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Asymptotic aspect of Jensen and Jensen type functional equations in multi-normed spaces

Zhihua Wang*

School of Science, Hubei University of Technology, Wuhan, Hubei 430068, P.R. China.

Abstract

In this paper, we investigate the Hyers-Ulam stability of additive functional equations of two forms: of "Jensen" and "Jensen type" in the framework of multi-normed spaces. We therefore provide a link between multi-normed spaces and functional equations. More precisely, we establish the Hyers-Ulam stability of functional equations of these types for mappings from Abelian groups into multi-normed spaces. We also prove the stability on a restricted domain and discuss an asymptotic behavior of functional equations of these types in the framework of multi-normed spaces.

Keywords: Hyers-Ulam stability, Jensen and Jensen type functional equations, Multi-normed spaces, Asymptotic behavior.
2010 MSC: 39B52, 58K55.

1. Introduction and preliminaries

The stability problem of functional equations was raised by Ulam [22] in 1940. He posed the following problem: under what conditions does there exist an additive mapping near an approximately additive mapping? In the next year, this problem was solved by Hyers [11] in the case of Banach space. Later, Hyers' result was generalized by Aoki [4] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. Găvruta [10] provided a further generalization of Rassias' result. During the last decades several stability problems for various functional equations have been investigated by many authors for mappings with more general domains and ranges [2, 5, 6, 9, 12, 13, 14, 15, 18, 19, 23]. These results have many applications in information theory, physics, economic theory and social and behavioral sciences [1, 3].

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Email address: matwzh2000@126.com (Zhihua Wang)

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Rassias et al. [21] introduced the following additive functional equations of two forms: of "Jensen" and "Jensen type"

$$f(x+y)+f(x-y)=2f(x), \tag{1.1}$$
$$f(x+y)-f(x-y)=2f(y). \tag{1.2}$$

Equations (1.1) and (1.2) are also called additive functional equations of the first and second form, respectively. In [21], they solved the Hyers-Ulam stability of equations (1.1) and (1.2) for mappings from real normed space into real Banach space, and discussed the stability on a restricted(unbounded) domain, and these results were applied to the study of an interesting asymptotic behavior of functional equations of these types.

The notion of multi-normed space was introduced by Dales and Polyakov [8] (or see [7, 16, 17]). This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples were given in [7, 8]. In 2007, stability of mappings on multi-normed spaces was given in [7], and asymptotic aspect of the quadratic functional equation in multi-normed spaces was investigated in [17]. More precisely, Moslehian, Nikodem and Popa [17] established the Hyers-Ulam stability of the quadratic functional equation for mappings from Abelian groups into multi-normed spaces. They also proved the stability on a restricted domain and applied these results to a study of an asymptotic behavior of the quadratic functional equation in the framework of multi-normed spaces.

In this paper, using some ideas from [7, 17, 21], we achieve the Hyers-Ulam stability of equations (1.1) and (1.2) for mappings from Abelian groups into multi-normed spaces. Furthermore, we also study the stability on balls under certain assumptions and discuss an asymptotic behavior of functional equations of these types in the framework of multi-normed spaces. Our results generalize those results of [21] to multi-normes spaces. We therefore provide a link between two various discipline: multi-normed spaces and functional equations. These circumstances can be applied to other significant functional equations. Throughout the paper $\mathcal{F}$ denotes a Banach space.

Following [7, 8, 17], we recall some basic facts concerning multi-normed spaces and some preliminary results.

Let $(\mathcal{E},\|\cdot\|)$ be a complex normed space, and let $k\in\mathbb{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E}\oplus\cdots\oplus\mathcal{E}$ consisting of k-tuples $(x_1,\cdots,x_k)$, where $x_1,\cdots,x_k\in\mathcal{E}$. The linear operations $\mathcal{E}^k$ are defined coordinatewise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1,2,\cdots,k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols.

