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On the stability of an affine functional equation

Liviu Cădariu*, Laura Găvruţa, Paşc Găvruţa

"Politehnica" University of Timişoara, Department of Mathematics, Piaţa Victoriei no.2, 300006 Timişoara, Romania.

Dedicated to the memory of Professor Viorel Radu

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Abstract

In this paper, we obtain the general solution and we prove the generalized Hyers-Ulam stability for an affine functional equation.

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1. Introduction and Preliminaries

The study of the functional equations stability originated from a question of S. M. Ulam ([29], 1940) in a talk at the University of Wisconsin, concerning the stability of group homomorphisms:

Let $(G_1, \circ)$ *be a group and* $(G_2, *)$ *a metric group with a metric* $d(\cdot, \cdot)$*. Given* $\varepsilon > 0$*, does there exist a* $\delta > 0$ *such that if* $f : G_1 \to G_2$ *satisfies*

$$d(f(x \circ y), f(x) * f(y)) \leq \delta, \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $h : G_1 \to G_2$ *with*

$$d(f(x), h(x)) \leq \varepsilon, \text{ for all } x \in G_1?$$

In 1941 D. H. Hyers [22] gave an affirmative answer to the question of Ulam for Cauchy functional equation in Banach spaces. The result of D. H. Hyers was generalized in 1950 by T. Aoki [1] for approximately additive mappings and in 1978 by Th. M. Rassias [27] for approximately linear mappings, by considering

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*Corresponding author

Email addresses: liviu.cadariu@mat.upt.ro, lcadariu@yahoo.com (Liviu Cădariu), laura.gavruta@mat.upt.ro (Laura Găvruţa), pgavruta@yahoo.com (Paşc Găvruţa)

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the unbounded Cauchy differences. A further generalization was obtained by P. Găvruţa [19] in 1994, by replacing the Cauchy differences by a control mapping φ satisfying a very simple condition of convergence. We refer the reader to the expository papers [15], [28] and to the books [12], [23] and [24] (see also the papers [14], [17], [20], [16], for supplementary details).

A large part of proofs in this topic used the *direct method* (of Hyers): the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution. On the other hand, in 1991 J. A. Baker [2] used the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, V. Radu [26] proposed a new method, successively developed in [6, 7, 8], to obtaining the existence of the exact solutions and the error estimations, based on the *fixed point alternative*. Subsequently, these results were generalized by D. Miheţ [25], L. Găvruţa [18] and by L. Cădariu & V. Radu [9, 10]. Lately, P. Găvruţa and L. Găvruţa introduced a new method in [21], called the *weighted space method*, for the generalized Hyers-Ulam stability (see, also [4]). Recently, a general fixed point result and some applications to the stability of a nonlinear functional equation were obtained in [5] (see also [3]).

In the paper [11] I.-S. Chang & H.-M. Kim obtained the general solution and the generalized Hyers-Ulam stability for the quadratic type functional equations:

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x)$$

and

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y).$$

In the present paper we obtain the general solution of the following affine functional equation

$$f(2x+y)+f(x+2y)+f(x)+f(y)=4f(x+y), \forall x,y\in G, \tag{1.1}$$

where $f: G\to X$, G is an abelian group and X is a normed space. After that, by using the *direct method* as well as *the fixed point method*, we prove some generalized Hyers-Ulam stability results for this equation.

2. Solution of the functional equation (1.1)

Theorem 2.1. *A mapping f is a solution of the functional equation (1.1) iff it is an affine mapping (i.e., it is the sum between a constant and an additive function).*

Proof. It is easy to see that any affine function f is a solution of the equation (1.1).

Conversely, we have two cases:

Case 1: $f(0)=0$.

If we take $y=-x$ in (1.1), we obtain

$$f(x)+f(-x)+f(x)+f(-x)=4f(0)=0, \forall x\in G,$$

which implies $f(-x)=-f(x)$, for all $x\in G$. It results that f is an odd mapping.

By replacing x with $x-y$ in (1.1), we have:

$$f(2x-y)+f(x+y)+f(x-y)+f(y)=4f(x), \forall x,y\in G.$$

If we substitute y by $-y$ in the last equation, the following relation holds:

$$f(2x+y)+f(x-y)+f(x+y)+f(-y)=4f(x), \forall x,y\in G, \tag{2.1}$$

Interchanging x with y in the above equation, it results

$$f(2y+x)+f(y-x)+f(y+x)+f(-x)=4f(y), \forall x,y\in G. \tag{2.2}$$

Now, we sum up the relations (2.1) and (2.2):

$$f(2x+y)+f(x+2y)+2f(x+y)-(f(x)+f(y))=4(f(x)+f(y)), \forall x,y\in G,$$

hence

$$f(2x+y)+f(x+2y)+2f(x+y)+f(x)+f(y)=6(f(x)+f(y))-2f(x+y) \tag{2.3}$$

for all $x,y\in G$.

