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Fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings in α-η-complete metric spaces

Preeyaluk Chuadchawna, Anchalee Kaewcharoen*, Somyot Plubtieng

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

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Abstract

In this paper, we introduce the concept of generalized α-η-ψ-Geraghty contraction type mappings and prove the unique fixed point theorems for such mappings in α-η-complete metric spaces without assuming the subadditivity of ψ. We also give an example for supporting the result and present an application using our main result to obtain a solution of the integral equation.

Keywords: α-η-complete metric spaces, α-η-continuous mappings, triangular α-orbital admissible mappings, generalized α-η-ψ-Geraghty contraction type mappings.
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1. Introduction and Preliminaries

One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach [1]. There were many authors have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and references contained therein). One of the remarkable result is Geraghty's theorem given by Geraghty [4]. In 2013, Cho *et al.* [3] introduced the notion of α-Geraghty contraction type mappings and assured the unique fixed point theorems for such mappings in complete metric spaces. Recently, Popescu [12] defined the concept of triangular α-orbital admissible mappings and proved the unique fixed point theorems for

*Corresponding author

Email addresses: Chuadchawna@hotmail.com (Preeyaluk Chuadchawna), anchaleeka@nu.ac.th (Anchalee Kaewcharoen), somyotp@nu.ac.th (Somyot Plubtieng)

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the mentioned mappings which are generalized α-Geraghty contraction type mappings. On the other hand, Karapinar [8] proved the existence of a unique fixed point for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contraction type mapping.
For the sake of convenience, we recall the Geraghty's theorem. Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Geraghty [4] proved the following unique fixed point theorem in a complete metric space.

Theorem 1.1 ([4]). *Let (X,d) be a complete metric space and $T : X \to X$. Suppose that there exists $\beta \in \mathcal{F}$ such that*

$$d(Tx,Ty) \le \beta(d(x,y))d(x,y) \text{ for all } x,y \in X.$$

Then T has a unique fixed point $x^* \in X$.

In 2012, Samet *et al.* [13] introduced the notion of α-admissible mappings.

Definition 1.2 ([13]). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then T is said to be α-admissible if

$$\alpha(x,y) \ge 1 \quad \text{implies} \quad \alpha(Tx,Ty) \ge 1.$$

Karapinar *et al.* [9] defined the concept of triangular α-admissible mappings.

Definition 1.3 ([9]). A mapping $T : X \to X$ is said to be triangular α-admissible if

(a) T is α-admissible;
(b) $\alpha(x,z) \ge 1$ and $\alpha(z,y) \ge 1$ imply $\alpha(x,y) \ge 1$.

The definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings are defined by Popescu [12] in 2014.

Definition 1.4 ([12]). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then T is said to be α-orbital admissible if

$$\alpha(x,Tx) \ge 1 \quad \text{implies} \quad \alpha(Tx,T^2x) \ge 1.$$

Definition 1.5 ([12]). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then T is said to be triangular α-orbital admissible if

(a) T is α-orbital admissible;
(b) $\alpha(x,y) \ge 1$ and $\alpha(y,Ty) \ge 1$ imply $\alpha(x,Ty) \ge 1$.

Remark 1.6. Every triangular α-admissible mapping is a triangular α-orbital admissible mapping. There exists a triangular α-orbital admissible mapping which is not a triangular α-admissible mapping. For more details see [12].

Popescu [12] gave the definition of generalized α-Geraghty contraction type mappings and proved the fixed point theorems for such mappings in complete metric spaces.

Definition 1.7 ([12]). Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$. A mapping $T : X \to X$ is said to be a generalized α-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty) \le \beta(M_T(x,y))M_T(x,y),$$

where

$$M_T(x,y) = \max\left\{d(x,y),\, d(x,Tx),\, d(y,Ty),\, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Theorem 1.8 ([12])**.** *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a generalized α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Recently, Karapinar [8] introduced the concept of α-ψ-Geraghty contraction type mappings in complete metric spaces.

Let Ψ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0$ if and only if $t = 0$;
(d) ψ is subadditive, that is $\psi(s + t) \leq \psi(s) + \psi(t)$.

Definition 1.9. Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized α-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty)\} \text{ and } \psi \in \Psi.$$

Theorem 1.10 ([8])**.** *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) *T is a generalized α-ψ-Geraghty contraction type mapping;*
(ii) *T is a triangular α-admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

On the other hand, Hussain *et al.* [6] introduced the concepts of α-η-complete metric spaces and α-η-continuous functions.