Definition 1.1 ([7, 8, 17])**.** A multi-norm on $\{\mathcal{E}^k : k\in\mathbb{N}\}$ is a sequence $(\|\cdot\|_k)=(\|\cdot\|_k : k\in\mathbb{N})$ such that $\|\cdot\|_k$ is a norm on $\mathcal{E}^k$ for each $k\in\mathbb{N}$, $\|x\|_1=\|x\|$ for each $x\in\mathcal{E}$, and the following axioms are satisfied for each $k\in\mathbb{N}$ with $k\geq 2$:
(N1) $\|(x_{\sigma(1)},\cdots,x_{\sigma(k)})\|_k=\|(x_1,\cdots,x_k)\|_k$, for $\sigma\in\mathfrak{S}_k$, $x_1,\cdots,x_k\in\mathcal{E}$;
(N2) $\|(\alpha_1x_1,\cdots,\alpha_kx_k)\|_k\leq(\max_{i\in\mathbb{N}_k}|\alpha_i|)\|(x_1,\cdots,x_k)\|_k$, for $\alpha_1,\cdots,\alpha_k\in\mathbb{C}$, $x_1,\cdots,x_k\in\mathcal{E}$;
(N3) $\|(x_1,\cdots,x_{k-1},0)\|_k=\|(x_1,\cdots,x_{k-1})\|_{k-1}$, for $x_1,\cdots,x_{k-1}\in\mathcal{E}$;
(N4) $\|(x_1,\cdots,x_{k-1},x_{k-1})\|_k=\|(x_1,\cdots,x_{k-1})\|_{k-1}$, for $x_1,\cdots,x_{k-1}\in\mathcal{E}$.
In this case, we say that $((\mathcal{E}^k,\|\cdot\|_k):k\in\mathbb{N})$ is a multi-normed space.

Suppose that $((\mathcal{E}^k,\|\cdot\|_k):k\in\mathbb{N})$ is a multi-normed space, and take $k\in\mathbb{N}$. We need the following two properties of multi-norms. They can be found in [8].
(a) $\|(x,\cdots,x)\|_k=\|x\|$, for $x\in\mathcal{E}$,
(b) $\max_{i\in\mathbb{N}_k}\|x_i\|\leq\|(x_1,\cdots,x_k)\|_k\leq\sum_{i=1}^{k}\|x_i\|\leq k\max_{i\in\mathbb{N}_k}\|x_i\|$, for $x_1,\cdots,x_k\in\mathcal{E}$.

It follows from (b) that, if $(\mathcal{E},\|\cdot\|)$ is a Banach space, then $(\mathcal{E}^k,\|\cdot\|_k)$ is a Banach space for each $k\in\mathbb{N}$; in this case $((\mathcal{E}^k,\|\cdot\|_k):k\in\mathbb{N})$ is a multi-Banach space.

Lemma 1.2 ([7, 8, 17]). *Suppose that $k \in \mathbb{N}$ and $(x_1, \cdots, x_k) \in \mathcal{E}^k$. For each $j \in \{1, \cdots, k\}$, let $(x_n^j)_{n=1,2,\cdots}$ be a sequence in $\mathcal{E}$ such that $\lim\limits_{n\to\infty} x_n^j = x_j$. Then*

$$\lim_{n\to\infty}(x_n^1 - y_1, \cdots, x_n^k - y_k) = (x_1 - y_1, \cdots, x_k - y_k)$$

holds for all $(y_1, \cdots, y_k) \in \mathcal{E}^k$.

Definition 1.3 ([7, 8, 17]). Let $((\mathcal{E}^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. A sequence (x_n) in $\mathcal{E}$ is a multi-null sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_n, \cdots, x_{n+k-1})\|_k \le \varepsilon \ (n \ge n_0).$$

Let $x \in \mathcal{E}$, we say that the sequence (x_n) is multi-convergent to x in $\mathcal{E}$ and write $\lim\limits_{n\to\infty} x_n = x$ if $(x_n - x)$ is a multi-null sequence.

2. Hyers-Ulam stability of Jensen and Jensen type functional equations

In this section, we start our work with the stability results of equations (1.1) and (1.2) for mappings from Abelian groups into multi-Banach spaces.

Theorem 2.1. *Let $\delta \ge 0$, $\mathcal{E}$ be an Abelian group and $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1), \cdots, f(x_k + y_k) + f(x_k - y_k) - 2f(x_k))\|_k \le \delta \tag{2.1}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the first form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) + f(0), \cdots, f(x_k) - A(x_k) + f(0))\|_k \le \delta \tag{2.2}$$

for all $x_1, \cdots, x_k \in \mathcal{E}$

Proof. Let $x_1, \cdots, x_k \in \mathcal{E}$. Putting $y_1 = x_1, \cdots, y_k = x_k$ in (2.1), we get

$$\sup_{k\in\mathbb{N}} \|(f(2x_1) + f(0) - 2f(x_1), \cdots, f(2x_k) + f(0) - 2f(x_k))\|_k \le \delta. \tag{2.3}$$

