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Attractive points and convergence theorems of generalized hybrid mapping

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Abstract

In this paper, by means of the concept of attractive points of a nonlinear mapping, we prove strong convergence theorem of the Ishikawa iteration for an $(\alpha, \beta)-$generalized hybrid mapping in a uniformly convex Banach space, and obtain weak convergence theorem of the Ishikawa iteration for such a mapping in a Hilbert space.

Keywords: Attractive points, generalized hybrid mapping, Ishikawa iteration, Mann iteration, Xu's inequality.
2010 MSC: 47H10, 54H25, 49J40, 47H05, 47H04, 65J15, 47H10.

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1. Introduction

Let E be a Banach space with the norm $\|\cdot\|$ and let K be a nonempty subset of E. In 2010, Kocourek, Takahashi and Yao [5] firstly introduced the concept of the generalized hybrid mapping, which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. A mapping $T : K \to K$ is called $(\alpha, \beta)-$*generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in K$, where $\mathbb{R}$ is the set of real numbers. T is said to be *nonexpansive* if T is $(1, 0)-$generalized hybrid; T is called *hybrid* (Takahashi [10]) if T is $(\frac{3}{2}, \frac{1}{2})-$generalized hybrid, i.e.

$$3\|Tx - Ty\|^2 \le \|x - Ty\|^2 + \|Tx - y\|^2 + \|x - y\|^2 \quad \forall x, y \in K;$$

T is called *nonspreading* (Kohsaka and Takahashi [6]) if T is $(2, 1)-$generalized hybrid, i.e.

$$2\|Tx - Ty\|^2 \le \|x - Ty\|^2 + \|Tx - y\|^2 \quad \forall x, y \in K.$$

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Baillon [1] proved the first nonlinear ergodic theorem: suppose that K is a nonempty closed convex subset of Hilbert space E and $T : K \to K$ is nonexpansive mapping such that $F(T) \neq \emptyset$, then $\forall x \in K$, the Cesàro means

$$T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x$$

weakly converges to a fixed point of T.

Bruck [2, 3] studied the property of Cesàro means for nonexpansive mapping in uniformly convex Banach space. Takahashi and Yao [11] proved the nonlinear ergodic theorem for both hybrid and nonspreading mappings in a Hilbert space. Kocourek, Takahashi and Yao [5] showed that both the nonlinear ergodic theorem and the weak convergence theorem of the Mann iteration for $(\alpha, \beta)-$generalized hybrid mapping. The Mann iteration is the original definition of Mann [7] for a nonexpansive mapping T,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n \ \ \{\alpha_n\} \subset (0, 1),\ x_1 \in K.$$

Takahashi and Takeuchi [12] obtained the nonlinear ergodic theorem without convexity for $(\alpha, \beta)-$generalized hybrid mappings. Hojo and Takahashi [4] showed the strong convergence of the Halpern iteration of Cesàro means for $(\alpha, \beta)-$generalized hybrid mapping T under some proper conditions,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n \ \ \{\alpha_n\} \subset (0, 1),\ u, x_1 \in K.$$

In this paper, we will deal with strong and weak convergence of the Ishikawa iteration for finding attractive points of $(\alpha, \beta)-$generalized hybrid mappings under some conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$,

$$\begin{cases} x_{n+1} & = \alpha_n x_n + (1 - \alpha_n)T y_n \\ y_n & = \beta_n x_n + (1 - \beta_n)T x_n \end{cases} \tag{1.2}$$

Our results obviously develop and complement the corresponding ones of Kocourek, Takahashi and Yao [5], Takahashi and Yao [11] , Takahashi and Takeuchi [12], Takahashi [10] and others.

2. Preliminaries and basic results

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let K be a nonempty subset of a Banach space E with the norm $\|\cdot\|$ and let T be a mapping T from K to E. A point $y \in E$ is called *an attractive point* of T if for all $x \in K$

$$\|Tx - y\| \leq \|x - y\|.$$

We denote by $A(T)$ the set of all attractive points of T, i.e.,

$$A(T) = \{y \in E; \|Tx - y\| \leq \|x - y\| \ \ \forall x \in K\}.$$

Takahashi and Takeuchi [12] used this concept and proved the closed and convex property of $A(T)$ in a Hilbert space H. For more details, see Takahashi and Takeuchi [12].