Definition 1.11 ([6])**.** Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, +\infty)$. Then X is said to be α-η-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converges in X.

Example 1.12. Let $X = (0, \infty)$ and define a metric on X by $d(x, y) = |x - y|$ for all $x, y \in X$. Therefore X is not complete. Let Y be a closed subset of X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} (x + y)^3, & \text{if } x, y \in Y \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x, y) = 3x^2 y.$$

We will prove that (X, d) is an α-η-complete metric space. Suppose that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is in Y. By the completeness of Y, there exists $x^* \in Y$ such that $x_n \to x^*$ as $n \to \infty$.

Definition 1.13 ([6])**.** Let (X,d) be a metric space and $\alpha,\eta : X \times X \to [0,+\infty)$. A mapping $T : X \to X$ is said to be an α-η-continuous mapping if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $Tx_n \to Tx$ as $n \to \infty$.

Example 1.14. Let $X = [0,\infty)$ and define a metric on X by $d(x,y) = |x-y|$ for all $x,y \in X$. Assume that $T : X \to X$ and $\alpha,\eta : X \times X \to [0,+\infty)$ are defined by

$$Tx = \begin{cases} x^4, & \text{if } x \in [0,1] \\ \cos \pi x + 3, & \text{if } x \in (1,\infty), \end{cases} \quad , \quad \alpha(x,y) = \begin{cases} x^3 + y^3 + 1, & \text{if } x,y \in [0,1] \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x,y) = x^3.$$

Therefore T is not continuous. We will prove that T is an α-η-continuous mapping. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. This implies that $x_n \in [0,1]$ and so $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} x_n^4 = x^4 = Tx$.

In this work, we introduce the notion of generalized α-η-ψ-Geraghty contraction type mappings in metric spaces. Moreover, we prove the unique fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings which are triangular α-orbital admissible mappings in the setting of α-η-complete metric spaces without assuming the subadditivity of ψ. Our results improve and generalize the results proved by Karapinar [8] and Poposcu [12]. Furthermore, we also give an example for supporting the result and present an application using our main result to obtain a solution of the integral equation.

2. Main results

Let Ψ' denote the class of the functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.1. Let $T : X \to X$ and $\alpha,\eta : X \times X \to [0,\infty)$. Then T is said to be α-orbital admissible with respect to η if

$$\alpha(x,Tx) \geq \eta(x,Tx) \text{ implies } \alpha(Tx,T^2x) \geq \eta(Tx,T^2x).$$

Definition 2.2. Let $T : X \to X$ and $\alpha,\eta : X \times X \to [0,\infty)$. Then T is said to be triangular α-orbital admissible with respect to η if

1. T is α-orbital admissible with respect to η;
2. $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$ imply $\alpha(x,Ty) \geq \eta(x,Ty)$.

Remark 2.3. If we suppose that $\eta(x,y) = 1$ for all $x,y \in X$, then Definition 2.1 reduces to Definition 1.4 and Definition 2.2 reduces to Definition 1.5.

We now prove the important lemma that will be used for proving our main results.

Lemma 2.4. *Let $T : X \to X$ be a triangular α-orbital admissible with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_n,x_m) \geq \eta(x_n,x_m)$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Proof. Since $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and T is α-orbital admissible with respect to η, we obtain that

$$\alpha(x_2,x_3) = \alpha(Tx_1,T(Tx_1)) \geq \eta(Tx_1,T(Tx_1)) = \eta(x_2,x_3).$$

By continuing the process as above, we have $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Suppose that

$$\alpha(x_n,x_m) \geq \eta(x_n,x_m) \tag{2.1}$$

and we will prove that $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$, where $m > n$. Since $\alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1})$, we obtain that

$$\alpha(x_m, Tx_m) = \alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1}) = \eta(x_m, Tx_m). \tag{2.2}$$

By (2.1), (2.2) and triangular α-orbital admissibility of T, we have

$$\alpha(x_n, Tx_m) \geq \eta(x_n, Tx_m).$$

This implies that

$$\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1}).$$

Hence $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$. □

We now introduce the concept of generalized α-η-ψ-Geraghty contraction type mappings and prove the fixed point theorems for such mappings.

