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Hybrid projection algorithms for approximating fixed points of asymptotically quasi-pseudocontractive mappings

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

The purpose of this paper is to modify Ishikawa iterative process to have strong convergence without any compact assumptions for asymptotically quasi-pseudocontractive mappings in the framework of real Hilbert spaces.

Keywords: Asymptotically pseudocontractive mapping; asymptotically nonexpansive mapping; fixed point; hybrid projection algorithm.
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1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$. Assume that C is a nonempty closed convex subset of H and $T : C \to C$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T.

T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \, n \geq 1. \tag{1.1}$$

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and (1.1) holds for every $x \in C$ but $y \in F(T)$. We remark here that the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk; see [3] for more details. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex.

T is said to be *pseudocontractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

We remark here that the class of asymptotically pseudocontractive mappings was introduced by Schu; see [16] for more details.

It is clear that (1.2) is equivalent to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

The class of asymptotically pseudocontractive mappings contains properly the class of asymptotically non-expansive mappings as a subclass, which can be seen from the following example.

Example. ([15]) For $x \in [0, 1]$, define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}.$$

Then T is asymptotically pseudocontractive but it is not asymptotically nonexpansive.

$T : C \to C$ is said to be *asymptotically quasi-pseudocontractive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - p, x - p \rangle \leq k_n \|x - p\|^2, \quad \forall x \in C, \, p \in F(T). \tag{1.4}$$

It is clear that (1.4) is equivalent to

$$\|T^n x - p\|^2 \leq (2k_n - 1)\|x - p\|^2 + \|x - T^n x\|^2, \quad \forall x \in C, \, p \in F(T). \tag{1.5}$$

In 1991, Schu [16] proved the following results for asymptotically pseudocontractive mappings in the framework of Hilbert spaces.

Theorem Schu. *Let C be a nonempty closed bounded convex subset of a Hilbert space H. Let $L > 0$ and $T : C \to C$ be completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive with sequence $\{k_n\} \subset [1, \infty)$, $q_n = 2k_n - 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$, $\epsilon \leq \alpha_n \leq \beta_n \leq b$ for all $n \geq 1$ and for some $\epsilon > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$. For given $x_1 \in K$, define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping and its extensions. The first one was introduced by Mann [7], which is defined as follows:

$$\begin{cases} x_0 \in C \quad \textit{arbitrary choosen}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$.

The second one was referred to as Ishikawa iteration process [4], which is defined recursively as follows:

$$\begin{cases} x_0 \in C \quad \textit{arbitrary choosen}, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $(0,1)$.

But both (1.6) and (1.7) have only weak convergence, in general; see [2] and [19]. Reich [14] shows that, if E is a uniformly convex and has a Fréchet differentiable norm, and the sequence $\{\alpha_n\}$ is such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the process (1.6) converges weakly to a point in $F(T)$ (an extension of the results to the process (1.7) can be found in [19]). Therefore, many authors have attempted to modify (1.6) and (1.7) to have strong convergence.

In 2006, Martinez-Yanes and Xu [9] modified (1.7) to have strong convergence by hybrid projection algorithms in Hilbert spaces. To be more precise, They proved the following result.

Theorem MYX. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0,1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)T x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Recently, Qin, Su and Shang [13] improved the results of Martinez-Yanes and Xu [9] from nonexpansive mappings to asymptotically nonexpansive mappings. More precisely, They proved the following theorem.

Theorem QSS. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \leq 1-\delta$ for all n and for some $\delta \in (0,1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n z_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1-\alpha_n)[k_n^2\|z_n\|^2 - \|x_n\|^2 \\ \qquad\qquad\qquad + (k_n^2 - 1)M + 2\langle x_n - k_n^2 z_n, v\rangle]\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where M is a appropriate constant such that $M > \|v\|^2$ for each $v \in C_n$, then $\{x_n\}$ converges to $P_{F(T)}x_0$.

Very recently, Zhou [20] improved the results of Martinez-Yanes and Xu [9] from nonexpansive mappings to Lipschitz pseudo-contractions. To be more precise, he proved the following theorem.

