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Convergence of iterative methods for solving random operator equations

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This paper is dedicated to the memory of Professor Viorel Radu

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Abstract

We discuss the concept of probabilistic quasi-nonexpansive mappings in connection with the mappings of Nishiura. We also prove a result regarding the convergence of the sequence of successive approximations for probabilistic quasi-nonexpansive mappings.

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1. Introduction

Consider the equation

$$Ax = y, \qquad y \in X \tag{1.1}$$

where A is a mapping on a metric space X endowed with the metric d. Suppose that (1.1) has a solution x^* and only one. An iterative method for solving (1.1) is defined by a mapping T from X into X having x^* as a fixed point, and consists of a sequence of successive approximations x_n, $n \geq 0$, defined by

$$x_{n+1} = Tx_n, \qquad n \geq 0. \tag{1.2}$$

The fundamental problem in approximating the solution x^* concerns the convergence of the sequence (1.2) to x^*. In the literature there are many results in this respect. Widely used are the methods for which the operator T is *quasi-nonexpansive* on a subset D of X, meaning that

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(a) T has a fixed point $x^* \in D$, and
(b) T satisfies the inequality

$$d(Tx, x^*) < d(x, x^*), \qquad x \in D, x \neq x^*. \tag{1.3}$$

This inequality implies the uniqueness of the fixed point x^* of T. If, moreover, T satisfies (1.3) in the form

$$d(Tx, x^*) \leq kd(x, x^*), \qquad x \in D, x \neq x^* \tag{1.4}$$

for a constant $k \in [0, 1)$, the T is said to be a *strict quasi-nonexpansive* mapping on D.

We note that the concept of quasi-nonexpansive mappings was first considered on the real line by F. Tricomi in 1941, and it was generalized and studied intensively in random spaces.

On the other hand, when the equation (1.1) is perturbed with a random noise, the mappings A and T depend on a parameter belonging to a probability space $(\Omega, \mathcal{A}, P)$. Thus, A and T can be considered as random mappings, and the fixed point formulation of the equation (1.1) becomes

$$T(\omega, \xi(\omega)) = \xi(\omega), \qquad \omega \in \Omega. \tag{1.5}$$

The fixed point is a mapping ξ^* from Ω into X. The problem of the measurability of ξ^* was studied by many authors (see e.g. [2]).

Gh. Bocşan [1] showed that the study of the random fixed point for the equation (1.5) can be connected with the study of fixed points for the Nemytskii operator induced by T on probabilistic metric spaces. This operator is defined on the space S of the X-valued random variables ξ by

$$(\tilde{T}\xi)(\omega) = T(\omega, \xi(\omega)), \qquad \omega \in \Omega. \tag{1.6}$$

The convergence in probability of the sequence of iterates of equation (1.5)

$$\xi_{n+1}(\omega) = T(\omega, \xi_n(\omega)), \qquad \omega \in \Omega, n \geq 0 \tag{1.7}$$

means the convergence of the sequence

$$\xi_{n+1} = \tilde{T}\xi_n, \qquad n \geq 0. \tag{1.8}$$

Based on this observation, it is justified to generalize the results on quasi-nonexpansive mappings to probabilistic metric spaces.

2. The order on Δ^+

Consider the set Δ^+ of all left-continuous distribution functions F satisfying $F(0) = 0$ and define an order on this set by $F \leq G$ iff $F(x) \leq G(x)$ for all $x > 0$; we write $F < G$ if $F \leq G$ and $F(x) < G(x)$ for some $x > 0$. As in [4], for $F \in \Delta^+$ define $F^\vee : [0, 1) \to [0, \infty)$ by

$$F^\vee(a) = \inf\{x > 0 : F(x) > a\}.$$

We adapt the considerations in [4] to our context. We first note that F can be recovered from $F^\vee$ by the formula

$$F(x) = \sup\{a \in [0, 1) : F^\vee(a) < x\}$$

because

$$F(x) > a \Leftrightarrow F^\vee(a) < x. \tag{2.1}$$

Indeed, if $F(x) > a$, then, since F is left continuous in x, we have $F(x - \delta) > a$ for some $\delta > 0$, therefore $F^\vee(a) \leq x - \delta < x$. Thus, $F(x) > a$ implies $F^\vee(a) < x$. Clearly, if $F^\vee(a) < x$, then $x \in \{x > 0 : F(x) > a\}$, hence $F(x) > a$. Therefore, $F^\vee(a) < x$ implies $F(x) > a$, and thus the equivalence (2.1) is proved.