Definition 2.5. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized α-η-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that $\alpha(x, y) \geq \eta(x, y)$ implies

$$\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where

$$M_T(x, y) = \max\left\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \quad \text{and} \quad \psi \in \Psi'.$$

Remark 2.6. In Definition 2.5, if we take $\eta(x, y) = 1$ and $\psi(t) = t$, then it reduces to Definition 1.7.

Theorem 2.7. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is a generalized α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Then T has a fixed point. Hence we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By Lemma 2.4, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since T is a generalized α-η-ψ-Geraghty contraction type mapping, we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_T(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}),\, d(x_n, Tx_n),\, d(x_{n+1}, Tx_{n+1}),\, \frac{1}{2}(d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n))\} \\ &= \max\left\{d(x_n, x_{n+1}),\, d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2}),\, \frac{d(x_n, x_{n+2})}{2} + \frac{d(x_{n+1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2}),\, \left[\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right]\right\} \\ &= \max\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $\max\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\le \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \\ &\le \beta(\psi(M_T(x_n, x_{n+1})))\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2})),\end{aligned}$$

which is a contradiction. Thus we conclude that

$$\max\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}).$$

By (2.3), we get that $\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. Since ψ is nondecreasing, we have $d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Hence we deduce that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing. Therefore, there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Suppose that $r > 0$. Then due to (2.3), we have

$$\psi(d(x_{n+1}, x_{n+2})) \le \beta(\psi(M_T(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})).$$

Therefore

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \le \beta(\psi(M_T(x_n, x_{n+1}))) < 1.$$

This implies that $\lim_{n\to\infty} \beta(\psi(M_T(x_n, x_{n+1}))) = 1$. Since $\beta \in \mathcal{F}$, we have $\lim_{n\to\infty} \psi(M_T(x_n, x_{n+1})) = 0$, which yields

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

This is a contradiction. Next, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $m(k) > n(k) > k$ with $d(x_{n(k)}, x_{m(k)}) \ge \varepsilon$. Let $m(k)$ be the smallest number satisfying the condition above. Then we have $d(x_{n(k)}, x_{m(k)-1}) < \varepsilon$. Therefore

$$\varepsilon \le d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < \varepsilon + d(x_{m(k)-1}, x_{m(k)}).$$

Letting $k \to \infty$, we have $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$. Since

$$|d(x_{n(k)}, x_{m(k)-1}) - d(x_{n(k)}, x_{m(k)})| \le d(x_{m(k)}, x_{m(k)-1}),$$

we have $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon$. Similarly, we obtain that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.$$

By Lemma 2.4, we have $\alpha(x_{n(k)-1}, x_{m(k)-1}) \ge \eta(x_{n(k)-1}, x_{m(k)-1})$. Thus we have

$$\begin{aligned}\psi(d(x_{n(k)}, x_{m(k)})) &= \psi(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\le \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1})))\psi(M_T(x_{n(k)-1}, x_{m(k)-1})),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} M_T(x_{n(k)-1}, x_{m(k)-1}) &= \max\{d(x_{n(k)-1}, x_{m(k)-1}),\, d(x_{n(k)-1}, Tx_{n(k)-1}),\, d(x_{m(k)-1}, Tx_{m(k)-1}), \\ &\quad \frac{1}{2}(d(x_{n(k)-1}, Tx_{m(k)-1}) + d(x_{m(k)-1}, Tx_{n(k)-1}))\} \\ &= \max\Big\{d(x_{n(k)-1}, x_{m(k)-1}),\, d(x_{n(k)-1}, x_{n(k)}),\, d(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{d(x_{n(k)-1}, x_{m(k)})}{2} + \frac{d(x_{m(k)-1}, x_{n(k)})}{2}\Big\}.\end{aligned}$$

Therefore

$$\lim_{k\to\infty} M_T(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.6}$$

By (2.5) and (2.6), we have

$$1 = \frac{\lim_{k\to\infty} \psi(d(x_{n(k)}, x_{m(k)}))}{\lim_{k\to\infty} \psi(M_T(x_{n(k)-1}, x_{m(k)-1}))} \le \lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1}))),$$

which implies $\lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1}))) = 1$. Consequently, we get $\lim_{k\to\infty} M_T(x_{n(k)-1}, x_{m(k)-1}) = 0$. Hence $\varepsilon = 0$ which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is an α-η-complete metric space and $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, there is $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is α-η-continuous, we get $\lim_{n\to\infty} Tx_n = Tx^*$ and so $x^* = Tx^*$. Hence T has a fixed point. □

In following theorem, we replace the continuity of T by some suitable conditions.