Replacing $x_1, \cdots, x_k$ by $2^n x_1, \cdots, 2^n x_k$ and dividing by 2^{n+1} in (2.3), we obtain

$$\sup_{k\in\mathbb{N}} \|(\frac{f(2^{n+1}x_1)}{2^{n+1}} - \frac{f(2^n x_1)}{2^n} + \frac{f(0)}{2^{n+1}}, \cdots, \frac{f(2^{n+1}x_k)}{2^{n+1}} - \frac{f(2^n x_k)}{2^n} + \frac{f(0)}{2^{n+1}})\|_k \le \frac{\delta}{2^{n+1}}. \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \quad & \| \quad (\frac{f(2^{n+m}x_1)}{2^{n+m}} - \frac{f(2^n x_1)}{2^n} + f(0)(\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m}}), \cdots, \frac{f(2^{n+m}x_k)}{2^{n+m}} \\
& - \quad \frac{f(2^n x_k)}{2^n} + f(0)(\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m}}))\|_k \\
& \le \quad \delta(\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m}}), \quad \forall n, m \in \mathbb{N}.
\end{aligned} \tag{2.5}$$

This shows that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence for each fixed $x \in \mathcal{E}$. Since $\mathcal{F}$ is the complete multi-norm, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges. Define $A : \mathcal{E} \to \mathcal{F}$ by $A(x) := \lim\limits_{n\to\infty} \frac{f(2^n x)}{2^n}$. Hence for each $\varepsilon > 0$ there exists n_0 such that

$$\sup_{k\in\mathbb{N}} \|(\frac{f(2^n x)}{2^n} - A(x), \cdots, \frac{f(2^{n+k-1}x)}{2^{n+k-1}} - A(x))\|_k \le \delta$$

for all $n \geq n_0$. By property (b) of multi-norm, we obtain

$$\lim_{n\to\infty} \|\frac{f(2^n x)}{2^n} - A(x)\| = 0, \quad \forall x \in \mathcal{E}. \tag{2.6}$$

Setting $n = 0$ in (2.5), we have

$$\sup_{k\in\mathbb{N}} \|(\frac{f(2^m x_1)}{2^m} - f(x_1) + f(0)\sum_{i=1}^{m} 2^{-i}, \cdots, \frac{f(2^m x_k)}{2^m} - f(x_k) + f(0)\sum_{i=1}^{m} 2^{-i})\|_k \leq \delta \sum_{i=1}^{m} 2^{-i}$$

Letting m tend to infinity and using Lemma 1.2 and (2.6) we obtain

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) + f(0), \cdots, f(x_k) - A(x_k) + f(0))\|_k \leq \delta.$$

Let $x, y \in \mathcal{E}$. Putting $x_1 = \cdots = x_k = 2^n x, y_1 = \cdots = y_k = 2^n y$ in (2.1) and divide both sides by 2^n, we obtain

$$\|2^{-n} f(2^n(x + y)) + 2^{-n} f(2^n(x - y)) - 2 \cdot 2^{-n} f(2^n x)\| \leq 2^{-n}\delta,$$

taking the limit as $n \to \infty$ we get $A(x + y) + A(x - y) = 2A(x)$. Hence A is an additive mapping of the first form.

To prove the uniqueness of A, assume that there is another additive mapping $A' : \mathcal{E} \to \mathcal{F}$ of the first form which satisfies (2.1), then

$$\begin{aligned} \|A'(x) - A(x)\| &\leq 2^{-n}\|A'(2^n x) - A(2^n x)\| \\ &\leq 2^{-n}\|A'(2^n x) - f(2^n x) + f(0)\| + 2^{-n}\|f(2^n x) - A(2^n x) + f(0)\| \\ &\leq 2^{-n}(\delta + \delta). \end{aligned}$$

Hence $A' = A$. This proves the uniqueness of A. This completes the proof of the theorem. □

Corollary 2.2. *Let $\delta \geq 0$, $\mathcal{E}$ be an Abelian group, and let $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1), \cdots, f(x_k + y_k) + f(x_k - y_k) - 2f(x_k))\|_k \leq \delta$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the first form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \delta + \|f(0)\|$$

for all $x_1, \cdots, x_k \in \mathcal{E}$.

