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Rectangular b-metric space and contraction principles

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Abstract

The concept of rectangular b-metric space is introduced as a generalization of metric space, rectangular metric space and b-metric space. An analogue of Banach contraction principle and Kannan's fixed point theorem is proved in this space. Our result generalizes many known results in fixed point theory.

Keywords: Fixed points, b-metric space, rectangular metric space, rectangular b-metric space.
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1. Introduction and Preliminaries

Since the introduction of Banach contraction principle in 1922, because of its wide applications, the study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction Principle in various generalized metric spaces. In the sequel Branciari [9] introduced the concept of rectangular metric space (RMS) by replacing the sum on the right hand side of the triangular inequality in the definition of a metric space by a three-term expression and proved an analogue of the Banach Contraction Principle in such space. Since then many fixed point theorems for various contractions on rectangular metric space appeared (see [1],[3],[4],[10],[15],[16],[17],[18],[19],[22],[23],[25],[26]).

On the other hand, in [5] Bakhtin introduced b-metric space as a generalization of metric space and proved analogue of Banach contraction principle in b-metric space. Since then, several papers have dealt

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with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [2],[6],[7],[8],[11],[12],[13],[14],[20] and the references therein).

In this paper we have introduced the concept of rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concept of metric space, rectangular metric space and b-metric space. Note that spaces with non Hausdorff topology plays an important role in Tarskian approach to programming language semantics used in computer science (For some details see [24]). Analog of the Banach contraction principle as well as the Kannan type fixed point theorem in rectangular b-metric space are proved. Some examples are included which shows that our generalizations are genuine.

Definition 1.1 ([5])**.** Let X be a nonempty set and the mapping $d\colon X \times X \to [0,\infty)$ satisfies:

(bM1) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(bM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(bM3) there exist a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a b-metric on X and (X,d) is called a b-metric space (in short bMS) with coefficient s.

Definition 1.2 ([9])**.** Let X be a nonempty set and the mapping $d\colon X \times X \to [0,\infty)$ satisfies:

(RM1) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(RM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(RM3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x,y\}$.

Then d is called a rectangular metric on X and (X,d) is called a rectangular metric space (in short RMS).

We define a rectangular b-metric space as follows :

Definition 1.3. Let X be a nonempty set and the mapping $d\colon X \times X \to [0,\infty)$ satisfies:

(RbM1) $d(x,y) = 0$ if and only if $x = y$;
(RbM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(RbM3) there exists a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x,y\}$.

Then d is called a rectangular b-metric on X and (X,d) is called a rectangular b-metric space (in short RbMS) with coefficient s.

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true.

Example 1.4. Let $X = \mathbb{N}$, define $d\colon X \times X \to X$ by

$$d(x,y) = \begin{cases} 0, & \text{if } \ x = y; \\ 4\alpha, & \text{if } \ x, y \in \{1,2\} \text{ and } x \neq y; \\ \alpha, & \text{if } \ x \text{ or } y \notin \{1,2\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X,d) is not a rectangular metric space, as $d(1,2) = 4\alpha > 3\alpha = d(1,3) + d(3,4) + d(4,2)$.

Example 1.5. Let $X = \mathbb{N}$, define $d\colon X \times X \to X$ such that $d(x,y) = d(y,x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 10\alpha, & \text{if } \ x = 1, y = 2; \\ \alpha, & \text{if } \ x \in \{1,2\} \text{ and } y \in \{3\}; \\ 2\alpha, & \text{if } \ x \in \{1,2,3\} \text{ and } y \in \{4\}; \\ 3\alpha, & \text{if } \ x \text{ or } y \notin \{1,2,3,4\} \text{ and } x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = 2 > 1$, but (X,d) is not a rectangular metric space, as $d(1,2) = 10\alpha > 5\alpha = d(1,3) + d(3,4) + d(4,2)$.

Note that every b-metric space with coefficient s is a $RbMS$ with coefficient s^2 but the converse is not necessarily true. (See Example 1.7 below).