From (1.1) and (2.3) we obtain

$$4f(x+y)=6(f(x)+f(y))-2f(x+y)\Leftrightarrow f(x+y)=f(x)+f(y), \forall x,y\in G.$$

so, f is an additive mapping.

Case 2: General case.

Let us consider the function $g(x):=f(x)-f(0)$. It is clear that $g(0)=0$ and $f(x)=g(x)+f(0)$.

Replacing f in (1.1), it results

$$g(2x+y)+g(x+2y)+g(x)+g(y)=4g(x+y), \forall x,y\in G.$$

Taking in account that $g(0)=0$, from *Case 1*, we obtain that g is an additive maping, hence $f(x)=g(x)+f(0)$ is an affine function. □

3. The direct method for the generalized Hyers-Ulam stability of the equation (1.1)

In this section we will obtain some properties of the generalized Hyers-Ulam stability for the affine functional equation (1.1). For the proof, we will use the direct method.

We denote by $(G,+)$ an abelian group, by $(X,||\cdot||)$ a Banach space and by $\varphi: G\times G\to[0,\infty)$ a mapping such that

$$\Phi(x):=\sum_{k=0}^{\infty}\frac{\varphi(2^k x,0)}{2^k}<\infty, \forall x\in G \tag{3.1}$$

and

$$\lim_{n\to\infty}\frac{\varphi(2^n x,2^n y)}{2^n}=0, \forall x,y\in G. \tag{3.2}$$

We formulate the main result of the paper:

Theorem 3.1. *Let $f:G\to X$, such that*

$$||f(2x+y)+f(x+2y)+f(x)+f(y)-4f(x+y)||\le\varphi(x,y), \forall x,y\in G. \tag{3.3}$$

Then there exists a unique mapping $A:G\to X$, which satisfies the equation (1.1) and

$$||f(x)-A(x)-f(0)||\le\frac{1}{2}\Phi(x), \tag{3.4}$$

for all $x\in G$.

Proof: For $y=0$ in (3.3), we obtain

$$||f(2x)-2f(x)+f(0)||\le\varphi(x,0), \forall x\in G.$$

If we define the function $g:G\to X$,

$$g(x):=f(x)-f(0), \tag{3.5}$$

we have

$$||g(2x)-2g(x)||\le\varphi(x,0), \forall x\in G.$$

Thus

$$\left\|\frac{g(2x)}{2}-g(x)\right\| \leq \frac{1}{2}\varphi(x,0), \forall x \in G. \tag{3.6}$$

If we replace x by $2x$ in the above relation and divide it by 2, it results

$$\left\|\frac{g(2^2x)}{2^2}-\frac{g(2x)}{2}\right\| \leq \frac{1}{2^2}\varphi(2x,0), \forall x \in G. \tag{3.7}$$

Using the triangle inequality, from (3.6) and (3.7), it follows that

$$\left\|\frac{g(2^2x)}{2^2}-g(x)\right\| \leq \frac{1}{2}\left(\varphi(x,0)+\frac{1}{2}\varphi(2x,0)\right), \forall x \in G.$$

It is easy to prove, by induction on n, that

$$\left\|\frac{g(2^nx)}{2^n}-g(x)\right\| \leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\varphi(2^kx,0)}{2^k}, \forall x \in G.$$

Now we claim that the sequence $\{2^{-n}g(2^nx)\}$ is a Cauchy sequence. Indeed, for $n > m > 0$, we have:

$$\begin{aligned}
\left\|2^{-n}g(2^nx)-2^{-m}g(2^mx)\right\| &= 2^{-m}\left\|2^{-(n-m)}g(2^{n-m}\cdot 2^mx)-g(2^mx)\right\| \leq \\
&\leq 2^{-m}\,2^{-1}\sum_{k=0}^{n-m-1}\frac{\varphi(2^{k+m}x,0)}{2^k} = \\
&= \frac{1}{2}\sum_{p=m}^{n-1}\frac{\varphi(2^px,0)}{2^p}, \forall x \in G.
\end{aligned}$$