Theorem 2.8. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

- (i) *(X, d) is an α-η-complete metric space;*
- (ii) *T is a generalized α-η-ψ-Geraghty contraction type mapping;*
- (iii) *T is a triangular α-orbital admissible mapping with respect to η;*
- (iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge \eta(x_1, Tx_1)$;*
- (v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \ge \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. By the analogous proof as in Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \ge \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\ &\le \beta(\psi(M_T(x_{n(k)}, x^*)))\psi(M_T(x_{n(k)}, x^*)), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M_T(x_{n(k)}, x^*) &= \max\{d(x_{n(k)}, x^*),\, d(x_{n(k)}, Tx_{n(k)}),\, d(x^*, Tx^*), \\ &\qquad \frac{1}{2}(d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)}))\} \\ &= \max\{d(x_{n(k)}, x^*),\, d(x_{n(k)}, x_{n(k)+1}),\, d(x^*, Tx^*), \\ &\qquad \frac{1}{2}(d(x_{n(k)}, Tx^*) + d(x^*, x_{n(k)+1})\}. \end{aligned}$$

Suppose that $Tx^* \neq x^*$. Letting $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} M_T(x_{n(k)}, x^*) = d(x^*, Tx^*).$$

From (2.7), we have

$$\frac{\psi(d(x_{n(k)+1}, Tx^*))}{\psi(M_T(x_{n(k)}, x^*))} \le \beta(\psi(M_T(x_{n(k)}, x^*))) < 1.$$

Letting $k \to \infty$ in the above inequality, we obtain that $\lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)}, x^*))) = 1$ and so $\lim_{k\to\infty} M_T(x_{n(k)}, x^*) = 0$. Hence $d(x^*, Tx^*) = 0$. This is a contradiction. It follows that $Tx^* = x^*$. □

For the uniqueness of a fixed point of a generalized α-η-ψ-contractive type mapping, we assume the suitable condition introduced by Popescu [12].

Theorem 2.9. *Suppose all assumptions of* Theorem 2.7 (*respectively* Theorem 2.8) *hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq \eta(x,v)$, $\alpha(y,v) \geq \eta(y,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*,v) \geq \eta(x^*,v)$, $\alpha(y^*,v) \geq \eta(y^*,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*,T^n v) \geq \eta(x^*,T^n v) \text{ and } \alpha(y^*,T^n v) \geq \eta(y^*,T^n v),$$

for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned}\psi(d(x^*,T^{n+1}v)) &= \psi(d(Tx^*,TT^n v))\\ &\leq \beta(\psi(M_T(x^*,T^n v)))\psi(M_T(x^*,T^n v)),\end{aligned}$$

for all $n \in \mathbb{N}$ where

$$\begin{aligned}M_T(x^*,T^n v) &= \max\{d(x^*,T^n v),\, d(x^*,Tx^*),\, d(T^n v,T^{n+1}v),\\ &\qquad \frac{1}{2}(d(x^*,T^{n+1}v) + d(T^n v,Tx^*)\}\\ &= \max\{d(x^*,T^n v),\, d(T^n v,T^{n+1}v),\, \frac{1}{2}(d(x^*,T^{n+1}v) + d(T^n v,x^*)\}.\end{aligned}$$

By Theorem 2.7, we deduce that $\{T^n v\}$ converges to a fixed point z^* of T. Taking $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} M_T(x^*,T^n v) = d(x^*,z^*).$$

We will prove that $x^* = z^*$. Suppose that $x^* \neq z^*$. Since

$$\frac{\psi(d(x^*,T^{n+1}v))}{\psi(M_T(x^*,T^n v))} \leq \beta(\psi(M_T(x^*,T^n v))),$$

we obtain that $\lim_{n\to\infty} \beta(\psi(M_T(x^*,T^n v))) = 1$. This implies that $\lim_{n\to\infty} M_T(x^*,T^n v) = 0$, and then $d(x^*,z^*) = 0$ which is a contradiction. Hence $x^* = z^*$. Similarly, we can prove that $y^* = z^*$. Thus $x^* = y^*$. It follows that T has a unique fixed point. □

In Theorem 2.7 and Theorem 2.8, if we put $\eta(x,y) = 1$ and $\psi(t) = t$, then we obtain the following result proved by Popescu [12].