Proposition 2.1. *The order on Δ^+ can be characterized in terms of the functions $F^\vee$ by*

$$F \leq G \Leftrightarrow F^\vee \geq G^\vee.$$

Proof. It is immediate that $F \leq G$ implies $F^\vee \geq G^\vee$. We now suppose $F^\vee \geq G^\vee$, therefore $F^\vee(a) \geq G^\vee(a)$ for all $a \in (0,1)$. For $x > 0$ let $a < F(x)$. From relation (2.1) it follows that $F^\vee(a) < x$, and from $F^\vee \geq G^\vee$ we obtain $G^\vee(a) < x$. Once again, by (2.1), $G(x) > a$. Therefore, for every $a < F(x)$ we have $a < G(x)$. Thus, $F(x) \leq G(x)$, for all $x > 0$. □

Remark 2.2. The following characterization also holds:

$$F < G \Leftrightarrow F^\vee > G^\vee.$$

Proof. We first note that $F < G$ implies $F^\vee \geq G^\vee$. Also, from $F < G$ it follows that there exists $x_0 > 0$ such that $F(x_0) < G(x_0)$. Let $a_0 \in (0,1)$ be such that $F(x_0) < a_0 < G(x_0)$. From the first inequality, we have $x_0 \notin \{x > 0 : F(x) > a_0\}$, i.e. $F^\vee(a_0) \geq x_0$. From the second inequality we have $G^\vee(a_0) < x_0$. Hence, $F < G$ implies $F^\vee > G^\vee$.

Conversely, suppose $F^\vee > G^\vee$. From Proposition 2.1, it is clear that $F \leq G$. Let a_0 and x_0 be such that $F^\vee(a_0) > x_0 > G^\vee(a_0)$. Then, by using (2.1), we obtain $F(x_0) \leq a_0 < G(x_0)$. □

3. The mappings of Nishiura

Consider a Menger space S with respect to a continuous t-norm, and let F_{pq} be the probabilistic distance between the points $p, q \in S$. We recall that the probabilistic distance defines the (ε, λ)-uniformity on S, for which the following family is a basis:

$$\mathcal{U} = \{U(x,a), x > 0, a \in [0,1)\}, \ U(x,a) = \{(p,q) \in S \times S : F_{pq}(x) > a\}.$$

The topology on S defined by the (ε, λ)-uniformity is defined by the family

$$\mathcal{U}_S = \{U_p(x,a), p \in S, x > 0, a \in [0,1)\}, \ U_p(x,a) = \{q \in S : F_{pq}(x) > a\}.$$

The convergence of a sequence and the continuity of a mapping on S are considered with respect to this topology. On the other hand, following E. Nishiura [3], for each $a \in [0,1)$ define a function d_a on $S \times S$ by the equality

$$d_a(p,q) = \inf\{x > 0 : F_{pq}(x) > a\} = F_{pq}^\vee(a), \qquad p, q \in S.$$

Lemma 3.1. *[3] The following properties of the family d_a, $a \in [0,1)$, hold:*

(i) *$p = q$ iff $d_a(p,q) = 0$ for all $a \in [0,1)$.*
(ii) *$F_{pq}(x) > a$ iff $d_a(p,q) < x$.*
(iii) *For each $(p,q) \in S \times S$, the function $a \mapsto d_a(p,q)$ is nondecreasing, right-continuous, and*

$$F_{pq}(x) = \sup\{a \in [0,1) : d_a(p,q) < x\}, \qquad x > 0.$$

Therefore

$$U(x,a) = \{(p,q) \in S \times S : d_a(p,q) < x\}$$

and

$$U_p(x,a) = \{q \in S : d_a(p,q) < x\},$$

and thus we obtain the following result:

Lemma 3.2. *(i) A sequence $\{p_n, n \geq 1\}$ converges to p in the space S if and only if for each $a \in [0,1)$, the sequence $\{d_a(p_n,p), n \geq 1\}$ converges to 0.*

(ii) For each $a \in [0,1)$, the function $d_a(p,q)$ is continuous on $S \times S$.

4. Probabilistic quasi-nonexpansive mappings

Definition 4.1. Let $T : D \to S$ be a mapping defined on a subset D of S. T is said to be a *probabilistic quasi-nonexpansive mapping* on D if

(i) T has a fixed point $p^* \in D$, and
(ii) T satisfies the inequality

$$F_{p^*Tp} > F_{p^*p}, \qquad p \in D, p \neq p^*.$$

Clearly, from (ii) it follows that T has a unique fixed point in D.

Lemma 4.2. *Suppose $T : D \to S$ has a fixed point $p^* \in D$. If T is a probabilistic quasi-nonexpansive mapping on D, then*

(i) *for each $a \in [0, 1)$ the following inequality holds:*

$$d_a(Tp, p^*) \leq d_a(p, p^*), \qquad p \in D, p \neq p^*$$

(ii) *for all $p \in D$, $p \neq p^*$, there exists $a_p \in [0, 1)$ such that*

$$d_{a_p}(Tp, p^*) < d_{a_p}(p, p^*).$$

We now consider a probabilistic quasi-nonexpansive mapping $T : D \subseteq S \to S$ and let $p_{n+1} = Tp_n$, $n \geq 0$, denote the sequence of successive approximations of T starting at $p_0 \in D$; of course, we suppose that $p_n \in D$ for all $n \geq 1$. Then, we prove the main result of this paper.