A Banach space E is said to be *uniformly convex* if for all

$$\varepsilon \in [0, 2],$$

$$\exists \delta_\varepsilon > 0$$

such that

$$\|x\| = \|y\| = 1 \ \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

The following lemmas are well-known which can be found in [13].

Lemma 2.1. (Xu [13, Theorem 2]) *Let $q > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^q + (1-\lambda)\|y\|^q - \omega_q(\lambda)g(\|x-y\|), \tag{2.1}$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \leq r\}$ and $\lambda \in [0,1]$, where $\omega_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

Note that the inequality in Lemma 2.1 is known as *Xu's inequality.*

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner produce $\langle \cdot, \cdot \rangle$. Obviously, the Xu's inequality is replaced by the following equality in a Hilbert space H, for $x, y \in H$ and $t \in \mathbb{R}$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2. \tag{2.2}$$

Lemma 2.2. *Let K be a nonempty closed and convex subset of a real uniformly convex Banach space E and let $T : K \to K$ be a $(\alpha, \beta)-$generalized hybrid mapping with $A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the Ishikawa iteration*

$$\begin{cases} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n \end{cases} \tag{2.3}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ such that

$$\liminf_{n\to\infty} \beta_n(1-\alpha_n)(1-\beta_n) > 0. \tag{2.4}$$

Then (i) the sequence $\{x_n\}$ is bounded;

(ii) the limit $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in A(T)$;

(iii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Take $u \in A(T)$.
By the definition of the attractive point, we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n(x_n - u) + (1-\alpha_n)(Ty_n - u)\| \\
&\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\|Ty_n - u\| \\
&\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\|y_n - u\| \\
&\leq \alpha_n\|x_n - u\| + (1-\alpha_n)(\beta_n\|x_n - u\| + (1-\beta_n)\|Tx_n - u\|) \\
&\leq (\alpha_n + (1-\alpha_n)(\beta_n + (1-\beta_n)))\|x_n - u\| \\
&\leq \|x_n - u\| \\
&\quad \vdots \\
&\leq \|x_1 - u\|.
\end{aligned}$$

So the sequence $\{x_n\}$ is bounded and the sequence $\{\|x_n - u\|\}$ is monotone non-increasing, and hence the limit $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in A(T)$.

Now we show (iii).
Let

$$r \geq \max_{n\in\mathbb{N}} \|x_n - u\|.$$

Then

$$\|Ty_n - u\| \leq \|y_n - u\| \leq \|x_n - u\| \leq r \text{ and } \|Tx_n - u\| \leq \|x_n - u\| \leq r.$$

It follows from Lemma 2.1($q=2$) that

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|\alpha_n(x_n-u)+(1-\alpha_n)(Ty_n-u)\|^2 \\
&\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Ty_n-u\|^2 \\
&\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|y_n-u\|^2 \\
&= \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|\beta_n(x_n-u)+(1-\beta_n)(Tx_n-u)\|^2 \\
&\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\beta_n\|x_n-u\|^2+(1-\beta_n)\|Tx_n-u\|^2 \\
&\quad -\beta_n(1-\beta_n)g(\|Tx_n-x_n\|)) \\
&\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\beta_n\|x_n-u\|^2+(1-\beta_n)\|x_n-u\|^2 \\
&\quad -\beta_n(1-\beta_n)g(\|Tx_n-x_n\|)) \\
&\le \|x_n-u\|^2-(1-\alpha_n)\beta_n(1-\beta_n)g(\|Tx_n-x_n\|).
\end{aligned} \tag{2.5}$$

Then we have

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|Tx_n-x_n\|)\le \|x_n-u\|^2-\|x_{n+1}-u\|^2,$$

and so,

$$\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n(1-\beta_n)g(\|Tx_n-x_n\|)\le \|x_1-u\|^2<+\infty.$$

From the condition ((2.4)), it follows that

$$\lim_{n\to\infty} g(\|Tx_n-x_n\|)=0.$$

By the property of the function g, we have

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

This completes the proof. □

When $\alpha_n=0$ for all n, the following conclusions hold obviously.