Corollary 2.10 ([12]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a generalized α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

By taking $\eta(x, y) = 1$ and the same techniques using in Theorem 2.7 and Theorem 2.8, we obtain the following result.

Corollary 2.11. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a triangular α-orbital admissible mapping;*
(ii) *if there exists $\beta \in \mathcal{F}$ such that*

$$\alpha(x, y) \geq 1 \text{ implies } \psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty)\} \text{ and } \psi \in \Psi';$$

(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Consequently, we obtain that the following result proved by Karapinar [8].

Corollary 2.12 ([8])**.** *Let (X, d) be a complete metric space. Assume that $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a triangular α-admissible mapping;*
(ii) *T is a generalized α-ψ-Geraghty contraction type mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

3. Consequences

Definition 3.1. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be an α-η-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that $\alpha(x, y) \geq \eta(x, y)$ implies

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)),$$

where $\psi \in \Psi'$.

Theorem 3.2. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. As in the proof of Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since T is α-η-continuous, we have

$$x_{n+1} = Tx_n \to Tx^* \text{ as } n \to \infty.$$

Hence T has a fixed point . □

Theorem 3.3. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. As in the proof of Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
\psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\
&\leq \beta(\psi(d(x_{n(k)}, x^*)))\psi(d(x_{n(k)}, x^*)) \\
&< \psi(d(x_{n(k)}, x^*)).
\end{aligned}$$

Letting $k \to \infty$ in above inequality, we obtain that $\psi(d(x^*, Tx^*)) \leq 0$. Thus $\psi(d(x^*, Tx^*)) = 0$. This implies that $d(x^*, Tx^*) = 0$. Hence $x^* = Tx^*$. □

Theorem 3.4. *Suppose all assumptions of* Theorem 3.2 (*respectively* Theorem 3.3) *hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x, v) \geq \eta(x, v)$, $\alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*, v) \geq \eta(x^*, v)$, $\alpha(y^*, v) \geq \eta(y^*, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*, T^n v) \geq \eta(x^*, T^n v) \quad \text{and} \quad \alpha(y^*, T^n v) \geq \eta(y^*, T^n v)$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
\psi(d(x^*, T^{n+1} v)) &= \psi(d(Tx^*, TT^n v)) \\
&\leq \beta(\psi(d(x^*, T^n v)))\psi(d(x^*, T^n v)) \\
&< \psi(d(x^*, T^n v))
\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. Consequently, the sequence $\{\psi(d(x^*, T^n v))\}$ is nonincreasing, then there exists $r \geq 0$ such that $\lim_{n\to\infty} \psi(d(x^*, T^n v)) = r$. By (3.1) we have

$$\frac{\psi(d(x^*, T^{n+1} v))}{\psi(d(x^*, T^n v))} \leq \beta(\psi(d(x^*, T^n v))).$$

Letting limit $n \to \infty$, we have $\lim_{n\to\infty} \beta(\psi(d(x^*, T^n v))) = 1$ and then $\lim_{n\to\infty} \psi(d(x^*, T^n v)) = 0$. It follows that $\lim_{n\to\infty} d(x^*, T^n v) = 0$. Hence $\lim_{n\to\infty} T^n v = x^*$. Similarly, we can prove that $\lim_{n\to\infty} T^n v = y^*$. Hence $x^* = y^*$. □

Corollary 3.5 ([8]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $T : X \to X$. Assume that the following conditions are satisfied:*

(i) *there exists $\beta \in \mathcal{F}$ such that*

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ with $x \preceq y$ where $\psi \in \Psi'$;

(ii) *there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;*

(iii) *T is nondecreasing;*

(iv) *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*. Further if for all $x \neq y \in X$, there exists $v \in X$ such that $x \preceq v$, $y \preceq v$ and $v \preceq Tv$, then T has a unique fixed point.

Proof. Define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ \frac{1}{4}, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x \preceq y \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$. By (i), we have

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)).$$

This implies that T is an α-η-ψ-Geraghty contraction type mapping. Since X is complete metric space, we have X is α-η-complete metric space. By (ii), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Let $\alpha(x, Tx) \geq \eta(x, Tx)$, we have $x \preceq Tx$. Since T is nondecreasing, we obtain that $Tx \preceq T(Tx)$. Then $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Let $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$, so we have $x \preceq y$ and $y \preceq Ty$. It follows that $x \preceq Ty$. Then $\alpha(x, Ty) \geq \eta(x, Ty)$. Thus all conditions of Theorem 3.2 and Theorem 3.3 are satisfied. Hence T has a fixed point. □

We now give an example for supporting Theorem 3.2.