For any $x \in X$ we define the open ball with center x and radius $r > 0$ by

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

The open balls in $RbMS$ are not necessarily open(See Example 1.7 below). Let $\mathcal{U}$ be the collection of all subsets $\mathcal{A}$ of X satisfying the condition that for each $x \in \mathcal{A}$ there exist $r > 0$ such that $B_r(x) \subseteq \mathcal{A}$. Then $\mathcal{U}$ defines a topology for the $RbMS$ (X, d), which is not necessarily Hausdorff(See Example 1.7 below).

We define convergence and Cauchy sequence in rectangular b-metric space and completeness of rectangular b-metric space as follows :

Definition 1.6. Let (X, d) be a rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) (X, d) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Note that, limit of a sequence in a $RbMS$ is not necessarily unique and also every convergent sequence in a $RbMS$ is not necessarily a Cauchy sequence. The following example illustrates this fact.

Example 1.7. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and B is the set of all positive integers. Define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2\alpha, & \text{if } \ x, y \in A; \\ \frac{\alpha}{2n}, & \text{if } \ x \in A \text{ and } y \in \{2, 3\}; \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = 2 > 1$. However we have the following :

1) (X, d) is not a rectangular metric space, as $d(\frac{1}{2}, \frac{1}{3}) = 2\alpha > \frac{17}{12} = d(\frac{1}{2}, 4) + d(4, 3) + d(3, \frac{1}{3})$ and hence not a metric space.

2) There does not exist $s > 0$ satisfying $d(x, y) \le s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, and so (X, d) is not a b-metric space.

3) $B_{\frac{\alpha}{2}}(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$ and there does not exist any open ball with center 2 and contained in $B_{\frac{\alpha}{2}}(\frac{1}{2})$. So $B_{\frac{\alpha}{2}}(\frac{1}{2})$ is not an open set.

4) The sequence $\{\frac{1}{n}\}$ converges to 2 and 3 in $RbMS$ and so limit is not unique. Also $d(\frac{1}{n}, \frac{1}{n+p}) = 2\alpha \not\to 0$ as $n \to \infty$, therefore $\{\frac{1}{n}\}$ is not a Cauchy sequence in $RbMS$.

5) There does not exist any $r_1, r_2 > 0$ such that $B_{r_1}(2) \bigcap B_{r_2}(3) = \phi$ and so (X, d) is not Hausdorff.

2. Main results

Following theorem is the analogue of Banach contraction principle in rectangular b-metric space.

Theorem 2.1. *Let (X, d) be a complete rectangular b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \le \lambda d(x, y) \tag{2.1}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s}]$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is Cauchy sequence. If $x_n = x_{n+1}$ then x_n is fixed point of T. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n, x_{n+1}) = d_n$, it follows from (2.1) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \lambda d(x_{n-1}, x_n) \\ d_n &\leq \lambda d_{n-1}. \end{aligned}$$

Repeating this process we obtain

$$d_n \leq \lambda^n d_0. \tag{2.2}$$

Also, we can assume that x_0 is not a periodic point of T. Indeed, if $x_0 = x_n$ then using (2.2), for any $n \geq 2$, we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ d_0 &= d_n \\ d_0 &\leq \lambda^n d_0, \end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Again setting $d(x_n, x_{n+2}) = d_n^*$ and using (2.1) for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \leq \lambda d(x_{n-1}, x_{n+1}) \\ d_n^* &\leq \lambda d_{n-1}^* \end{aligned}$$

Repeating this process we obtain

$$d(x_n, x_{n+2}) \leq \lambda^n d_0^*. \tag{2.3}$$

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.
If p is odd say $2m+1$ then using (2.2) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m+1})] \\ &\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^m d_{n+2m} \\ &\leq s[\lambda^n d_0 + \lambda^{n+1} d_0] + s^2[\lambda^{n+2} d_0 + \lambda^{n+3} d_0] + s^3[\lambda^{n+4} d_0 + \lambda^{n+5} d_0] \\ &\quad + \cdots + s^m \lambda^{n+2m} d_0 \\ &\leq s\lambda^n[1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 + s\lambda^{n+1}[1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 \\ &= \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 \qquad (\text{note that } s\lambda^2 < 1). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0. \tag{2.4}$$

If p is even say $2m$ then using (2.2) and (2.3) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\ &\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m})] \\ &\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \end{aligned}$$

$$\begin{aligned} &\leq s[\lambda^n d_0 + \lambda^{n+1} d_0] + s^2[\lambda^{n+2} d_0 + \lambda^{n+3} d_0] + s^3[\lambda^{n+4} d_0 + \lambda^{n+5} d_0] \\ &\quad + \cdots + s^{m-1}[\lambda^{2m-4} d_0 + \lambda^{2m-3} d_0] + s^{m-1}\lambda^{n+2m-2} d_0^* \\ &\leq s\lambda^n[1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 + s\lambda^{n+1}[1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 \\ &\quad + s^{m-1}\lambda^{n+2m-2} d_0^*, \end{aligned}$$

i.e.