Theorem Zhou. *Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ be a Lipschitz pseudo-contraction such that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0, 1)$ satisfying the conditions:*

(a) $\beta_n \le \alpha_n$, $\forall n \ge 0$;

(b) $\liminf_{n\to\infty} \alpha_n > 0$;

(c) $\limsup_{n\to\infty} \alpha_n \le \alpha \le \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \ge 0$, *where $L \ge 1$ is the Lipschitzian constant of T.*

Let a sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges strongly to a fixed point v of T, where $v = P_{F(T)}x_0$.

In this paper, motivated by Acedo and Xu [1], Kim and Xu [5, 6], Marino and Xu [8], Martinez-Yanes and Xu [9], Nakajo and Takahashi [10], Qin et al. [11], Qin, Cho and Zhou [12], Qin, Su and Shang [13], Su and Qin [17, 18] and Zhou [20, 21], we modify Ishikawa iterative process (1.7) to have strong convergence for asymptotically quasi-pseudocontractive mappings in the framework of Hilbert spaces without any compact assumption.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. ([8]) *Let H be a real Hilbert space. Then the following equations hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all $x, y \in H$.*

(b) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for all $t \in [0, 1]$ and $x, y \in H$.*

Lemma 1.2. *Let C be a closed convex subset of real Hilbert space H and P_C be the metric projection from H onto C (i.e., for $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$, $z = P_C x$ if and only if there holds the relations: $\langle x - z, y - z\rangle \le 0$ for any $y \in C$.*

The following lemma can be found in Zhou and Su [22], we still give the proof for the completeness of the paper.

Lemma 1.3. *Let C be a nonempty bounded closed convex subset of H and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically quasi-pseudocontractive mapping. Then $F(T)$ is a closed convex subset of C.*

Proof. From the continuity of T, we can conclude that $F(T)$ is closed.

Next, we show that $F(T)$ is convex. If $F(T) = \emptyset$, then the conclusion is always true. Let $p_1, p_2 \in F(T)$. We prove $p \in F(T)$, where $p = tp_1 + (1-t)p_2$, for $t \in (0, 1)$. Put $y_{(\alpha,n)} = (1-\alpha)p + \alpha T^n p$, where $\alpha \in (0, \frac{1}{1+L})$.

For all $w \in F(T)$, we see that

$$\begin{aligned}
&\|p - T^n p\|^2 \\
&= \langle p - T^n p, p - T^n p\rangle \\
&= \frac{1}{\alpha}\langle p - y_{(\alpha,n)}, p - T^n p\rangle \\
&= \frac{1}{\alpha}\langle p - y_{(\alpha,n)}, p - T^n p - (y_{(\alpha,n)} - T^n y_{(\alpha,n)})\rangle + \frac{1}{\alpha}\langle p - y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle \\
&= \frac{1}{\alpha}\langle p - y_{(\alpha,n)}, p - T^n p - (y_{(\alpha,n)} - T^n y_{(\alpha,n)})\rangle + \frac{1}{\alpha}\langle p - w + w - y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle \\
&\le \frac{1+L}{\alpha}\|p - y_{(\alpha,n)}\|^2 + \frac{1}{\alpha}\langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle + \frac{1}{\alpha}\langle w - y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle \\
&\le (1+L)\alpha\|p - T^n p\|^2 + \frac{1}{\alpha}\langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle + \frac{1}{\alpha}(k_n - 1)\|w - y_{(\alpha,n)}\|^2.
\end{aligned}$$

This implies that

$$\alpha[1 - (1+L)\alpha]\|p - T^n p\|^2 \le \langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)}\rangle + (k_n - 1)\|w - y_{(\alpha,n)}\|^2, \quad \forall w \in F(T). \tag{1.8}$$

Taking $w = p_i$, $i = 1, 2$ in (1.8), multiplying t and $(1-t)$ on the both sides of (1.8), respectively and adding up, we see that

$$\alpha[1 - (1+L)\alpha]\|p - T^n p\|^2 \le (k_n - 1)\|w - y_{(\alpha,n)}\|^2.$$

This shows that $T^n p - p \to 0$ as $n \to \infty$. Note that T is uniformly L-Lipschitzian. It follows that $T^{n+1}p - Tp \to 0$ as $n \to \infty$. This is, $p \in F(T)$. This completes the proof. □