Theorem 2.3. *Let $\delta \geq 0$, $\mathcal{E}$ be an Abelian group and $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(f(x_1 + y_1) - f(x_1 - y_1) - 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \leq \delta \tag{2.7}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the second form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) - f(0), \cdots, f(x_k) - A(x_k) - f(0))\|_k \leq \delta \tag{2.8}$$

for all $x_1, \cdots, x_k \in \mathcal{E}$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\delta \geq 0$, $\mathcal{E}$ be an Abelian group, and let $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(f(x_1+y_1) - f(x_1-y_1) - 2f(y_1), \cdots, f(x_k+y_k) - f(x_k-y_k) - 2f(y_k))\|_k \leq \delta$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the second form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \frac{3}{2}\delta$$

for all $x_1, \cdots, x_k \in \mathcal{E}$.

Proof. Setting $x_1 = \cdots = x_k = y_1 = \cdots = y_k = 0$ in (2.7), we have

$$\|f(0)\| = \sup_{k\in\mathbb{N}} \|(f(0), \cdots, f(0))\|_k \leq \frac{\delta}{2}.$$

According to Theorem 2.3, there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the second form such that (2.8) is satisfied. We obtain

$$\begin{aligned}\sup_{k\in\mathbb{N}} \| & (f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \\ \leq & \sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1) - f(0), \cdots, f(x_k) - A(x_k) - f(0))\|_k + \|(f(0), \cdots, f(0)\|_k \\ \leq & \delta + \|f(0)\|\end{aligned}$$

for $x_1, \cdots, x_k \in \mathcal{E}$. Hence, we get

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \frac{3}{2}\delta.$$

This completes the proof of the corollary. □

3. Stability on a restricted(bounded) domain

Throughout this section, we denote by $B_r(\mathcal{E}^k)$ the closed ball in $\mathcal{E}^k$ of radius r around the origin. we study the stability results of equations (1.1) and (1.2) on balls under certain assumptions.

Theorem 3.1. *Let $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space, $p > 1$, $r > 0$, $\phi_k : \mathcal{E}^{2k} \to [0, \infty) (k \in \mathbb{N})$ be a family of functions such that $\sup_{k\in\mathbb{N}} \phi_k(\mathbf{x}, \mathbf{x}) < \infty$ and $\phi_k(\frac{\mathbf{x}}{\mathbf{2}}, \frac{\mathbf{y}}{\mathbf{2}}) \leq \frac{1}{2^p}\phi_k(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in B_r(\mathcal{E}^k)$ and $k \in \mathbb{N}$. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying $f(0) = 0$ and*

$$\|(f(x_1+y_1) + f(x_1-y_1) - 2f(x_1), \cdots, f(x_k+y_k) + f(x_k-y_k) - 2f(x_k))\|_k \leq \phi_k(\mathbf{x}, \mathbf{y}) \tag{3.1}$$

for all $k \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in B_r(\mathcal{E}^k)$ with $\mathbf{x} \pm \mathbf{y} \in B_r(\mathcal{E}^k)$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the first form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \frac{\sup_{k\in\mathbb{N}} \phi_k(\mathbf{x}, \mathbf{x})}{2^{2p-1} - 2^p} \tag{3.2}$$

where $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$.

Proof. Let $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$. Replacing $\mathbf{x}, \mathbf{y}$ in (3.1) by $\frac{\mathbf{x}}{2}$, we obtain

$$\|(f(x_1) - 2f(\frac{x_1}{2}), \cdots, f(x_k) - 2f(\frac{x_k}{2}))\|_k \leq \phi_k(\frac{\mathbf{x}}{2}, \frac{\mathbf{x}}{2}) \leq \frac{1}{2^p}\phi_k(\mathbf{x}, \mathbf{x}). \tag{3.3}$$

Replacing $\mathbf{x}$ by $\frac{\mathbf{x}}{2^n} \in B_r(\mathcal{E}^k)$ and multiplying with 2^n in (3.3), we get

$$\|(2^n f(\frac{x_1}{2^n}) - 2^{n+1} f(\frac{x_1}{2^{n+1}}), \cdots, 2^n f(\frac{x_k}{2^n}) - 2^{n+1} f(\frac{x_k}{2^{n+1}}))\|_k \leq (\frac{1}{2^{p-1}})^n \frac{1}{2^p}\phi_k(\mathbf{x}, \mathbf{x}).$$

Using the same reasoning as in the proof of Theorem 2.1 we conclude that the sequence $\{2^n f(2^{-n}x)\}$ is Cauchy and so is convergent in the complete multi-norm $\mathcal{F}$. In addition, the mapping

$$\tilde{A}(x) := \lim_{n\to\infty} 2^n f(2^{-n}x), \quad x \in B_r(\mathcal{E})$$

satisfies

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \tilde{A}(x_1), \cdots, f(x_k) - \tilde{A}(x_k))\|_k \leq \frac{\sup_{k\in\mathbb{N}} \phi_k(\mathbf{x}, \mathbf{x})}{2^{2p-1} - 2^p},$$

where $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$.