Example 3.6. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\beta(t) = \frac{1}{1+2t}$ for all $t > 0$ and $\beta(0) = 0$. Then $\beta \in \mathcal{F}$. Let $\psi(t) = \frac{1}{4}t$ and a mapping $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{2}{3}x, & \text{if } 0 \leq x \leq 1 \\ 2x, & \text{if } x > 1. \end{cases}$$

Also, we define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{otherwise,} \end{cases} \quad , \qquad \eta(x, y) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq x, y \leq 1 \\ 2, & \text{otherwise.} \end{cases}$$

First, we will prove that (X, d) is an α-η-complete metric space. If $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0, 1]$. Since $([0, 1], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[0, 1] \subseteq X$. Let $\alpha(x, Tx) \geq \eta(x, Tx)$. Thus $x \in [0, 1]$ and $Tx \in [0, 1]$ and so $T^2x = T(Tx) \in [0, 1]$. Then $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Thus T is α-orbital admissible with respect to η. Let $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$. We have $x, y, Ty \in [0, 1]$. This implies that $\alpha(x, Ty) \geq \eta(x, Ty)$. Hence T is triangular α-orbital admissible with respect to η. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0, 1]$ for all $n \in \mathbb{N}$. This implies that

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{2}{3}x_n = \frac{2}{3}x = Tx$. That is T is α-η -continuous. It is clear that condition (iv) of Theorem 3.2 is satisfied with $x_1 = 1$ since $\alpha(1, T(1)) = \alpha(1, \frac{2}{3}) = 1 > \frac{1}{4} = \eta(1, \frac{2}{3}) = \eta(1, T(1))$. Finally, we will prove that T is an α-η-ψ-Geraghty contraction type mapping. Let $\alpha(x, y) \geq \eta(x, y)$. Therefore $x, y \in [0, 1]$. It follows that

$$\begin{aligned}
&\beta(\psi(d(x,y)))\psi(d(x,y)) - \psi(d(Tx,Ty))\\
&\qquad = \beta(\frac{1}{4}(d(x,y))) \cdot \frac{1}{4}(d(x,y)) - \frac{1}{4}(d(Tx,Ty))\\
&\qquad = \beta(\frac{1}{4}|x-y|) \cdot \frac{1}{4}|x-y| - \frac{1}{4}|Tx-Ty|\\
&\qquad = \frac{1}{1+\frac{1}{2}|x-y|} \cdot \frac{1}{4}|x-y| - \frac{1}{4}|\frac{2}{3}x - \frac{2}{3}y|\\
&\qquad = \frac{\frac{1}{4}|x-y|}{1+\frac{1}{2}|x-y|} - \frac{1}{6}|x-y|\\
&\qquad = \frac{|x-y|(3-2+|x-y|)}{6(2+|x-y|)}\\
&\qquad \geq 0.
\end{aligned} \tag{3.2}$$

Then we have $\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$. Thus all assumptions of Theorem 3.2 are satisfied. Hence T has a fixed point $x^* = 0$.

4. Applications to ordinary differential equations

The following ordinary differential equation is taken from Karapinar [8]:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), & t \in [0,1] \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The Green function associated to (4.1) is defined by

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $C(I)$ be the space of all continuous functions defined on I where $I = [0, 1]$. Suppose that $d(x, y) = \|x - y\|_\infty = \sup_{t\in I} |x(t) - y(t)|$ for all $x, y \in C(I)$. It is well known that $(C(I), d)$ is a complete metric space.

Assume that the following conditions hold:

(i) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$, we have $|f(t, a)-f(t, b)| \leq 8\ln(|a - b| + 1)$ for all $t \in I$;

(ii) there exists $x_1 \in C(I)$ such that for all $t \in I$,

$$\xi\left(x_1(t), \int_0^1 G(t,s)f(s,x_1(s))ds\right) \geq 0;$$

(iii) for all $t \in I$ and for all $x, y, z \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \text{ imply } \xi(x(t), z(t)) \geq 0;$$

(iv) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi\left(\int_0^1 G(t,s)f(s,x(s))ds, \int_0^1 G(t,s)f(s,y(s))ds\right) \geq 0;$$

(v) if $\{x_n\}$ is a sequence in $C([0,1])$ such that $x_n \to x \in C([0,1])$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

We now assure the existence of a solution of the above second order differential equation. The method for proving the following result is taken from [8] but is slightly different.

