positive constant N such that

$$||\Phi(t,t_0)x|| \leq N||x||,$$

for all $x \in X$ and $t \geq t_0 \geq 0$.

Let $t_0 \geq 0$, $\delta > 0$, $x \in X$ such that $\Phi(t,t_0)x \neq 0$ for all $t \geq t_0$ and the first input function $f : \mathbb{R}_+ \to X$,

$$f(t) = \varphi_{[t_0,t_0+\delta]}(t)\frac{\Phi(t,t_0)x}{||\Phi(t,t_0)x||},$$

where $\varphi_{[t_0,t_0+\delta]}$ is the characteristic function of the interval $[t_0, t_0 + \delta]$.

We have that

$$\int_0^\infty ||f(\tau)||d\tau = \int_0^\infty \varphi_{[t_0,t_0+\delta]}(\tau)d\tau = \delta < \infty,$$

therefore f is in $L^1(X)$ and $||f||_1 = \delta$.

Since the pair of spaces $(L^1(X), L^\infty(X))$ is admissible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, then the output function $x_f : \mathbb{R}_+ \to X$ is in $L^\infty(X)$, where

$$x_f(t) = \int_0^t \Phi(t,\tau)f(\tau)d\tau.$$

But

$$x_f(t) = \int_{t_0}^{t_0+\delta} \frac{1}{||\Phi(\tau,t_0)x||}d\tau||\Phi(t,t_0)x||,$$

for all $t \geq t_0 + \delta$ and by Theorem 3.1 we have that

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \frac{1}{||\Phi(\tau,t_0)x||}d\tau||\Phi(t,t_0)x|| \leq k.$$

If $\delta \to 0$, then

$$||\Phi(t,t_0)x|| \leq k||x||,$$

for all $t \geq t_0 \geq 0$.

Now, if there exists $t_1 > t_0$ such that

$$\Phi(t_1,t_0)x = 0,$$

then obviously

$$\Phi(t, t_0)x = 0, \text{ for all } t \geq t_1.$$

We denote $\sigma = \inf_{t\geq t_0}\{\Phi(t, t_0)x = 0\}$, so $\Phi(\sigma, t_0)x = 0$ and $\Phi(t, t_0)x \neq 0$ for all $t \in [t_0, \sigma)$.

Therefore

$$||\Phi(t, t_0)x|| \leq k||x|| \text{ for all } t \in [t_0, \sigma)$$

as seen above and obviously

$$||\Phi(t, t_0)x|| \leq k||x|| \text{ for all } t \geq \sigma.$$

We can conclude that

$$||\Phi(t, t_0)x|| \leq k||x||$$

for all $t \geq t_0 \geq 0$ and x in X.

Next we show that the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$ is uniform exponentially stable. Let $g : \mathbb{R}_+ \to X$,

$$g(t) = \varphi_{[t_0,t_0+1]}(t)\Phi(t, t_0)x$$

be the second input function, where $\varphi_{[t_0,t_0+1]}$ is the characteristic function of $[t_0, t_0 + 1]$.

We have that

$$\int_0^\infty ||g(\tau)||^p d\tau \leq k^p||x||^p < \infty,$$

so g is in $L^p(X)$ and $||g||_p \leq k||x||$.

Since the pair of spaces $(L^p(X), L^q(X))$ is admissible to $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$, then the output function $x_g : \mathbb{R}_+ \to X$,

$$x_g(t) = \int_0^t \Phi(t, \tau)g(\tau)d\tau$$

is in $L^q(X)$.

But

$$x_g(t) = \Phi(t, t_0)x$$

for all $t \geq t_0 + 1$, so by Theorem 3.1 we have that

$$\Big(\int_{t_0+1}^\infty ||\Phi(\tau, t_0)x||^q d\tau\Big)^{\frac{1}{q}} \leq k||g||_p \leq k^2||x||.$$

We have that

$$\Big(\int_{t_0}^\infty ||\Phi(\tau, t_0)x||^q d\tau\Big)^{\frac{1}{q}} \leq k\sqrt[q]{1 + k^q}||x||$$

for all x in X.

If $q < \infty$, by Theorem 1.1 in [6] (also known as the Theorem of Datko) it follows that the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$ is uniformly exponential stable.

If $q = \infty$, then $p > 1$ because $(p, q) \neq (1, \infty)$.

We consider the third input function $h : \mathbb{R}_+ \to X$,

$$h(t) = \varphi_{[t_0,t_0+\delta]}(t)\Phi(t, t_0)x.$$

We have that

$$||h||_p \leq \delta^{\frac{1}{p}}k||x||,$$

so h is in $L^p(X)$ and since the pair of spaces $(L^p(X), L^\infty(X))$ is admissible to $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$, then the output function $x_h : \mathbb{R}_+ \to X$,

$$x_h(t) = \int_0^t \Phi(t, \tau)h(\tau)d\tau$$

is in $L^\infty(X)$.

By Theorem 3.1, it follows that

$$||x_h(t)|| \leq ||x_h||_\infty \leq k||h||_p \leq \delta^{\frac{1}{p}} k^2 ||x||$$

for all $t \geq 0$.

Therefore

$$||\Phi(t, t_0)x|| \leq \delta^{\frac{1}{p}-1} k^2 ||x||,$$

for all $t \geq t_0 + \delta$.

Now there exists $\delta_0 > 0$ such that

$$||\Phi(t_0 + \delta_0, t_0)x|| \leq \frac{1}{2}||x||$$

for all $t_0 \geq 0$ and x in X. Since we have already proved that the evolution family $\{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ is uniformly stable, by Lemma 5.3, page 539 in [9] (also known as the Lemma of Massera and Schäffer), it follows that the evolution family is uniformly exponential stable. □

Remark 3.3. The converse of the theorem above is true if and only if $p \leq q$.

For the proof of the remark above see [7], Theorem 6.4, page 477.

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