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + s^{m-1}\lambda^{n+2m-2} d_0^* \\ &< \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + (s\lambda)^{2m}\lambda^{n-2} d_0^* \quad (\text{as } 1 < s) \\ &\leq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^* \quad (\text{as } \lambda \leq \tfrac{1}{s}). \end{aligned}$$

Therefore

$$d(x_n, x_{n+2m}) \leq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^*. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all } p > 0. \tag{2.6}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{2.7}$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u, Tu) &\leq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= s[d(u, x_n) + d_n + d(Tx_n, Tu)] \\ &\leq s[d(u, x_n) + d_n + \lambda d(x_n, u)]. \end{aligned}$$

Using (2.6) and (2.7) it follows from above inequality that $d(u, Tu) = 0$, i.e., $Tu = u$. Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (2.1) that $d(u, v) = d(Tu, Tv) \leq \lambda d(u, v) < d(u, v)$, a contradiction. Therefore, we must have $d(u, v) = 0$, i.e., $u = v$. Thus fixed point is unique. □

Example 2.2. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5\}\}$ and $B = [1, 2]$. Define $d\colon X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$\begin{cases} d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.03 \\ d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.02 \\ d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6 \\ d(x, y) = |x - y|^2 & \text{otherwise} \end{cases}$$

Then (X, d) is a rectangular b-metric space with coefficient $s = 4 > 1$. But (X, d) is neither a metric space nor a rectangular metric space. Let $T : X \to X$ be defined as :

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x \in A \\ \frac{1}{5} & \text{if } x \in B \end{cases}$$

Then T satisfies the condition of Theorem 2.1 and has a unique fixed point $x = \frac{1}{4}$.

Remark 2.3. We say that $T : X \to X$ has property P if $F(T) = F(T^n)$ (see [21]) where $F(T) = \{x \in X : Tx = x\}$. It is an easy exercise to see that under the assumptions of Theorem 2.1, T has property P.

Following theorem is the analogue of Kannan type contraction in rectangular b-metric space.

Theorem 2.4. *Let (X, d) be a complete rectangular b-metric space with coefficient $s > 1$ and $T\colon X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \le \lambda[d(x, Tx) + d(y, Ty)] \tag{2.8}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s+1}]$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. We shall show that $\{x_n\}$ is Cauchy sequence. If $x_n = x_{n+1}$ then x_n is fixed point of T. So, suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$. Setting $d(x_n, x_{n+1}) = d_n$, it follows from (2.8) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \le \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ d(x_n, x_{n+1}) &= \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ d_n &= \lambda[d_{n-1} + d_n] \\ d_n &\le \frac{\lambda}{1-\lambda} d_{n-1} = \beta d_{n-1}, \end{aligned}$$

where $\beta = \frac{\lambda}{1-\lambda} < \frac{1}{s}$ (as, $\lambda < \frac{1}{s+1}$). Repeating this process we obtain

$$d_n \le \beta^n d_0. \tag{2.9}$$

Also, we can assume that x_0 is not a periodic point of T. Indeed, if $x_0 = x_n$ then using (2.9), for any $n \ge 2$, we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ d_0 &= d_n \\ d_0 &\le \beta^n d_0, \end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T. Thus we assume that $x_n \ne x_m$ for all distinct $n, m \in \mathbb{N}$. Again using (2.8) and (2.9) for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \le \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})] \\ &= \lambda[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})] = \lambda[d_{n-1} + d_{n+1}] \\ &\le \lambda[\beta^{n-1} d_0 + \beta^{n+1} d_0] \\ &= \lambda \beta^{n-1}[1 + \beta^2] d_0 \\ &= \gamma \beta^{n-1} d_0. \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2}) \le \gamma \beta^{n-1} d_0, \tag{2.10}$$