Theorem 4.3. *Suppose that the probabilistic quasi-nonexpansive mapping T is continuous and the set D is closed. If the sequence of successive approximations of T starting at $p_0 \in D$ has a convergent subsequence, then it converges to $p^* \in D$, the unique fixed point of T.*

Proof. From Lemma 4.2, it follows that the sequence $\{d_a(p_n, p^*), n \geq 0\}$ is nonincreasing. We denote $d_a^* = \lim_{n\to\infty} d_a(p_n, p^*)$. Let $\{p_{n_j}\} \subseteq \{p_n\}$ be a convergent subsequence of $\{p_n\}$, and let $q^* = \lim_{j\to\infty} p_{n_j}$. Since $\{p_{n_j}\} \subseteq D$ and D is closed, we have $q^* \in D$. Moreover, from the continuity of d_a,

$$d_a^* = \lim_{j\to\infty} d_a(p_{n_j}, p^*) = d_a(q^*, p^*).$$

Also, from the continuity of T it follows that

$$d_a^* = \lim_{j\to\infty} d_a(p_{n_j+1}, p^*) = \lim_{j\to\infty} d_a(Tp_{n_j}, p^*) = d_a(Tq^*, p^*).$$

Thus, we have $d_a(Tq^*, p^*) = d_a(q^*, p^*)$ for all a. But, if $q^* \neq p^*$, then by Lemma 4.2 (ii) it follows that there exists a_0 such that $d_{a_0}(Tq^*, p^*) < d_{a_0}(q^*, p^*)$. Therefore, $q^* = p^*$ and then $d_a^* = 0$. This implies that the sequence $\{d_a(p_n, p^*)\}$ converges to 0 for each a. By Lemma 3.2 (i), the sequence $\{p_n\}$ converges to p^* and the theorem is proved. □

Definition 4.4. A mapping T is said to be strict quasi-nonexpansive on D if for a constant $k \in (0, 1)$ the following inequality holds:

$$F_{p^*Tp} \geq k \circ F_{p^*p},$$

or equivalently

$$F_{p^*Tp}(x) \geq F_{p^*p}\left(\frac{x}{k}\right), \qquad x > 0, p \in D, p \neq p^*.$$

We note that for the sequence of successive approximations of T starting at $p_0 \in D$ we have

$$F_{p^*Tp_n} \geq k \circ F_{p^*Tp_{n-1}} \geq k^2 \circ F_{p^*Tp_{n-2}} \geq \cdots \geq k^n \circ F_{p^*Tp_0}.$$

Consider a probability measure space $(\Omega, \mathcal{A}, P)$. Then, the space S of all X-valued random variables which are equal with probability one can be endowed with a probabilistic metric, with respect to the t-norm $T_m(u,v) = \max(u+v-1,0)$, defined by

$$F_{\xi\eta}(x) = P(d(\xi,\eta) < x), \qquad x \geq 0.$$

This probabilistic metric defines the convergence in probability on S. Let $T(\cdot,\cdot)$ be a continuous random operator on random domain $\omega \to D(\omega)$ for which the trajectories $x \to T(x,\omega)$ satisfy the condition of quasi-nonexpansivity in the following form: for a.a $\omega \in \Omega$ the mapping $x \to T(x,\omega)$ is strict quasi-nonexpansive with the constant $k \in [0,1)$, i.e.

$$d(T(x,\omega), \xi^*(\omega)) < d(x, \xi^*(\omega)), x \in D(\omega), x \neq \xi^*(\omega).$$

Suppose that $\omega \to D(\omega)$ is a closed random set, and let D be the set of all measurable selectors of $\omega \to D(\omega)$. We also denote by $T(\cdot)$ the mapping defined by $\omega \to T(\xi(\omega),\omega)$ for $\xi \in D$. It is easy to verify that $T(\cdot)$ is a continuous mapping on S and it is also strict quasi-nonexpansive with the constant k on $D \subset S$. Since $\omega \to D(\omega)$ is closed in probability, it follows that Theorem 4.3 can be applied, and that we can also obtain an estimation of the approximation error in terms of the probabilistic metric defined above.

References

[1] Gh. Bocşan, *On random operators on separable Banach spaces*, Sem. on Probab. Theory Appl. Univ. Timişoara 38, 1978. 1

[2] Gh. Constantin, I. Istrăţescu, *Elements of probabilistic analysis with applications*, Mathematics and its Applications (East European Series), 36, Kluwer Academic Publishers, Dordrecht, 1989. 1

[3] E. Nishiura, *Constructive methods in probabilistic metric spaces*, Fundamenta Mathematicae 67 (1970), 115–124. 3, 3.1

[4] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland Series in Probability and Applied Mathematics, New York, Amsterdam, Oxford, 1983. 2