Corollary 2.3. *Let K be a nonempty closed and convex subset of a real uniformly convex Banach space E and let $T:K\to K$ be a $(\alpha,\beta)-$generalized hybrid mapping with $A(T)\ne\emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the following iteration*

$$x_{n+1}=T(\beta_n x_n+(1-\beta_n)Tx_n) \tag{2.6}$$

where the sequence $\{\beta_n\}$ in $(0,1)$ such that

$$\liminf_{n\to\infty}\beta_n(1-\beta_n)>0. \tag{2.7}$$

Then (i) the sequence $\{x_n\}$ is bounded;

(ii) the limit $\lim_{n\to\infty}\|x_n-u\|$ exists for each $u\in A(T)$;

(iii) $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

3. Strongly Convergent Theorems

Let K be a nonempty subset of a Banach space E. A mapping $T:K\to K$ is said to satisfy *Condition I* if there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0, f(r)>0$ for $r\in(0,\infty)$ such that

$$\|x-Tx\|\ge f(d(x,A(T))) \text{ for all } x\in K,$$

where $d(x,A(T))=\inf\{\|x-y\|; y\in A(T)\}$. This concept was introduced by Senter and Dotson [9] and the examples of mappings that satisfy Condition I was given.

Theorem 3.1. *Let K be a nonempty closed and convex subset of a uniformly convex Banach space E and and let $T : K \to K$ be a $(\alpha, \beta)-$generalized hybrid mapping with $A(T) \neq \emptyset$ and satisfying Condition I. Suppose that the sequence $\{x_n\}$ is defined by the Ishikawa iteration*

$$\begin{cases} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ such that

$$\liminf_{n\to\infty} \beta_n(1-\alpha_n)(1-\beta_n) > 0. \tag{3.2}$$

Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

Proof. It follows from Lemma 2.2 that the sequence $\{x_n\}$ is bounded and

$$\|x_{n+1} - u\| \leq \|x_n - u\| \text{ for each } u \in A(T) \text{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.3}$$

Then Condition I implies $\lim_{n\to\infty} f(d(x_n, A(T))) = 0$, and hence

$$\lim_{n\to\infty} d(x_n, A(T)) = 0. \tag{3.4}$$

Next we show that the sequence $\{x_n\}$ is a Cauchy sequence of E. In fact, for any $n, m \in \mathbb{N}$, without loss of generality, we may set $m > n$, then $\|x_m - u\| \leq \|x_n - u\|$ for each $u \in A(T)$ by ((3.3)), and so

$$\|x_n - x_m\| \leq \|x_n - u\| + \|u - x_m\| \leq 2\|x_n - u\|. \tag{3.5}$$

Since u is arbitrary, then we may take the infimum for u in ((3.5)),

$$\|x_n - x_m\| \leq 2\inf\{\|x_n - u\|; u \in A(T)\} = 2d(x_n, A(T)).$$

From ((3.4)), it follows that as $\lim_{n\to\infty} \|x_n - x_m\| = 0$, which means that $\{x_n\}$ is a Cauchy sequence. So there exists $z \in E$ such that

$$\lim_{n\to\infty} \|x_n - z\| = 0.$$

By ((3.3)), we have

$$\lim_{n\to\infty} \|Tx_n - z\| = 0.$$

Now we prove $z \in A(T)$. In fact, it follows from the definition of $(\alpha, \beta)-$generalized hybrid mapping that for all $x \in K$,

$$\alpha\|Tx_n - Tx\|^2 + (1-\alpha)\|x_n - Tx\|^2 \leq \beta\|Tx_n - x\|^2 + (1-\beta)\|x_n - x\|^2. \tag{3.6}$$

Let $n \to \infty$ in ((3.6)). Then by the continuity of the norm $\|\cdot\|$ and the function $g(t) = t^2$, we have

$$\alpha\|z - Tx\|^2 + (1-\alpha)\|z - Tx\|^2 \leq \beta\|z - x\|^2 + (1-\beta)\|z - x\|^2,$$

and hence

$$\|z - Tx\| \leq \|z - x\| \text{ for all } x \in K.$$

So $z \in A(T)$ and $\lim_{n\to\infty} \|x_n - z\| = 0$. The proof is completed. □

A mapping $T : K \to E$ is said to be *demicompact* (Petryshyn [8]) provided whenever a sequence $\{x_n\} \subset K$ is bounded and the sequence $\{x_n - Tx_n\}$ strongly converges, then there is a subsequence $\{x_{n_k}\}$ which strongly converges.