Let x be an arbitrary fixed element of $B_r(\mathcal{E})$. Because of $\frac{x}{2} \in B_r(\mathcal{E})$, we have

$$2\tilde{A}(\frac{x}{2}) = \lim_{n\to\infty} 2^{n+1} f(2^{-n-1}x) = \lim_{n\to\infty} 2^n f(2^{-n}x) = \tilde{A}(x).$$

Therefore $2^{n+m}\tilde{A}(\frac{x}{2^{n+m}}) = \tilde{A}(x)$ and so the mapping $A : \mathcal{E} \to \mathcal{F}$ is given by $A(x) := 2^n\tilde{A}(2^{-n}x)$, where n is the least non-negative integer such that $2^{-n}x \in B_r(\mathcal{E})$, is well-defined. It is easy to show that

$$A(x) := \lim_{n\to\infty} 2^n f(2^{-n}x) \quad (x \in \mathcal{E})$$

and $\tilde{A}|_{B_r(\mathcal{E})} = A$.

Let $x, y \in \mathcal{E}$. There is a large enough n such that $2^{-n}x, 2^{-n}y, 2^{-n}(x + y), 2^{-n}(x - y) \in B_r(\mathcal{E})$. Putting $x_1 = \cdots = x_k = 2^{-n}x, y_1 = \cdots = y_k = 2^{-n}y$ in (3.1) and multiplying both sides with 2^n, we obtain

$$\begin{aligned} \| \quad & 2^n f(2^{-n}(x + y)) + 2^n f(2^{-n}(x - y)) - 2 \cdot 2^n f(2^{-n}x)\| \\ \leq \quad & (\frac{1}{2^{p-1}})^n \frac{1}{2^p}\phi_k(x, \cdots, x, y, \cdots, y), \end{aligned}$$

whence, by taking the limit as $n \to \infty$, we get $A(x+y)+A(x-y) = 2A(x)$. Hence A is an additive mapping of the first form. Uniqueness of A can be proved by using the strategy used in the proof of Theorem 2.1. □

Corollary 3.2. *Let $\delta > 0$, $p > 1$, $r < 1$, $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space and $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} \|(f(x_1 + y_1) + f(x_1 - y_1) \quad &- \quad 2f(x_1), \cdots, f(x_k + y_k) + f(x_k - y_k) - 2f(x_k))\|_k \\ &\leq \quad \delta\|x_1\|^{\frac{p}{2k}} \cdots \|x_k\|^{\frac{p}{2k}}\|y_1\|^{\frac{p}{2k}} \cdots \|y_k\|^{\frac{p}{2k}} \end{aligned}$$

for all $k \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in B_r(\mathcal{E}^k)$ with $\mathbf{x} \pm \mathbf{y} \in B_r(\mathcal{E}^k)$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the first form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \frac{1}{2^{2p-1} - 2^p}$$

where $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$.

Proof. In Theorem 3.1, let $\phi_k(\mathbf{x}, \mathbf{y}) = \delta\|x_1\|^{\frac{p}{2k}} \cdots \|x_k\|^{\frac{p}{2k}}\|y_1\|^{\frac{p}{2k}} \cdots \|y_k\|^{\frac{p}{2k}}$. □

Theorem 3.3. *Let $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space, $p > 1$, $r > 0$, $\varphi_k : \mathcal{E}^{2k} \to [0, \infty)(k \in \mathbb{N})$ be a family of functions such that $\sup_{k\in\mathbb{N}} \varphi_k(\mathbf{x}, \mathbf{x}) < \infty$ and $\varphi_k(\frac{\mathbf{x}}{\mathbf{2}}, \frac{\mathbf{y}}{\mathbf{2}}) \le \frac{1}{2^p}\varphi_k(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in B_r(\mathcal{E}^k)$ and $k \in \mathbb{N}$. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying $f(0) = 0$ and*

$$\|(f(x_1 + y_1) - f(x_1 - y_1) - 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \le \varphi_k(\mathbf{x}, \mathbf{y}) \tag{3.4}$$

for all $k \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in B_r(\mathcal{E}^k)$ with $\mathbf{x} \pm \mathbf{y} \in B_r(\mathcal{E}^k)$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the second form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \le \frac{\sup_{k\in\mathbb{N}} \varphi_k(\mathbf{x}, \mathbf{x})}{2^{2p-1} - 2^p} \tag{3.5}$$

where $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$.