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly L-Lipschitz and asymptotically quasi-pseudocontractive mapping such that $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following algorithm:*

$$\begin{cases}
x_0 \in H \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = P_{C_1}x_0, \\
y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\
z_n = (1 - \beta_n)x_n + \beta_n T^n y_n, \\
C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \le \|x_n - z\|^2 + \beta_n\theta_n - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2\}, \\
x_{n+1} = P_{C_{n+1}}x_0,
\end{cases}$$

where

$$\theta_n = 2(k_n - 1)[2k_n + 1 + (1+L)^2]\left(\sup_{z \in F(T)}\|x_n - z\|\right)^2 \to 0.$$

Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfy the restrictions:
(a) $\beta_n \le \alpha_n$, $\forall n \ge 1$;
(b) $\liminf_{n\to\infty} \alpha_n > 1$;
(c) $\limsup_{n\to\infty} \alpha_n \le \alpha < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \ge 0$.
Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. We divide the proof into five parts.

Step 1. Show that C_n is closed and convex for all $n \geq 1$.

It is obvious that C_1 is closed and convex. Assume that C_m is closed and convex. Next, we show that C_{m+1} is closed and convex for the same m. For all $z \in C_m$, we see that

$$\|z_m - z\|^2 \leq \|x_m - z\|^2 + \beta_m\theta_m - \alpha_m\beta_m(1 - 2\alpha_m - L^2\alpha_m^2)\|x_m - T^m x_m\|^2$$

is equivalent to the following inequality

$$2\langle x_m - z_m, z\rangle \leq \|x_m\|^2 - \|z_m\|^2 + \beta_m\theta_m - \alpha_m\beta_m(1 - 2\alpha_m - L^2\alpha_m^2)\|x_m - T^m x_m\|^2.$$

This shows that C_{m+1} is closed and convex. We, therefore, obtain that C_n is convex for every $n \geq 1$.

Step 2. Show that $F(T) \subset C_n$, $\forall n \geq 1$.

It is obvious that $F(T) \subset C_1$. Assume that $F(T) \subset C_m$ for some m. Next, we show that $F(T) \subset C_{m+1}$ for the same m. In view of Lemma 1.1, for all $u \in F(T) \subset C_m$, we see from (1.3) that

$$\begin{aligned}
\|z_m - u\|^2 &= \|(1-\beta_m)(x_m - u) + \beta_m(T^m y_m - u)\|^2 \\
&= (1-\beta_m)\|x_m - u\|^2 + \beta_m\|T^m y_m - u\|^2 - \beta_n(1-\beta_m)\|x_m - T^m y_m\|^2 \\
&\leq (1-\beta_m)\|x_m - u\|^2 + \beta_m\big((2k_m - 1)\|y_m - u\|^2 + \|y_m - T^m y_m\|^2\big) \\
&\quad - \beta_m(1-\beta_m)\|x_m - T^m y_m\|^2
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\|y_m - T^m y_m\|^2 \\
&= \|(1-\alpha_m)(x_m - T^m y_m) + \alpha_m(T^m x_m - T^m y_m)\|^2 \\
&= (1-\alpha_m)\|x_m - T^m y_m\|^2 + \alpha_m\|T^m x_n - T^m y_m\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq (1-\alpha_m)\|x_m - T^m y_m\|^2 + L^2\alpha_m\|x_m - y_m\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq (1-\alpha_m)\|x_m - T^m y_m\|^2 + \alpha_m(L^2\alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2.
\end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned}
\|y_m - u\|^2 &= (1-\alpha_m)\|x_m - u\|^2 + \alpha_m\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq (1-\alpha_m)\|x_m - u\|^2 + \alpha_m(2k_m - 1)\|x_m - u\|^2 + \alpha_m\|x_m - T^m x_m\|^2 \\
&\quad - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq [1 + 2\alpha_m(k_m - 1)]\|x_m - u\|^2 + \alpha_m^2\|x_m - T^m x_m\|^2.
\end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), we arrive at