where $\gamma = \lambda[1 + \beta^2] > 0$.
For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.
If p is odd say $2m + 1$ then using (2.9) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\le s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m+1})] \\ &\le s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^m d_{n+2m} \end{aligned}$$

$$\begin{aligned}
&\leq s[\beta^n d_0 + \beta^{n+1} d_0] + s^2[\beta^{n+2} d_0 + \beta^{n+3} d_0] + s^3[\beta^{n+4} d_0 + \beta^{n+5} d_0] \\
&\quad + \cdots + s^m \beta^{n+2m} d_0 \\
&\leq s\beta^n[1 + s\beta^2 + s^2\beta^4 + \cdots]d_0 + s\beta^{n+1}[1 + s\beta^2 + s^2\beta^4 + \cdots]d_0 \\
&= \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 \qquad (\text{note that } s\beta^2 < 1).
\end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0. \tag{2.11}$$

If p is even say $2m$ then using (2.9) and (2.10) we obtain

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
&\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+2m})] \\
&\leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\
&\quad + \cdots + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
&\leq s[\beta^n d_0 + \beta^{n+1} d_0] + s^2[\beta^{n+2} d_0 + \beta^{n+3} d_0] + s^3[\beta^{n+4} d_0 + \beta^{n+5} d_0] \\
&\quad + \cdots + s^{m-1}[\beta^{2m-4} d_0 + \beta^{2m-3} d_0] + s^{m-1}\gamma\beta^{n+2m-3} d_0 \\
&\leq s\beta^n[1 + s\beta^2 + s^2\beta^4 + \cdots]d_0 + s\beta^{n+1}[1 + s\beta^2 + s^2\beta^4 + \cdots]d_0 \\
&\quad + s^{m-1}\gamma\beta^{n+2m-3} d_0,
\end{aligned}$$

i.e.

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + s^{m-1}\gamma\beta^{n+2m-3} d_0 \\
&< \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma(s\beta)^{2m}\beta^{n-3} d_0 \qquad (\text{as } 1 < s) \\
&\leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma\beta^{n-3} d_0 \qquad (\text{as } \beta \leq \tfrac{1}{s}).
\end{aligned}$$

Therefore

$$d(x_n, x_{n+2m}) \leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma\beta^{n-3} d_0. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \ \text{ for all } \ p > 0. \tag{2.13}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{2.14}$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
d(u, Tu) &\leq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\
&= s[d(u, x_n) + d_n + d(Tx_n, Tu)] \\
&\leq s[d(u, x_n) + d_n + \lambda\{d(x_n, Tx_n) + d(u, Tu)\}] \\
&= s[d(u, x_n) + d_n + \lambda\{d(x_n, x_{n+1}) + d(u, Tu)\}] \\
(1 - s\lambda) d(u, Tu) &\leq s[d(u, x_n) + \beta^n d_0 + \lambda d(x_n, x_{n+1})]
\end{aligned}$$

Using (2.13) and (2.14) and the fact that $\lambda < \frac{1}{s+1}$, it follows from above inequality that $d(u, Tu) = 0$, i.e., $Tu = u$. Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (2.8) that $d(u, v) = d(Tu, Tv) \leq \lambda[d(u, Tu) + d(v, Tv)] = \lambda[d(u, u) + d(v, v)] = 0$. Therefore, we have $d(u, v) = 0$, i.e., $u = v$. Thus fixed point is unique. □

Remark 2.5. On the basis of discussion contained in this paper, we have the following:
1) The open ball defined in b-metric space, RMS and RbMS are not necessarily open set.
2) The collection of open balls in RbMS, RMS and b-metric space do not necessarily form a basis for a topology.
3) RbMS, RMS and b-metric space are not necessarily Hausdorff.

OpenProblems :

1) In Theorem 2.1, can we extent the range of λ to the case $\frac{1}{s} < \lambda < 1$.

2) Prove analogue of Chatterjee contraction, Reich contraction, Ciric contraction and Hardy-Rogers contraction in $RbMS$.

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