Proof. The proof is similar to the proof of Theorem 3.1. □ □

Corollary 3.4. *Let $\delta > 0$, $p > 1$, $r < 1$, $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space and $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}\|(f(x_1 + y_1) - f(x_1 - y_1) &- 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \\ &\le \delta\|x_1\|^{\frac{p}{2k}} \cdots \|x_k\|^{\frac{p}{2k}}\|y_1\|^{\frac{p}{2k}} \cdots \|y_k\|^{\frac{p}{2k}}\end{aligned}$$

for all $k \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in B_r(\mathcal{E}^k)$ with $\mathbf{x} \pm \mathbf{y} \in B_r(\mathcal{E}^k)$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the second form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \le \frac{1}{2^{2p-1} - 2^p}$$

where $\mathbf{x} = (x_1, \cdots, x_k) \in B_r(\mathcal{E}^k)$.

4. Asymptotic behavior of Jensen and Jensen type functional equations

The Hyers-Ulam stability of equations (1.1) and (1.2) on restricted(unbounded) domain is investigated, and the results are applied to the study of an interesting asymptotic behavior of those equations in the framework of multi-normed spaces.

Lemma 4.1. *Let $\delta \ge 0$, $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space and $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that (d_k) is a sequence of positive numbers and $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\|(f(x_1 + y_1) + f(x_1 - y_1) - 2f(x_1), \cdots, f(x_k + y_k) + f(x_k - y_k) - 2f(x_k))\|_k \le \delta \tag{4.1}$$

for all $k \in \mathbb{N}$ and all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$ with $\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \ge d_k$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ of the first form such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \le \frac{5\delta}{2} + \|f(0)\| \tag{4.2}$$

for all $x_1, \cdots, x_k \in \mathcal{E}$.

Proof. Fix $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k)$. Assume $\|\mathbf{x}\|_k + \|\mathbf{y}\|_k < d_k$. If $\mathbf{x} = \mathbf{y} = 0$, then we choose $\mathbf{z} = (z_1, \cdots, z_k) \in \mathcal{E}^k$ to be an element of $\mathcal{E}$ with $\|\mathbf{z}\|_k = d_k$. Otherwise, let us choose

$$\mathbf{z} = \mathbf{x} + \tfrac{d_k\mathbf{x}}{\|\mathbf{z}\|_k} \quad \text{if } \|\mathbf{x}\|_k \ge \|\mathbf{y}\|_k, \quad \mathbf{z} = \mathbf{y} + \tfrac{d_k\mathbf{y}}{\|\mathbf{y}\|_k} \quad \text{if } \|\mathbf{x}\|_k \le \|\mathbf{y}\|_k.$$

We note that $\|\mathbf{z}\|_k = \|\mathbf{x}\|_k + d_k > d_k$ if $\|\mathbf{x}\|_k \geq \|\mathbf{y}\|_k$, $\|\mathbf{z}\|_k = \|\mathbf{y}\|_k + d_k > d_k$ if $\|\mathbf{x}\|_k \leq \|\mathbf{y}\|_k$. Clearly, we see that

$$\begin{aligned} &\|\mathbf{x}-\mathbf{z}\|_k + \|\mathbf{y}+\mathbf{z}\|_k \geq d_k, && \|\mathbf{x}-\mathbf{y}\|_k + \|2\mathbf{z}\|_k \geq d_k,\\ &\|\mathbf{x}+\mathbf{z}\|_k + \|-\mathbf{y}+\mathbf{z}\|_k \geq d_k, && \|\mathbf{x}\|_k + \|\mathbf{z}\|_k \geq d_k. \end{aligned} \tag{4.3}$$

From (4.1) and (4.3), we get

$$\begin{aligned} \| & \; 2(f(x_1+y_1)+f(x_1-y_1)-2f(x_1),\cdots,f(x_k+y_k)+f(x_k-y_k)-2f(x_k))\|_k \\ \leq & \; \|(f(x_1+y_1)+f(x_1-y_1-2z_1)-2f(x_1-z_1),\cdots,f(x_k+y_k) \\ + & \; f(x_k-y_k-2z_k)-2f(x_k-z_k))\|_k \\ + & \; \|(f(x_1-y_1-2z_1)+f(x_1-y_1+2z_1)-2f(x_1-y_1),\cdots,f(x_k-y_k-2z_k) \\ + & \; f(x_k-y_k+2z_k)-2f(x_k-y_k))\|_k \\ + & \; \|(f(x_1-y_1+2z_1)+f(x_1+y_1)-2f(x_1+z_1),\cdots,f(x_k-y_k+2z_k) \\ + & \; f(x_k+y_k)-2f(x_k+z_k))\|_k \\ + & \; \|2(f(x_1+z_1)+f(x_1-z_1)-2f(x_1),\cdots,f(x_k+z_k) \\ + & \; f(x_k-z_k)-2f(x_k))\|_k. \end{aligned}$$