$$\begin{aligned}
\|z_m - u\|^2 &\leq (1-\beta_m)\|x_m - u\|^2 + \beta_m(2k_m - 1)[1 + 2\alpha_m(k_m - 1)]\|x_m - u\|^2 \\
&\quad + (2k_m - 1)\alpha_m^2\beta_m\|x_m - T^m x_m\|^2 + \alpha_m\beta_m(L^2\alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2 \\
&\quad + \beta_m(\beta_m - \alpha_m)\|x_m - T^m y_m\|^2 \\
&\leq (1-\beta_m)\|x_m - u\|^2 + \beta_m(2k_m - 1)[1 + 2\alpha_m(k_m - 1)]\|x_m - u\|^2 \\
&\quad + 2(k_m - 1)\alpha_m^2\beta_m\|x_m - T^m x_m\|^2 + \alpha_m\beta_m(L^2\alpha_m^2 + 2\alpha_m - 1)\|x_m - T^m x_m\|^2 \\
&\quad + \beta_m(\beta_m - \alpha_m)\|x_m - T^m y_m\|^2 \\
&\leq \|x_m - u\|^2 + 2(k_m - 1)\beta_m[2\alpha_m k_m + 1 - \alpha_m + \alpha_m^2(1+L)^2]\|x_m - u\|^2 \\
&\quad + \alpha_m\beta_m(L^2\alpha_m^2 + 2\alpha_m - 1)\|x_m - T^m x_m\|^2 + \beta_m(\beta_m - \alpha_m)\|x_m - T^m y_m\|^2 \\
&\leq \|x_m - u\|^2 + 2(k_m - 1)\beta_m[2k_m + 1 + (1+L)^2]\|x_m - u\|^2 \\
&\quad + \alpha_m\beta_m(L^2\alpha_m^2 + 2\alpha_m - 1)\|x_m - T^m x_m\|^2 + \beta_m(\beta_m - \alpha_m)\|x_m - T^m y_m\|^2.
\end{aligned}$$

From the condition (a), we obtain that

$$\|z_m - u\|^2 \le \|x_m - u\|^2 + \beta_m\theta_m - \alpha_m\beta_m(1 - 2\alpha_m - L^2\alpha_m^2)\|x_m - T^m x_m\|^2.$$

Therefore, we obtain that $u \in C_{m+1}$. This concludes that $F(T) \subset C_n$, $\forall n \ge 1$.

Step 3. Show that $\{x_n\}$ is a Cauchy sequence in C.

In view of $x_n = P_{C_n}x_0$ and $P_{F(T)}x_0 \in F(T) \subset C_n$ for each $n \ge 1$, we see that

$$\|x_0 - x_n\| \le \|x_0 - P_{F(T)}x_0\|.$$

This proves that the sequence $\{x_n\}$ is bounded. From $x_n = P_{C_n}x_0$, we see that

$$\langle x_0 - x_n, x_n - y\rangle \ge 0, \quad \forall y \in C_n. \tag{2.4}$$

In view of $x_{n+1} \in C_{n+1} \subset C_n$, we see that

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned}$$

that is, $\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|$. This together with the boundedness of $\{x_n\}$ implies that $\lim_{n\to\infty}\|x_0 - x_n\|$ exists. By the construction of C_n, we see that $C_m \subset C_n$ and $x_m = P_{C_m}x_0 \in C_n$ for any positive integer $m \ge n$. From $x_n = P_{C_n}x_0$, we see that

$$\langle x_0 - x_n, x_n - x_m\rangle \ge 0. \tag{2.5}$$

It follows that

$$\begin{aligned} \|x_m - x_n\|^2 &= \|x_m - x_0 + x_0 - x_n\|^2 \\ &= \|x_m - x_0\|^2 + \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_0 - x_m\rangle \\ &\le \|x_m - x_0\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_n - x_m\rangle \\ &\le \|x_m - x_0\|^2 - \|x_0 - x_n\|^2. \end{aligned} \tag{2.6}$$

Letting $m, n \to \infty$ in (2.6), we have $\lim_{m,n\to\infty}\|x_n - x_m\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4. Show that $Tx_n - x_n \to 0$ as $n \to \infty$.