Theorem 3.2. *Let K be a nonempty closed and convex subset of a uniformly convex Banach space E and and let $T : K \to K$ be $(\alpha, \beta)-$generalized hybrid and demicompact with $A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the Ishikawa iteration ((3.1)) and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy ((3.2)). Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.*

Proof. It follows from Lemma 2.2 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.7}$$

Then the demicompactness of T implies there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $z \in E$ such that

$$\lim_{k\to\infty} \|x_{n_k} - z\| = 0. \tag{3.8}$$

By ((3.7)), we also have

$$\lim_{k\to\infty} \|Tx_{n_k} - z\| = 0.$$

From the definition of $(\alpha, \beta)-$generalized hybrid mapping, it follows that for all $x \in K$,

$$\alpha\|Tx_{n_k} - Tx\|^2 + (1-\alpha)\|x_{n_k} - Tx\|^2 \leq \beta\|Tx_{n_k} - x\|^2 + (1-\beta)\|x_{n_k} - x\|^2. \tag{3.9}$$

Let $k \to \infty$ in ((3.9)). Then by the continuity of the norm $\|\cdot\|$ and the function $g(t) = t^2$, we have

$$\|z - Tx\| \leq \|z - x\| \text{ for all } x \in K.$$

So $z \in A(T)$. Since $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in A(T)$ by Lemma 2.2 (ii), then we have

$$\lim_{n\to\infty} \|x_n - z\| = 0.$$

The proof is completed. □

The condition (3.2) contains $\alpha_n \equiv 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ as special cases. So the following result is obtained easily.

Corollary 3.3. *Let K be a nonempty closed and convex subset of a real uniformly convex Banach space E and let $T : K \to K$ be a $(\alpha, \beta)-$generalized hybrid mapping with $A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the following iteration*

$$x_{n+1} = T(\beta_n x_n + (1-\beta_n)Tx_n) \tag{3.10}$$

where the sequence $\{\beta_n\}$ in $(0, 1)$ such that

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0. \tag{3.11}$$

Assume that T either satisfies Condition I or is demicompact. Then the sequence $\{x_n\}$ converges strongly to an attractive point z of T.

4. Weakly Convergent Theorems

Let $\{x_n\}$ is a sequence in E, then $x_n \rightharpoonup x$ will denote weak convergence of the sequence $\{x_n\}$ to x.

Theorem 4.1. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $T : K \to K$ be a $(\alpha, \beta)-$generalized hybrid mapping with $A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the Ishikawa iteration*

$$\begin{cases} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \end{cases} \tag{4.1}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ such that

$$\liminf_{n\to\infty} \beta_n(1-\alpha_n)(1-\beta_n) > 0. \tag{4.2}$$

Then the sequence $\{x_n\}$ converges weakly to an attractive point z of T.

Proof. It follows from Lemma 2.2 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - u\| \text{ exists for each } u \in A(T) \text{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.3}$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$. We claim $z \in A(T)$. In fact, it follows from the definition of $(\alpha, \beta)-$generalized hybrid mapping that for all $x \in K$,

$$\alpha\|Tx_{n_k} - Tx\|^2 + (1-\alpha)\|x_{n_k} - Tx\|^2 \leq \beta\|Tx_{n_k} - x\|^2 + (1-\beta)\|x_{n_k} - x\|^2. \tag{4.4}$$

Then we have

$$\begin{aligned} &\alpha(\|x_{n_k} - Tx\|^2 + 2\langle x_{n_k} - Tx, Tx_{n_k} - x_{n_k}\rangle + \|Tx_{n_k} - x_{n_k}\|^2) + (1-\alpha)\|x_{n_k} - Tx\|^2 \\ &= \alpha\|Tx_{n_k} - Tx\|^2 + (1-\alpha)\|x_{n_k} - Tx\|^2 \\ &\leq \beta\|Tx_{n_k} - x\|^2 + (1-\beta)\|x_{n_k} - x\|^2 \\ &\leq \beta(\|Tx_{n_k} - x_{n_k}\|^2 + 2\langle Tx_{n_k} - x_{n_k}, x_{n_k} - x\rangle + \|x_{n_k} - x\|^2) + (1-\beta)\|x_{n_k} - x\|^2. \end{aligned}$$