We get

$$\|(f(x_1+y_1)+f(x_1-y_1)-2f(x_1),\cdots,f(x_k+y_k)+f(x_k-y_k)-2f(x_k))\|_k \leq \frac{5\delta}{2}.$$

This inequality holds for all $k \in \mathbb{N}$ and all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathcal{E}$. Now the result is deduced from Theorem 2.1. □

Theorem 4.2. *Let $((\mathcal{E}^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $((\mathcal{F}^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that $f : \mathcal{E} \to \mathcal{F}$ is a mapping. Then f is additive mapping of the first form if and only if for all $k \in \mathbb{N}$*

$$\|(f(x_1+y_1)+f(x_1-y_1)-2f(x_1),\cdots,f(x_k+y_k)+f(x_k-y_k)-2f(x_k))\|_k \to 0 \tag{4.4}$$

as $\|(x_1,\cdots,x_k)\|_k + \|(y_1,\cdots,y_k)\|_k \to \infty$.

Proof. On account of (4.4) we can find all $n \in \mathbb{N}$ a sequence (d_{n_k}) such that

$$\|(f(x_1+y_1)+f(x_1-y_1)-2f(x_1),\cdots,f(x_k+y_k)+f(x_k-y_k)-2f(x_k))\|_k \leq \frac{1}{n}$$

for all $k \in \mathbb{N}$ and all $x_1,\cdots,x_k,y_1,\cdots,y_k \in \mathcal{E}$ with

$$\|(x_1,\cdots,x_k)\|_k + \|(y_1,\cdots,y_k)\|_k \geq d_{n_k}.$$

By Lemma 4.1 for all $n \in \mathbb{N}$ there exists a unique additive mapping of the first form A_n such that

$$\|f(x) - A_n(x)\| \leq \frac{5}{2n} + \|f(0)\| \tag{4.5}$$

for all $x \in \mathcal{E}$. Since $\|f(x) - A_1(x)\| \leq \frac{5}{2} + \|f(0)\|$ and $\|f(x) - A_n(x)\| \leq \frac{5}{2n} + \|f(0)\| \leq \frac{5}{2} + \|f(0)\|$, by the uniqueness of A_1 we conclude that $A_n = A_1$ for all n. Hence, by letting $n \to \infty$ in (4.5), we conclude that f is additive mapping of the first form. The reverse assertion is trivial. □

Lemma 4.3. *Let $\theta, \delta \geq 0$, $((\mathcal{E}^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space and $((\mathcal{F}^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Suppose that (d_k) is a sequence of positive numbers and $f : \mathcal{E} \to \mathcal{F}$ is a mapping satisfying*

$$\|(f(x_1+y_1)-f(x_1-y_1)-2f(y_1),\cdots,f(x_k+y_k)-f(x_k-y_k)-2f(y_k))\|_k \leq \delta \tag{4.6}$$

for all $k \in \mathbb{N}$ *and all* $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$ *with* $\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \geq d_k$*, and*

$$\|(f(-s_1) + f(s_1), \cdots, f(-s_k) + f(s_k))\|_k \leq \theta \tag{4.7}$$

for all $s_1, \cdots, s_k \in \mathcal{E}$ *with* $\|(s_1, \cdots, s_k)\|_k \geq d_k$*. Then there exists a unique additive mapping* $A : \mathcal{E} \to \mathcal{F}$ *of the second form such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - A(x_1), \cdots, f(x_k) - A(x_k))\|_k \leq \frac{3}{2}(5\delta + 2\theta) \tag{4.8}$$

for all $x_1, \cdots, x_k \in \mathcal{E}$*.*

Proof. Fix $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k)$. Assume $\|\mathbf{x}\|_k + \|\mathbf{y}\|_k < d_k$. If $\mathbf{x} = \mathbf{y} = 0$, then we choose $\mathbf{z} = (z_1, \cdots, z_k) \in \mathcal{E}^k$ to be an element of $\mathcal{E}$ with $\|\mathbf{z}\|_k = d_k$. Otherwise, let us choose

$$\mathbf{z} = \mathbf{x} + \tfrac{d_k \mathbf{x}}{\|\mathbf{z}\|_k} \ \text{ if } \|\mathbf{x}\|_k \geq \|\mathbf{y}\|_k, \quad \mathbf{z} = \mathbf{y} + \tfrac{d_k \mathbf{y}}{\|\mathbf{y}\|_k} \ \text{ if } \|\mathbf{x}\|_k \leq \|\mathbf{y}\|_k.$$