Taking the limit as $m \to \infty$, it results that

$$\lim_{m\to\infty}\left\|2^{-n}g(2^nx)-2^{-m}g(2^mx)\right\| = 0, \forall x \in G.$$

Since X is a Banach space, then we obtain that the sequence $\{2^{-n}g(2^nx)\}$ converges. We define

$$A(x) := \lim_{n\to\infty}\frac{g(2^nx)}{2^n},$$

for each x in G. From (3.5) it is clear that

$$A(x) = \lim_{n\to\infty}\frac{f(2^nx)}{2^n}, \forall x \in G. \tag{3.8}$$

We claim that A satisfies (1.1). Replace x and y by 2^nx and 2^ny, respectively, in relation (3.3) and divide by 2^n. It follows that

$$||2^{-n}f(2^n(2x+y))+2^{-n}f(2^n(x+2y))+2^{-n}f(2^n(x))+2^{-n}f(2^n(y))-2^{-n}\cdot 4f(2^n(x+y))|| \leq 2^{-n}\varphi(2^nx,2^ny),$$

for all $x, y \in G$. Taking on the limit as $n \to \infty$ in the above relation and using (3.2) and (3.8), it results

$$A(2x+y)+A(x+2y)+A(x)+A(y) = 4A(x+y).$$

In order to show that A is the unique function defined on G, with the properties (1.1) and (3.4), let $B : G \to X$ be another affine mapping such that

$$||f(x)-f(0)-B(x)|| \leq \frac{1}{2}\Phi(x), \forall x \in G,$$

It follows that

$$A(2^n x) + A(0) = 2^n A(x), \quad B(2^n x) + B(0) = 2^n B(x),$$

for all x in G. Then

$$\begin{aligned}
||A(x) - B(x)|| &= \left|\left|\frac{(A(2^n x) + A(0)) - (B(2^n x) + B(0))}{2^n}\right|\right| \leq \\
&\leq \left|\left|\frac{A(2^n x) - f(0) - f(2^n x)}{2^n}\right|\right| + \left|\left|\frac{B(2^n x) - f(0) - f(2^n x)}{2^n}\right|\right| + \left|\left|\frac{A(0) - B(0)}{2^n}\right|\right| \leq \\
&\leq 2^{-n} \cdot \frac{1}{2}\, \Phi(2^n x) + 2^{-n} \cdot \frac{1}{2}\, \Phi(2^n x) + 2^{-n}\, ||A(0) - B(0)|| = \\
&= 2^{-n} \Phi(2^n x) + 2^{-n}\, ||A(0) - B(0)|| = \\
&= \sum_{k=0}^{\infty} \frac{\varphi(2^{k+n} x, 0)}{2^k \cdot 2^n} + 2^{-n}\, ||A(0) - B(0)|| = \\
&= \sum_{p=n}^{\infty} \frac{\varphi(2^p x, 0)}{2^p} + 2^{-n}\, ||A(0) - B(0)||\,, \forall x \in G.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above relation we obtain that A coincides with B. This completes the proof of the theorem. □

From the Theorem 3.1 we obtain the following corollary concerning the stability of type Aoki-Th.M. Rassias for the equation (1.1).

Corollary 3.2. *Let G be an abelian group and X be a Banach space, respectively. Let p, q, ε be real numbers such that $\varepsilon > 0$, $p, q \in [0, 1)$. Suppose that a function $f : G \to X$ satisifies*

$$||f(2x + y) + f(x + 2y) + f(x) + f(y) - 4f(x + y)|| \leq \varepsilon(||x||^p + ||y||^q), \forall x, y \in G.$$

Then there exists a unique mapping $A : G \to X$, which satisfies the equation (1.1) and the estimation

$$||f(x) - A(x) - f(0)|| \leq \frac{\varepsilon}{2 - 2^p}||x||^p, \forall x \in G.$$

To prove this result, it is enough to take in the Theorem 3.1 $\varphi(x, y) := \varepsilon(||x||^p + ||y||^q)$, with $\varepsilon > 0$ and $p, q \in [0, 1)$. Obviously, the relation (3.2) holds and $\Phi(x) = \frac{\varepsilon}{1 - 2^{p-1}}||x||^p$.

Remark 3.3. For $p = q = 0$ in the above corollary, properties of stability in Ulam-Hyers sense for the equation (1.1) are obtained.

Remark 3.4. It seems that in the case $p = q = 1$ the affine functional equation (1.1) is unstable.

4. Fixed points and generalized Hyers-Ulam stability of the affine functional equation (1.1)

In this section we will use our recent result in [5] to prove the properties of stability from the Theorem 3.1.