Let $k \to \infty$. Then by ((4.3)) ($\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$), we have

$$\limsup_{k\to\infty} \|x_{n_k} - Tx\|^2 \leq \limsup_{k\to\infty} \|x_{n_k} - x\|^2. \tag{4.5}$$

Since $x_{n_k} \rightharpoonup z$ and

$$\|x_{n_k} - x\|^2 = \|x_{n_k} - Tx\|^2 + 2\langle x_{n_k} - Tx, Tx - x\rangle + \|Tx - x\|^2,$$

then

$$\limsup_{k\to\infty} \|x_{n_k} - x\|^2 = \limsup_{k\to\infty} \|x_{n_k} - Tx\|^2 + 2\langle z - Tx, Tx - x\rangle + \|Tx - x\|^2.$$

Since $\|z - x\|^2 = \|z - Tx\|^2 + 2\langle z - Tx, Tx - x\rangle + \|Tx - x\|^2$, we have

$$2\langle z - Tx, Tx - x\rangle + \|Tx - x\|^2 = \|z - x\|^2 - \|z - Tx\|^2,$$

and hence

$$\limsup_{k\to\infty} \|x_{n_k} - x\|^2 = \limsup_{k\to\infty} \|x_{n_k} - Tx\|^2 + \|z - x\|^2 - \|z - Tx\|^2.$$

From ((4.5)), it follows that

$$\begin{aligned} \limsup_{k\to\infty} \|x_{n_k} - Tx\|^2 &\leq \limsup_{k\to\infty} \|x_{n_k} - x\|^2 \\ &= \limsup_{k\to\infty} \|x_{n_k} - Tx\|^2 + \|z - x\|^2 - \|z - Tx\|^2, \end{aligned}$$

and so

$$\|z - Tx\| \leq \|z - x\| \text{ for all } x \in K.$$

That is, $z \in A(T)$.

Now we prove $\{x_n\}$ converges weakly to z. Suppose not, then there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which weakly converges to some $y \neq z$. Again in the same way, we have $y \in A(T)$.

From ((4.3)), it follows that both $\lim_{n\to\infty} \|x_n - z\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exist. By the elementary properties in Hilbert space, we easily obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\|^2 &= \lim_{k\to\infty} \|x_{n_k} - z\|^2 \\
&= \lim_{k\to\infty} (\|x_{n_k} - y\|^2 + 2\langle x_{n_k} - y, y - z\rangle + \|y - z\|^2) \\
&= \lim_{k\to\infty} \|x_{n_k} - y\|^2 + 2\langle z - y, y - z\rangle + \|y - z\|^2 \\
&= \lim_{k\to\infty} \|x_{n_k} - y\|^2 - \|y - z\|^2 \\
&= \lim_{n\to\infty} \|x_n - y\|^2 - \|y - z\|^2 \\
&= \lim_{i\to\infty} \|x_{n_i} - y\|^2 - \|y - z\|^2 \\
&= \lim_{i\to\infty} (\|x_{n_i} - z\|^2 + 2\langle x_{n_i} - z, z - y\rangle + \|z - y\|^2) - \|y - z\|^2 \\
&= \lim_{i\to\infty} \|x_{n_i} - z\|^2 - 2\|z - y\|^2 \\
&= \lim_{n\to\infty} \|x_n - z\|^2 - 2\|z - y\|^2,
\end{aligned}$$

which implies $z = y$, a contradiction. Thus, $\{x_n\}$ converges weakly to an attractive point z of T. □

Take $\alpha_n \equiv 0$. We also obtained easily the following.

Corollary 4.2. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $T : K \to K$ be a $(\alpha, \beta)-$generalized hybrid mapping with $A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the following iteration*

$$x_{n+1} = T(\beta_n x_n + (1 - \beta_n)Tx_n) \tag{4.6}$$

where the sequence $\{\beta_n\}$ in $(0, 1)$ such that

$$\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0. \tag{4.7}$$

Then the sequence $\{x_n\}$ converges weakly to an attractive point z of T.

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