We note that $\|\mathbf{z}\|_k = \|\mathbf{x}\|_k + d_k > d_k$ if $\|\mathbf{x}\|_k \geq \|\mathbf{y}\|_k$, $\|\mathbf{z}\|_k = \|\mathbf{y}\|_k + d_k > d_k$ if $\|\mathbf{x}\|_k \leq \|\mathbf{y}\|_k$. Clearly, we see that

$$\begin{array}{ll} \|\mathbf{x} - \mathbf{z}\|_k + \|\mathbf{y} + \mathbf{z}\|_k \geq d_k, & \|\mathbf{x} - \mathbf{z}\|_k + \|\mathbf{y} - \mathbf{z}\|_k \geq d_k, \\ \|\mathbf{x} - 2\mathbf{z}\|_k + \|\mathbf{z}\|_k \geq d_k, & \|\mathbf{z}\|_k + \|\mathbf{y}\|_k \geq d_k, \end{array} \tag{4.9}$$

and $\|\mathbf{z} - \mathbf{y}\|_k \geq \|\mathbf{z}\|_k + \|\mathbf{y}\|_k = (\|\mathbf{y}\|_k + d_k) - \|\mathbf{y}\|_k = d_k$, because $\|\mathbf{z}\|_k = \|\mathbf{y}\|_k + d_k$

From (4.6), (4.7) and (4.9), we get

$$\begin{array}{rl} \| & (f(x_1 + y_1) - f(x_1 - y_1) - 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \\ \leq & \|(f(x_1 + y_1) - f(x_1 - y_1 - 2z_1) - 2f(y_1 + z_1), \cdots, f(x_k + y_k) \\ - & f(x_k - y_k - 2z_k) - 2f(y_k + z_k))\|_k \\ + & \|(f(x_1 + y_1 - 2z_1) - f(x_1 - y_1) - 2f(y_1 - z_1), \cdots, f(x_k + y_k - 2z_k) \\ - & f(x_k - y_k) - 2f(y_k - z_k))\|_k \\ + & \|(f(x_1 + y_1 - 2z_1) - f(x_1 - y_1 - 2z_1) - 2f(y_1), \cdots, f(x_k + y_k - 2z_k) \\ - & f(x_k - y_k - 2z_k) - 2f(y_k))\|_k \\ + & \|2(f(z_1 + y_1) - f(z_1 - y_1) - 2f(y_1), \cdots, f(z_k + y_k) \\ - & f(z_k - y_k) - 2f(y_k))\|_k \\ + & \|2(f(z_1 - y_1) + f(-(z_1 - y_1)), \cdots, f(z_k - y_k) + f(-(z_k - y_k)))\|_k. \end{array}$$

Thus , we get

$$\|(f(x_1 + y_1) - f(x_1 - y_1) - 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \leq 5\delta + 2\theta.$$

This inequality holds for all $k \in \mathbb{N}$ and all $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{E}$. Now the result is deduced from Theorem 2.3. □

Theorem 4.4. *Let* $((\mathcal{E}^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, and let* $((\mathcal{F}^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-Banach space. Suppose that* $f : \mathcal{E} \to \mathcal{F}$ *is a mapping. Then* f *is an additive mapping of the second form if and only if for all* $k \in \mathbb{N}$

$$\|(f(-s_1) + f(s_1), \cdots, f(-s_k) + f(s_k))\|_k \to 0 \tag{4.10}$$

and

$$\|(f(x_1 + y_1) - f(x_1 - y_1) - 2f(y_1), \cdots, f(x_k + y_k) - f(x_k - y_k) - 2f(y_k))\|_k \to 0 \tag{4.11}$$

as $\|(s_1, \cdots, s_k)\|_k \to \infty$ *and* $\|(x_1, \cdots, x_k)\|_k + \|(y_1, \cdots, y_k)\|_k \to \infty$ *hold, respectively.*

Proof. The proof is similar to the proof of Theorem 4.2 and the result follows from Lemma 4.3. □

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