We consider a nonempty set G, a complete metric space (X, d) and the mappings $\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G$ and $\mathcal{T} : X^G \to X^G$. We remember that X^G is the space of all mappings from G into X. In the following, we suppose that Λ satisfies the condition:

for every sequence $(\delta_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^G$, with $\delta_n(t) \underset{n\to\infty}{\longrightarrow} 0, t \in G \Longrightarrow (\Lambda\delta_n)(t) \underset{n\to\infty}{\longrightarrow} 0, t \in G.$ (C_1)

Proposition 4.1 ([5], Corollary 2.3)**.** *Let G be a nonempty set, (X, d) a complete metric space and $\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G$ be a non-decreasing operator satisfying the hypothesis (C_1). If $\mathcal{T} : X^G \to X^G$ is an operator satisfying the inequality.*

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \leq \Lambda(d(\xi(x), \mu(x))), \quad \xi, \mu \in X^G, x \in G, \tag{4.1}$$

and the functions $\varepsilon : G \to \mathbb{R}_+$ *and* $g : G \to X$ *are such that*

$$d((\mathcal{T}g)(x), g(x)) \le \varepsilon(x),\ x \in G, \tag{4.2}$$

and

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) < \infty,\ x \in G, \tag{C_2}$$

then, for every $x \in G$*, the limit*

$$A(x) := \lim_{n\to\infty} (\mathcal{T}^n g)(x)$$

exists and the function $A \in X^G$*, defined in this way, is a fixed point of* $\mathcal{T}$*, with*

$$d(g(x), A(x)) \le \varepsilon^*(x),\ x \in G.$$

Moreover, if the condition

$$\lim_{n\to\infty} (\Lambda^n \varepsilon^*)(x) = 0, \forall x \in G, \tag{C_3}$$

holds, then A *is the unique fixed point of* $\mathcal{T}$ *with the property*

$$d(g(x), A(x)) \le \varepsilon^*(x),\ x \in G.$$

The proof of Theorem 3.1. We apply the above proposition taking the mapping

$$\Lambda : \mathbb{R}_+^G \to \mathbb{R}_+^G, (\Lambda\delta)(x) := \frac{\delta(2x)}{2},\ (\delta : G \to \mathbb{R}_+),$$

and the operator

$$\mathcal{T} : X^G \to X^G, (\mathcal{T}\psi)(x) := \frac{\psi(2x)}{2},\ (\psi : G \to X).$$

From the definition of Λ, the relation (C_1) is obvious and (4.1) holds with equality.

If we take $\varepsilon(x) := \frac{\varphi(x,0)}{2}$, where the mapping φ is defined in Theorem 3.1, the relation (3.1) implies that the series

$$\varepsilon^*(x) = \sum_{k=0}^{\infty} \left(\Lambda^k \varepsilon\right)(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varphi(2^k x, 0)}{2^k} = \frac{\Phi(x)}{2}, \forall x \in G$$

is convergent, so (C_2) is verified.

As in the first part of the initial proof of Theorem 3.1, we have that

$$\left\|\frac{g(2x)}{2} - g(x)\right\| \le \frac{1}{2}\varphi(x,0), \forall x \in G,$$

where $g(x) := f(x) - f(0)$ and f satisfied the hypotheses of Theorem 3.1. This means that (4.2) holds.

Also

$$(\Lambda^n \varepsilon^*)(x) = \frac{(\Lambda^n \Phi)(x)}{2} = \frac{\Phi(2^n x)}{2^{n+1}} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varphi(2^{n+k}x, 0)}{2^{n+k}} = \frac{1}{2}\sum_{p=n}^{\infty} \frac{\varphi(2^p x, 0)}{2^p}, \forall x \in G.$$

Taking on the limit in the above relation as $n \to \infty$, we obtain that (C_3) is verified.

From Proposition 4.1, it results that the limit

$$\lim_{n\to\infty} (\mathcal{T}^n g)(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for every $x \in G$. Moreover, the mapping $A : G \to X$,

$$A(x) = \lim_{n\to\infty} (\mathcal{T}^n g)(x)$$

is the unique fixed point of $\mathcal{T}$, with

$$d(g(x), A(x)) \leq \varepsilon^*(x), \forall x \in G,$$

which implies that

$$||f(x) - f(0) - A(x)|| \leq \frac{1}{2}\Phi(x), \forall x \in G.$$

To prove that the function A is a solution of the affine equation (1.1) we use (3.2) and the definition of A. □

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