Since H is a Hilbert space and C is closed and convex, we may assume that

$$x_n \to q \in C \quad \text{as } n \to \infty. \tag{2.7}$$

Next, we show that $q = P_{F(T)}x_0$. To end this, we first show that $q \in F(T)$. By taking $m = n + 1$ in (2.6), we arrive at

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0, \tag{2.8}$$

In view of $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1}$, we obtain that

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \beta_n\theta_n - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} \|z_n - x_{n+1}\|^2 &= \|z_n - x_n + x_n - x_{n+1}\|^2 \\ &= \|z_n - x_n\|^2 + 2\langle x_n - z_n, x_{n+1} - x_n\rangle + \|x_n - x_{n+1}\|^2. \end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10) and noting that $z_n = (1 - \beta_n)x_n + \beta_n T^n y_n$, we see that

$$\beta_n^2\|x_n - T^n y_n\|^2 + 2\beta_n\langle x_n - T^n y_n, x_{n+1} - x_n\rangle \le \beta_n\theta_n - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2.$$

That is,

$$\beta_n\|x_n - T^n y_n\|^2 + 2\langle x_n - T^n y_n, x_{n+1} - x_n\rangle \le \theta_n - \alpha_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2.$$

It follows that

$$\alpha_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^n x_n\|^2 \le \theta_n - 2\langle x_n - T^n y_n, x_{n+1} - x_n\rangle.$$

From the assumptions on $\{\alpha_n\}$, we can choose $a \in (\alpha, \frac{1}{\sqrt{1+L^2}+1})$. For such chosen a, there exists a positive integer $N \ge 1$ such that $\alpha_n < a$ for all $n \ge N$. It follows that $1 - 2a - L^2a^2 > 0$. On the other hand, one can choose $b \in (0, c)$, where $c = \liminf_{n\to\infty} \alpha_n$. we obtain that $\alpha_n > b$ for n large enough. It follows that

$$b(1 - 2a - L^2a^2)\|x_n - T^n x_n\|^2 \le \theta_n + M\|x_{n+1} - x_n\|$$

for $n \ge 0$ large enough, where $M = 2\sup_{n\ge0}\{\|x_n - T^n y_n\|\}$. From (2.8), we obtain that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}\|x_n - Tx_n\| &= \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &\quad + \|T^{n+1}x_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|.\end{aligned}$$

From (2.8) and (2.11), we arrive at

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.12}$$

Step 5. Show that $x_n \to q = P_{F(T)}x_0$ as $n \to \infty$.

Notice that

$$\begin{aligned}\|q - Tq\| &\le \|q - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tq\| \\ &\le (1 + L)\|q - x_n\| + \|x_n - Tx_n\|.\end{aligned}$$

It follows from (2.7) and (2.12) that $q \in F(T)$. From (2.4), we see that

$$\langle x_0 - x_n, x_n - y\rangle \ge 0, \quad \forall y \in F(T) \subset C_n. \tag{2.13}$$

Taking the limit in (2.13), we obtain that $\langle x_0 - q, q - y\rangle \ge 0$, $\forall y \in F(T)$. In view of Lemma 1.2, we see that $q = P_{F(T)}x_0$. This completes the proof. □

Remark 2.2. Theorem 2.1 includes Theorem 4.1 of Kim and Xu [6] a as special case. It also improves the results of Kim and Xu [5] and Qin, Su and Shang [13] from asymptotically nonexpansive mappings to asymptotically quasi-pseudocontractive mappings.

For the class of Lipschitz quasi-pseudocontractive mappings, we have from Theorem 2.1 the following result.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a L-Lipschitz and quasi-pseudocontractive mapping such that $F(T) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated in the following algorithm:*

$$\begin{cases} x_0 \in H \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ z_n = (1 - \beta_n)x_n + \beta_n Ty_n, \\ C_n = \{z \in C_n : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0. \end{cases}$$

Assume that the control sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$ *satisfy the restrictions:*

(a) $\beta_n \leq \alpha_n$, $\forall n \geq 1$;

(b) $\liminf_{n\to\infty} \alpha_n > 1$;

(c) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \geq 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Remark 2.4. Comparing Corollary 2.3 with Theorem 3.6 of Zhou [20], we do not require that the mapping $I - T$ is demi-closed at zero. From the computation point of view, we remove the iterative step Q_n, see [20] for more details.

Remark 2.5. Corollary 2.3 also gives an affirmative answer to the problem proposed by Marino and Xu [8].

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