Theorem 4.1. *Suppose that conditions* (i)-(v) *are satisfied. Then* (4.1) *has at least one solution* $x^* \in C^2(I)$.

Proof. It is well known that $x^* \in C^2(I)$ is a solution of (4.1) if and only if $x^* \in C(I)$ is a solution of the integral equation (see [8]). Define a mapping $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s)) ds \quad \text{for all } t \in I.$$

Therefore the problem (4.1) is equivalent to finding $x^* \in C(I)$ that is a fixed point of T. Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. From (i), we obtain that

$$\begin{aligned} |Tx(t) - Ty(t)| &= \Big|\int_0^1 G(t,s)[f(s,x(s)) - f(s,y(s))]ds\Big| \\ &\leq \int_0^1 G(t,s)\big|f(s,x(s)) - f(s,y(s))\big|ds \\ &\leq 8\int_0^1 G(t,s)\ln(|x(s) - y(s)| + 1)ds \\ &\leq 8\int_0^1 G(t,s)\ln(d(x,y) + 1)ds \\ &\leq 8\ln(d(x,y)+1)\Big(\sup_{t\in I}\int_0^1 G(t,s)ds\Big). \end{aligned}$$

Since $\int_0^1 G(t,s)ds = -(t^2/2) + t/2$ for all $t \in I$, we have $\sup\limits_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}$. This implies that

$$d(Tx, Ty) \leq \ln(d(x,y) + 1).$$

Therefore

$$\ln(d(Tx,Ty)+1) \leq \ln(\ln(d(x,y)+1)+1) = \frac{\ln(\ln(d(x,y)+1)+1)}{\ln(d(x,y)+1)}\ln(d(x,y)+1).$$

Define mappings $\psi : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ by

$$\psi(x) = \ln(x+1) \text{ and } \beta(x) = \begin{cases} \frac{\psi(x)}{x}, & \text{if } x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and ψ is positive in $(0,\infty)$ with $\psi(0) = 0$ and also $\psi(x) < x$. Moreover, we obtain that $\beta \in \mathcal{F}$ and

$$\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$.
Define $\alpha, \eta : C(I) \times C(I) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t),y(t)) \geq 0, t \in I \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} \frac{1}{2}, & \xi(x(t),y(t)) \geq 0, t \in [0,1] \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ such that $\alpha(x, y) \geq \eta(x, y)$. It follows that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. This yields

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y).$$

Therefore T is an α-η-ψ-Geraghty contraction type mapping. Using (iv), for each $x \in C(I)$ such that $\alpha(x, Tx) \geq \eta(x, Tx)$, we obtain that $\xi(Tx(t), T^2x(t)) \geq 0$. This implies that $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Let $x, y \in C(I)$ such that $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$. Thus

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), Ty(t)) \geq 0 \text{ for all } t \in I.$$

By applying (iii), we obtain that $\xi(x(t), Ty(t)) \geq 0$ and so $\alpha(x, Ty) \geq \eta(x, Ty)$. It follows that T is triangular α-orbital admissible with respect to η. Using (ii), there exists $x_1 \in C(I)$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$. This implies that $\alpha(x_{n(k)}, x) \geq \eta(x_{n(k)}, x)$. Therefore all assumptions in Theorem 3.2 are satisfied. Hence T has a fixed point in $C(I)$. It follows that there exists $x^* \in C(I)$ such that $Tx^* = x^*$ is a solution of (4.1). □

Corollary 4.2. *Assume that the following conditions hold:*

(i) $f : [0, 1] \times \mathbb{R} \to [0, \infty)$ *is continuous and nondecreasing;*

(ii) *for all* $t \in [0, 1]$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *we have*

$$|f(t, a) - f(t, b)| \leq 8 \ln(|a - b| + 1);$$

(iii) *there exists* $x_1 \in C([0, 1])$ *such that for all* $t \in [0, 1]$, *we have*

$$x_1(t) \leq \int_0^1 G(t, s) f(s, x_1(s)) ds.$$

Then (4.1) *has a solution in* $C^2([0, 1])$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a, b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

By the analogous proof as in Theorem 4.1, we obtain that (4.1) has a solution. □

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